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On the Existence and Unicity of Solutions of Stochastic Integral Equations

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Let M be a local martingale, A be an adapted process with finite variation on each finite interval and H be an adapted cadlag process (i.e. H is continuous on the right and has finite left limits). We shall prove that the equation

$$X_t = H_t + \int_0^t f(s, X_{s-})\, dM_s + \int_0^t g(s, X_{s-})\, dA_s \tag{1}$$

has one and only one solution, provided the random functions f and g satisfy the properties (L) given below, i.e. a Lipschitz condition

$$|g(s, \omega, x) - g(s, \omega, y)| + |f(s, \omega, x) - f(s, \omega, y)| \leqq K\,|x-y|,$$

and two less stringent properties.

Results of this kind were proved recently by Kazamaki (3) and Protter (7) under much more restrictive continuity conditions on M and A.[1]

1. Notations

Let $(\Omega, \mathscr{F}, P)$ be a complete probability space and $(\mathscr{F}_t)_{t \geqq 0}$ an increasing family of sub-σ-fields of $\mathscr{F}$. We shall assume, as usual, that $\mathscr{F}_0$ contains all the null sets of $\mathscr{F}$, and that the family $(\mathscr{F}_t)_{t \geqq 0}$ is continuous on the right.

We shall say that a process $(X_t)_{t \geqq 0}$ is *cadlag* (corlol if one wants to stick to English) if, for almost all ω, the function $t \to X_t(\omega)$ is finite, right continuous and has finite left limits for all $t \in \mathbb{R}_+$.

Let $\mathscr{L}$ be the set of all local martingales $(M_t)_{t \geqq 0}$ with respect to the family $(\mathscr{F}_t)_{t \geqq 0}$. We assume, as usual, that M is cadlag and that $M_0 = 0$. For each $M \in \mathscr{L}$, let $[M, M]$ be the increasing process defined in [2]. Let T be a finite stopping time. And let (T_n) be an increasing sequence of stopping times such that $\lim T_n = +\infty$ a.e., and each T_n reduces the local martingales M and $M^2 - [M, M]$. We recall that T_n reduces M if and only if $M_{t \wedge T_n}$ is a uniformly integrable martingale. We

[1] Professor Protter has told us that he has independently proved the same result

have, using Doob's inequality

$$\mathbf{E}[\sup_{t \leqq T \wedge T_n} |M_t|^2] \leqq 4\mathbf{E}[M_{T\wedge T_n}^2] = 4\mathbf{E}([M, M]_{T\wedge T_n}) \leqq 4\mathbf{E}([M, M]_T),$$

and by Fatou's lemma

$$\mathbf{E}[\sup_{t \leqq T} |M_t|^2] \leqq 4\mathbf{E}([M, M]_T).$$

Let $\mathscr{V}^+$ be the set of all increasing, adapted, cadlag processes $(A_t)_{t \geqq 0}$ such that $A_0 = 0$. And let $\mathscr{V} = \mathscr{V}^+ - \mathscr{V}^+$. For any process B in $\mathscr{V}$, let us denote by $|B|_t = \int_0^t |dB_s|$ the variation of B on $[0, t]$. The process $|B|$ belongs to $\mathscr{V}^+$.

We shall talk later on of the unique solution of a stochastic integral equation. By "unique" we mean that, if X and Y are both solutions, they are indistinguishable ($P[\exists t; X_t \neq Y_t] = 0$). All our equalities of processes should also be understood in that sense.

If X is a process and T a stopping time, we shall denote by X^T the process X stopped at time T: $X_t^T = X_{t \wedge T}$.

2. Existence and Unicity of the Solution of the Stochastic Integral Equation (1)

Let f and g be two functions mapping $\mathbb{R}_+ \times \Omega \times \mathbb{R}$ to $\mathbb{R}$. We shall assume that f and g satisfy the following three properties, which we shall refer to, collectively, as the L-properties.

(L1) For all ω, s, x, y

$$|g(s, \omega, x) - g(s, \omega, y)| + |f(s, \omega, x) - f(s, \omega, y)| \leqq K\,|x - y|,$$

(L2) For all ω, and all x, the functions f and g are *left* continuous and have finite right limits in t.

(L3) For all (t, x) fixed the functions $f(t, \cdot, x)$ and $g(t, \cdot, x)$ are $\mathscr{F}_t$ measurable.

Since $\mathscr{F}_0$ contains all the null sets of $\mathscr{F}$, we might allow an exceptional subset of Ω on which (L1) and (L2) would fail to hold. However, it is more convenient to assume they hold every where.

Theorem 1. *Let $M \in \mathscr{L}$, $A \in \mathscr{V}$, and let H be an adapted cadlag process. Let f and g be two functions satisfying the L-properties. Then there exists one and only one adapted cadlag process satisfying the stochastic integral equation*

$$X_t = H_t + \int_0^t f(s, X_{s-})\, dM_s + \int_0^t g(s, X_{s-})\, dA_s. \tag{1}$$

Note 1. Throughout this paper $\int_0^t$ means $\int_{]0,t]}$.

Theorem 1 will follow from the following three lemmas. Before stating them, let us point out a few measurability properties which will be needed. Because of the uniform Lipschitz condition in x, the functions $(\omega, x) \to f(t, \omega, x)$ and $(\omega, x) \to g(t, \omega, x)$ are $\mathscr{F}_t \times \mathscr{B}(\mathbb{R})$-measurable. This implies that, if X is an adapted cadlag process, the functions $\omega \to f(t, \omega, X_{t-})$ and $\omega \to g(t, \omega, X_{t-})$ are $\mathscr{F}_t$-measurable. The processes $f(t, \omega, X_{t-})$ and $g(t, \omega, X_{t-})$ are adapted, left continuous, and have finite right limits. Therefore they are predictable and locally bounded in the sense of [2] p. 98. For any adapted cadlag process X, the stochastic integrals $\int_0^t f(s, X_{s-})\, dM_s$ and $\int_0^t g(s, X_{s-})\, dA_s$ exist, and we can always write the right term of equality (1).

Using the left continuity of f and g with respect to t, one sees that the processes $(t, \omega) \to f(t, \omega, x)$ and $(t, \omega) \to g(t, \omega, x)$ are progressively measurable; this implies that the functions $\omega \to f(t+T(\omega), \omega, x)$ and $\omega \to g(t+T(\omega), \omega, x)$ are $\mathscr{F}_{t+T}$-measurable for any finite stopping time T.

Lemma 1. *Let δ be such that $\alpha = 3K^2 \delta(4+\delta) < 1$. Assume that the local martingale M can be decomposed into*

$$M = N + B$$

where $N \in \mathscr{L}$, $B \in \mathscr{V}$ and $[N, N]_\infty + |B|_\infty + |A|_\infty \leqq \delta$. Then the Equation (1) has a unique solution on $[0, \infty)$.

Proof. Step 1. Let $Z_t = X_t - H_t$, where X_t is a solution of (1). The process Z verifies

$$Z_t = \int_0^t f'(s, Z_{s-})\, dM_s + \int_0^t g'(s, Z_{s-})\, dA_s \tag{2}$$

where

$$f'(s, \omega, x) = f(s, \omega, x + H_{s-}(\omega)),$$

$$g'(s, \omega, x) = g(s, \omega, x + H_{s-}(\omega)).$$

The functions f' and g' verify the L-properties. And the Equation (1) has a unique solution if and only if the Equation (2) has one. We can therefore assume that $H = 0$.

Step 2. Let $T_p(\omega) = \inf(t; |f(t, \omega, 0)| \geqq p$ or $|g(t, \omega, 0)| \geqq p) \wedge p$.

These stopping times T_p are finite and go to $+\infty$ a.e. as p goes to infinity. Suppose we can show that for each p the equation

$$X_t^p = \int_0^t f(s, X_{s-}^p)\, dM_s^{T_p} + \int_0^t g(s, X_{s-}^p)\, dA_s^{T_p}$$

has a unique solution on $[0, \infty)$. The processes X_t^p and X_t^{p+1} will be indistinguishable on $[0, T_p]$. The process $X_t = \sum_p X_t^p I_{\{T_{p-1} \leqq t < T_p\}}$ will then be, obviously, the unique adapted, cadlag solution of (1).

On the set $\{T_p > 0\}$ we have

$$\sup_{s \leqq T_p} |f(s, \omega, x)| \leqq p + K|x| \leqq (1+|x|)(p+k),$$

$$\sup_{s \leqq T_p} |f(s, \omega, x)|^2 \leqq (1+|x|^2)\, 2(p+K)^2.$$

The same growth conditions hold for g.

Step 3. With Step 1 and Step 2 we have reduced the problem to the study of the equation

$$X_t = \int_0^t f(s, X_{s-})\, dM_s^T + \int_0^t g(s, X_{s-})\, dA_s^T \tag{3}$$

where T is a finite stopping time, and the following inequalities hold on $\{T > 0\}$:

$$\sup_{s \leqq T} |f(s, \omega, x)| \leqq D(1+|x|),$$

$$\sup_{s \leqq T} |g(s, \omega, x)| \leqq D(1+|x|),$$

$$\sup_{s \leqq T} |f(s, \omega, x)|^2 \leqq C(1+|x|^2),$$

$$\sup_{s \leqq T} |g(s, \omega, x)|^2 \leqq C(1+|x|^2).$$

Step 3a. Existence of the solution of (3). Let us recall that we have assumed: $M = N + B$, $N \in \mathscr{L}$, $B \in \mathscr{V}$, $[N, N]_\infty + |B|_\infty + |A|_\infty \leqq \delta$. Define by recurrence the following processes

$$Y_t^0 = 0,$$

$$Y_t^n = \int_0^t f(s, Y_{s-}^{n-1})\, dN_s^T + \int_0^t f(s, Y_{s-}^{n-1})\, dB_s^T + \int_0^t g(s, Y_{s-}^{n-1})\, dA_s^T.$$

We easily see by induction that the processes Y_t^n are semimartingales (therefore cadlag). The processes $f(s, Y_{s-}^{n-1})$ being predictable and locally bounded, this is enough to build the Y_t^n by recurrence. Moreover we can prove by induction that $\mathbf{E}[\sup |Y_t^k|^2] < +\infty$ for all k. It is true for $k=0$. If it is true for $k \leqq n-1$, then we have

$$\begin{aligned} \mathbf{E}[\sup_t |Y_t^n|^2] \leqq 3\mathbf{E}\left[\sup_t \left|\int_0^t f(s, Y_{s-}^{n-1})\, dN_s^T\right|^2\right] \\ + 3\mathbf{E}\left[\sup_t \left|\int_0^t f(s, Y_{s-}^{n-1})\, dB_s^T\right|^2\right] \\ + 3\mathbf{E}\left[\sup_t \left|\int_0^t g(s, Y_{s-}^{n-1})\, dA_s^T\right|^2\right]. \end{aligned}$$

Look at the first term. The process $K_t = \int_0^t f(s, Y_{s-}^{n-1})\, dN_s^T$ is a local martingale, and $[K, K]_t = \int_0^t |f(s, Y_{s-}^{n-1})|^2\, d[N^T, N^T]_s$. Therefore

$$\begin{aligned} \mathbf{E}[\sup_t |K_t|^2] = \mathbf{E}[\sup_{s \leqq T} |K_t|^2] &\leqq 4\mathbf{E}\left[\int_0^T |f(s, Y_{s-}^{n-1})|^2\, d[N, N]_s\right] \\ &\leqq 4\mathbf{E}\left[\int_0^T C(1+|Y_{s-}^{n-1}|^2)\, d[N, N]_s\right] \\ &\leqq 4C\delta\mathbf{E}[1+\sup_s |Y_s^{n-1}|^2] < +\infty. \end{aligned}$$

Consider now the second term $\mathbf{E}\left[\sup_t \left|\int_0^t f(s, Y_{s-}^{n-1})\, dB_s^T\right|^2\right]$. We have using the Schwartz inequality.

$$\sup_t \left|\int_0^t f(s, Y_{s-}^{n-1})\, dB_s^T\right|^2 \leqq \left|\int_0^T |f(s, Y_{s-}^{n-1})|\,|dB_s^T|\right|^2 \leqq |B|_T \int_0^T |f(s, Y_{s-}^{n-1})|^2\,|dB_s^T|.$$

And therefore

$$\mathbf{E}\left[\sup_t \left|\int_0^t f(s, Y_{s-}^{n-1})\, dB_s^T\right|^2\right] \leqq C\delta^2\, \mathbf{E}\left[1+\sup_s |Y_s^{n-1}|^2\right] < +\infty.$$

One shows similarly that $\mathbf{E}\left[\sup_t \left|\int_0^t g(s, Y_{s-}^{n-1})\, dA_s^T\right|^2\right] < +\infty$. Hence we have $\mathbf{E}[\sup_t |Y_t^n|^2] < +\infty$.

Using the same method we obtain

$$\mathbf{E}[\sup_t |Y_t^{n+1} - Y_t^n|^2] \leqq 3K^2(4+\delta)\delta\, \mathbf{E}[\sup_t |Y_t^n - Y_t^{n-1}|^2].$$

Taking $\alpha = 3K^2(4+\delta)\delta$, we get by recurrence

$$\mathbf{E}[\sup_t |Y_t^{n+1} - Y_t^n|^2] \leqq \alpha^n\, \mathbf{E}[\sup_t |Y_t^1 - Y_t^0|^2] = K_1 \alpha^n.$$

For almost all ω the Y_t^n converge uniformly on $[0, \infty)$ to a finite process Y_t $\left(\text{use the fact that } \sum_n P\left(\sup_t |Y_t^{n+1} - Y_t^n| > \frac{1}{n^2}\right) \leqq \sum_n K_1 n^4 \alpha^n < +\infty\right)$. Let us take $Y_t = 0$ on the set $\{\omega;\ Y_t^n(\omega)$ does not converge uniformly$\}$. The process Y is cadlag and adapted. We still have to show

$$Y_t = \int_0^t f(s, Y_{s-})\, dN_s^T + \int_0^t f(s, Y_{s-})\, dB_s^T + \int_0^t g(s, Y_{s-})\, dA_s^T.$$

Using the same method again, we get

$$\mathbf{E}\left[\sup_t \left|Y_t - \int_0^t f(s, Y_{s-})\, dN_s^T - \int_0^t f(s, Y_{s-})\, dB_s^T - \int_0^t g(s, Y_{s-})\, dA_s^T\right|^2\right]$$
$$\leqq 2\mathbf{E}[\sup_t |Y_t - Y_t^n|^2] + 6K^2(4+\delta)\delta\, \mathbf{E}[\sup_t |Y_t - Y_t^{n-1}|^2].$$

And everything will be finished if we show that

$$\lim_n \mathbf{E}[\sup_t |Y_t - Y_t^n|^2] = 0.$$

Just note that $\sup_t |Y_t - Y_t^n|^2 \leqq \liminf_k \sup_t |Y_t^n - Y_t^{n+k}|^2$, and use Fatou's lemma to get

$$\|\sup_t |Y_t^n - Y_t|\,\|_{L^2} \leqq \liminf_k \|\sup_t |Y_t^n - Y_t^{n+k}\|_{L^2} \leqq \sqrt{K_1} \sum_{k=n}^{\infty} (\sqrt{\alpha})^k.$$

That finishes the proof of the existence of a solution of (3).

Step 3b. Unicity of the solution of Equation (3). Let Z_t be another adapted, cadlag solution of (3). For $m > 0$, take $S_m = \inf\{f;\ |Y_t| \geqq m$ or $|Z_t| \geqq m\}$. We have

$S_m > 0$ and $\lim_m S_m = +\infty$ a.e. The processes Y and Z are bounded on $[0, S_m]$ as the jumps at time S_m verify

$$\begin{aligned} |\Delta Y_{S_m}| &= |f(s, Y_{S_m^-})\, \Delta N_{S_m}^T + f(s, Y_{S_m^-})\, \Delta B_{S_m}^T + g(s, Y_{S_m^-})\, \Delta A_{S_m}^T| \\ &\leqq D(1+m)(\sqrt{\delta} + 2\delta) \end{aligned}$$

and similarly $|\Delta Z_{S_m}| \leqq D(1+m)(\sqrt{\delta}+2\delta)$.

Using again the same method, one obtains

$$\mathbf{E}[\sup_{t \leqq S_m} |Y_t - Z_t|^2] \leqq \alpha^n \mathbf{E}[\sup_{t \leqq S_m} |Y_t - Z_t|^2] \to 0$$

and Lemma 1 is proved!

Lemma 2. *Let δ satisfy $\alpha = 3K^2\delta(4+\delta) < 1$. Suppose that the local martingale M can be decomposed into*

$$M = N + B,$$

where $N \in \mathscr{L}$, $B \in \mathscr{V}$ and, for almost all ω, $|\Delta N_s|^2 \leqq \frac{\delta}{2}$, for all $s \in \mathbb{R}_+$. Then the Equation (1) has a unique solution on $[0, \infty)$.

Proof. Step 1. Let $|B|_t$ and $|A|_t$ be the variations of B and A on $[0, t]$, and set

$$D_s = [N, N]_s + |B|_s + |A|_s.$$

The time $T_1 = \inf\left(t; D_t \geqq \frac{\delta}{2}\right)$ is a strictly positive stopping time, and $D_{T_1-} \leqq \frac{\delta}{2}$. Define the following processes

$$\begin{aligned} N'_t &= N_{t \wedge T_1} \\ A'_t &= A_{t \wedge T_1} - \Delta A_{T_1} I_{\{t \geqq T_1\}}, \\ B'_t &= B_{t \wedge T_1} - \Delta B_{T_1} I_{\{t \geqq T_1\}}. \end{aligned}$$

The processes N', B' and A' verify the conditions of Lemma 1, since

$$[N', N']_\infty + |A'|_\infty + |B'|_\infty = D_{T_1-} + (\Delta N_{T_1})^2 \leqq \delta.$$

So there exists a unique cadlag solution for the equation

$$Y_t = H_t^{T_1} + \int_0^t f(s, Y_{s-})(dN'_s + dB'_s) + \int_0^t g(s, Y_{s-})\, dA'_s. \tag{4}$$

The process $X_t^1 = Y_t + [f(s, Y_{T_1-})\, \Delta B_{T_1} + g(s, Y_{T_1-})\, \Delta A_{T_1}]\, I_{\{t \geqq T_1\}}$ is a solution of

$$X_t^1 = H_t^{T_1} + \int_0^t f(s, X_{s-}^1)\, dM_s^{T_1} + \int_0^t g(s, X_{s-}^1)\, dA_s^{T_1}.$$

It is the unique solution. For, if Z_t is another solution, then

$$Z_t - [f(s, Z_{T_1-})\, \Delta B_{T_1} + g(s, Z_{T_1-})\, \Delta A_{T_1}]\, I_{\{t \geqq T_1\}}$$

is the solution of (4) and $Z_t = Y_t$ on $[0, T_1[$.

Step 2. How to go beyond the time T_1. Let $\mathscr{G}_t = \mathscr{F}_{t+T_1}$. If $K \in \mathscr{L}$ or $\mathscr{V}$, we define

$$K_t^1 = (K_{t+T_1} - K_{T_1}) I_{\{T_1 < +\infty\}}.$$

If $K \in \mathscr{L}$, and if an $(\mathscr{F}_t)$-stopping time S reduces K, it is easy to show that $R = (S - T_1)^+$ is a $(\mathscr{G}_t)$-stopping time, and that $K_{t \wedge R}^1$ is a uniformly integrable $(\mathscr{G}_t)$-martingale. This implies that N^1 is a $(\mathscr{G}_t)$-local martingale, and that $[N^1, N^1]_t = ([N, N]_{t+T_1} - N_{T_1}) I_{\{T_1 < +\infty\}}$. Define the functions f^1 and g^1 as follows

$$f^1(t, \omega, x) = f(t + T_1(\omega), \omega, x) I_{\{T_1 < +\infty\}},$$
$$g^1(t, \omega, x) = g(t + T_1(\omega), \omega, x) I_{\{T_1 < +\infty\}}.$$

These functions verify the L-properties with respect to $(\mathscr{G}_t)$.

We can apply Step 1 again, with respect to the family $(\mathscr{G}_t)$. Let

$$D_t^1 = |A^1|_t + |B^1|_t + [N^1, N^1]_t = (D_{t+T_1} - D_{T_1}) I_{\{T_1 < +\infty\}},$$

and $S_2 = \inf\left(t;\, D_t^1 \geqq \frac{\delta}{2}\right)$. There exists a unique solution of the equation

$$X_t^2 = H_{t+T_1}^{T_1+S_2} + X_{T_1} - H_{T_1} + \int_0^t f^1(s, X_{s-}^2)\, d(M^1)_s^{S_2} + \int_0^t g^1(s, X_{s-}^2)\, d(A^1)_s^{S_2}.$$

The time $U_2 = T_1 + S_2$ is an $(\mathscr{F}_t)$-stopping time. Using the fact that $(t - T_1)^+$ is a $(\mathscr{G}_t)$-stopping time, one sees easily that the process defined by

$$\begin{aligned} X_t &= X_t^1 && \text{on } [0, T_1] \\ &\; X_{t-T_1}^2 && \text{on }]T_1, T_1 + S_2] \\ &\; X_{S_2}^2 && \text{on } [T_1 + S_2, +\infty) \end{aligned}$$

is the unique adapted cadlag solution of

$$X_t = H_t^{U_2} + \int_0^t f(s, X_{s-})\, dM_s^{U_2} + \int_0^t g(s, X_{s-})\, dA_s^{U_2}$$

Step 3. One can go on, and define by recurrence

$$S_1 = T_1,\ S_2, \ldots, S_n, \ldots, U_1 = S_1,\ U_2 = S_1 + S_2, \ldots, U_n = S_1 + \cdots + S_n, \ldots$$

where $S_n = \inf\left(t;\, D_{t+U_{n-1}} - D_{U_{n-1}} \geqq \frac{\delta}{2}\right)$. For each n we shall have a unique solution of the stochastic integral equation

$$X_t = H_t^{U_n} + \int_0^t f(s, X_{s-})\, dM_s^{U_n} + \int_0^t g(s, X_{s-})\, dA_s^{U_n}.$$

Now we have $D_{U_n} \geqq \frac{n\delta}{2}$. So the time U_n is bigger than $T_n = \inf\left(t;\, D_t \geqq \frac{n\delta}{2}\right)$. As $\lim T_n = +\infty$, we have finished the proof of Lemma 2.

Lemma 3. *Let M be a local martingale, and β be an arbitrary number such that $\beta > 0$. Then M can be decomposed into*

$$M = N + B$$

where $N \in \mathscr{L}$, $B \in \mathscr{V}$ and, for almost all ω, $|\Delta M_s| \leqq 2\beta$ for all $s \in \mathbb{R}_+$.

Proof. Let $S_1, S_2, \ldots, S_n \ldots$ be the following stopping times

$$S_1 = \inf(t; |\Delta M_t| \geqq \beta)$$

$$S_n = \inf(t; t > S_{n-1}, |\Delta M_t| \geqq \beta)$$

and set

$$C_t = \sum_n \Delta M_{S_n} I_{\{t \geqq S_n\}}.$$

The process C_t belongs to $\mathscr{V}$, its variation is locally integrable. Let $\tilde{C}$ be the predictable process belonging to $\mathscr{V}$ such that $C - \tilde{C}$ is a local martingale. The martingale $H = M - C + \tilde{C}$ satisfies

$$|\Delta H_s| \leqq |\Delta M_s - \Delta C_s| + |\Delta \tilde{C}_s| \leqq \beta + |\Delta \tilde{C}_s|.$$

So we cannot have $|\Delta H_s| \geqq 2\beta$ unless $|\Delta \tilde{C}_s| \geqq \beta$. Look at the stopping times $U_1, U_2, \ldots, U_n, \ldots$

$$U_1 = \inf\{t; |\Delta \tilde{C}_s| \geqq \beta\}$$

$$U_n = \inf\{t; t > U_{n-1}, |\Delta \tilde{C}_s| \geqq \beta\}.$$

These stopping times are predictable. The process $D_t^p = \Delta H_{U_p} I_{\{t \geqq U_p\}}$ is a local martingale. The process $D_t = \sum_p D_t^p$ is in $\mathscr{V}$, and $D_t^{U_p}$ is a local martingale for each p. Therefore D_t is a local martingale (see [2]). Taking $N = H - D$ and $B = C - \tilde{C} + D$, we have the desired decomposition.

Note 2. We have, as a particular case of Theorem 1 (take $f = g$), the following result. Let Z be a semimartingale and H be an adapted cadlag process. If the random function f satisfies the L-properties, then the equation

$$X_t = H_t + \int_0^t f(s, X_{s-})\, dZ_s$$

has one and only one cadlag adapted solution.

This might even be a better way of stating Theorem 1: for, let Q be a probability equivalent to P; a P-local martingale is not in general a Q-local martingale. But a P-semimartingale Z is a Q-semimartingale and the stochastic integrals $\int Y_s\, dZ_s$ of locally bounded, predictable processes Y are the same for the two probabilities P and Q.

Theorem 1 extends trivially to systems of equations.

Theorem 2. *Let $\underset{\sim}{F}_j(t, \omega, \underset{\sim}{x}) = (F_j^1(t, \omega, \underset{\sim}{x}), \ldots, F_j^d(t, \omega, \underset{\sim}{x}))$, $1 \leqq j \leqq p$, be $\mathbb{R}^d$-valued functions mapping $\mathbb{R}_+ \times \Omega \times \mathbb{R}^d$ to $\mathbb{R}^d$. We suppose that the F_j^h verify the L-properties. Let $M^1, \ldots, M^p$ be p semimartingales and let $H^1, \ldots, H^d$ be some adapted cadlag processes. Then there exist one and only one $\mathbb{R}^d$-valued, adapted, cadlag process $\underset{\sim}{X} = (X^1, \ldots, X^d)$ such that*

$$X_t^i = H_t^i + \sum_{j=1}^p \int_0^t F_j^i(s, \underset{\sim}{X}_{s-})\, dM_s^j.$$

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