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Stopping Times for Stochastic Approximation Procedures*

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1. Introduction

The pioneering paper in the field of stochastic approximation was published in 1951 by Robbins and Monro [6]. That paper dealt with the following situation. Suppose that, for every point x belonging to the real line, a random variable $Y(x)$ can be observed. The distribution function of $Y(x)$ and the expected value of $Y(x)$, denoted by $M(x)$ and assumed to exist, are both unknown. Assuming that the equation $M(x)=\alpha$ has a unique root, denoted by $x=\theta$, it is desired to estimate θ by making observations on Y at points $x_1, x_2, x_3, \ldots$ which are generated sequentially in accordance with some definite experimental procedure in such a way that $x_n \to \theta$ in probability as $n \to \infty$.

The Robbins-Monro procedure (RM) for generating the sequence $\{x_n\}$ is to take x_1 to be any constant and define $x_2, x_3, \ldots$ in accordance with the recurrence relation

$$x_{n+1}=x_n+a_n(\alpha-y_n) \tag{1.1}$$

where y_n is a random variable whose conditional distribution for given x_n coincides with the distribution of the random variable $Y(x_n)$ and is independent of $x_1, \ldots, x_{n-1}$ and the sequence $\{a_n\}$ is a sequence of positive constants which converge to zero as $n \to \infty$.

Robbins and Monro [6] established sufficient conditions for $x_n \to \theta$ in probability as $n \to \infty$. Later researchers gave results on the rate of convergence, convergence with probability one, convergence in mean square, and asymptotic normality of the sequence x_n in both RM and various generalizations of RM—see [5, 8], or [10] for specific results and references.

In many practical situations it is desirable to terminate the successive approximation procedure when x_n is sufficiently close to θ with high probability. This paper indicates stopping times $N_{d,\gamma}$ and $T_{d,\gamma}$ which terminate RM in such a way that, for any given γ in the open interval $(0, \frac{1}{2})$,

$$\lim_{d\to 0} P(|x_{N_{d,\gamma}+1}-\theta| \leqq d)=1-2\gamma$$

and

$$\lim_{d\to 0} P[|M(x_{T_{d,\gamma}+1})-\alpha| \leqq d]=1-2\gamma.$$

The empirical behaviors of some of these stopping times have been investigated by the author [8] in a Monte Carlo study.

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2. Notation

The following conditions with $Z(x) \equiv Y(x) - M(x)$ are used in all of the theorems in this paper:

C1: γ is a positive constant less than $\frac{1}{2}$.

C2: The sequence $\{c_n\}$ is a sequence of positive constants such that $c_n n^{\lambda} \to c$ as $n \to \infty$, $c > 0$, and $0 < \lambda < \frac{1}{2}$.

C3: The sequence $\{a_n\}$ has the form $\{A/n\}$ where A is a constant such that $2A\alpha_1 > 1$.

C4: M is a Borel measurable function.

C5: For each $\varepsilon > 0$, $\inf_{\varepsilon < x - \theta < \varepsilon^{-1}} M(x) - \alpha > 0$ and $\sup_{\varepsilon < \theta - x < \varepsilon^{-1}} M(x) - \alpha < 0$.

C6: For some constants K_1 and K_2, $|M(x) - \alpha| \leqq K_1 + K_2 |x - \theta|$ for all x.

C7: $\sup_x E[|Z(x)|^2] \equiv w < \infty$.

C8: $\lim_{x \to \theta} E[|Z(x)|^2] = E[|Z(\theta)|^2] \equiv \sigma^2 > 0$.

C9: $\lim_{R \to \infty} \lim_{\varepsilon \to 0^+} \sup_{|x - \theta| \leqq \varepsilon} \int_{\{|Z(x)| > R\}} |Z(x)|^2 \, dP = 0$.

C10: For some positive constants g and α_1, if $|x - \theta| < g$, then

$$M(x) = \alpha + \alpha_1 (x - \theta) + \delta(x)$$

where $\delta(x) = o(|x - \theta|)$ as $|x - \theta| \to 0$.

C11: The distribution function of $Y(x)$, denoted by $H(\cdot|x)$, is such that, for every y, $H(y|\cdot)$ is Borel measurable.

C12: There exists $\varepsilon > 0$ such that, for every positive integer r,

$$\sup_{|x - \theta| < \varepsilon} E[|Z(x)|^r] < \infty .$$

The majority of this paper is concerned with results about the behavior of the sequence $\{x_n\}$. All of these results hold for any initial value x_1.

The procedures in the next section apply directly to the case in which the random variables $Y(x)$ are such that the α_1 referred to in condition C10 is positive. If the experimenter is actually observing random variables $Y^*(x)$ with a corresponding α_1 which is negative and he is seeking the value of x such that $E[Y^*(x)] = \alpha^*$, then when he carries out the procedures to follow he should let $Y(x) = -Y^*(x)$ and $\alpha = -\alpha^*$.

3. Sequentially Determined Bounded Length Confidence Intervals for RM

Blum [2] gave sufficient conditions, slightly weaker than C3–C7, for the sequence $\{x_n\}$ in RM to be such that

$$x_n \to \theta \quad \text{as } n \to \infty \text{ w.p.1}. \tag{3.1}$$

Then Sacks [7] showed that conditions C3–C10 are sufficient for

$$\begin{aligned} &n^{\frac{1}{2}}(x_{n+1} - \theta) \text{ in RM to be asymptotically normally} \\ &\text{distributed with mean zero and variance } A^2 \sigma^2 / (2A\alpha_1 - 1). \end{aligned} \tag{3.2}$$

Furthermore, Burkholder [3] proposed estimators of α_1 and σ^2 and obtained sufficient conditions for these estimators to converge to α_1 and σ^2 respectively with probability one. The form of Burkholder's estimator of α_1 requires that at the n-th step in RM an observation is taken not only on $Y(x_n)$ but also on $Y(x_n+c_n)$ where the sequence $\{c_n\}$ is a sequence of positive constants such that $c_n n^\lambda \to c$ as $n\to\infty$, $c>0$, and $0<\lambda<\frac{1}{2}$. Let the sequence $\{y_n''\}$ be a sequence of random variables such that the conditional distribution of y_n'' for given x_n coincides with the distribution of $Y(x_n+c_n)$ and is independent of $x_1, x_2, \ldots, x_{n-1}, y_1, y_2, \ldots, y_n, y_1'', y_2'', \ldots, y_{n-1}''$. Burkholder's results imply that, if conditions C2–C12 are satisfied, then as $n\to\infty$

$$t_n \equiv \max\left[1/2A, \sum_{j=1}^{n} (y_j''-y_j)/c_j n\right] \to \alpha_1 \quad \text{w.p.} 1 \tag{3.3}$$

and

$$s_n^2 \equiv (\tfrac{1}{2})\left\{\left[\sum_{j=1}^{n} (y_j''-\alpha)^2/n\right] + \left[\sum_{j=1}^{n} (y_j-\alpha)^2/n\right]\right\} \to \sigma^2 \quad \text{w.p.} 1. \tag{3.4}$$

If, for each n, $n^{\frac{1}{2}}(x_{n+1}-\theta)$ is normally distributed with mean zero and variance $A^2\sigma^2/(2A\alpha_1-1)$ and if K_γ is the $100(1-\gamma)$-th percentile of a standard normal random variable, then

$$P\{x_{n+1}-K_\gamma[A^2\sigma^2/n(2A\alpha_1-1)]^{\frac{1}{2}} \leqq \theta \leqq x_{n+1}+K_\gamma[A^2\sigma^2/n(2A\alpha_1-1)]^{\frac{1}{2}}\} = 1-2\gamma.$$

Thus, if a $100(1-2\gamma)\%$ confidence interval on θ of length $2d$ is desired, n cold be chosen as the smallest integer such that

$$d \geqq K_\gamma[A^2\sigma^2/n(2A\alpha_1-1)]^{\frac{1}{2}}$$

or, equivalently,

$$n \geqq K_\gamma^2 A^2\sigma^2/(2A\alpha_1-1)\,d^2. \tag{3.5}$$

Thus, when a $100(1-2\gamma)\%$ confidence interval on θ of length $2d$ is desired, the proposed stopping time for RM is $N_{d,\gamma}$ defined as the smallest positive integer n such that

$$n \geqq K_\gamma^2 A^2 s_n^2/(2At_n-1)\,d^2. \tag{3.6}$$

The principle results concerning this stopping time are:

Theorem 1. *If conditions* C1–C12 *are satisfied, then*

$$\lim_{d\to 0} N_{d,\gamma}/[K_\gamma^2 A^2\sigma^2/(2A\alpha_1-1)\,d^2] = 1 \quad \textit{w.p.1}. \tag{3.7}$$

Theorem 2. *If conditions* C1–C12 *are satisfied, then*

$$\lim_{d\to 0} P(|x_{N_{d,\gamma}+1}-\theta| \leqq d) = 1-2\gamma. \tag{3.8}$$

If $n_{d,\gamma} = K_\gamma^2 A^2\sigma^2/[(2A\alpha_1-1)\,d^2]$, then $n_{d,\gamma}$ would be the number of observations required to determine a $100(1-2\gamma)\%$ confidence interval on θ of length $2d$ if, for each n, $n^{\frac{1}{2}}(x_n-\theta)$ were exactly normally distributed with mean zero and known variance $A^2\sigma^2/(2A\alpha_1-1)$. Thus, Theorem 1 implies that $N_{d,\gamma}$ is an asymptotically efficient stopping time in the sense that $N_{d,\gamma}/n_{d,\gamma}\to 1$ as $d\to 0$ with probability one.

If $Y(x)$ is not a continuous random variable for all x, then the sequence $\{s_n^2\}$ in the definition of $N_{d,\gamma}$ should be replaced by the sequence $\{s_n^{2*}\}$ where

$$s_n^{2*} = \max(k_n, s_n^2)$$

and the sequence $\{k_n\}$ is a sequence of positive constants which converge to zero as $n\to\infty$. This substitution would not affect the results which follow and would only need to be done in order to insure that RM would not be stopped because the estimator of σ^2 was zero.

The experimenter might want to be able to determine stopping times $T_{d,\gamma}$ with the property that, for any given γ in $(0, \frac{1}{2})$,

$$\lim_{d\to 0} P[|M(x_{T_{d,\gamma}+1})-\alpha|\leqq d]=1-2\gamma. \tag{3.9}$$

Since

$$n^{\frac{1}{2}}[M(x_n)-\alpha]=n^{\frac{1}{2}}(x_n-\theta)\{[M(x_n)-\alpha]/(x_n-\theta)\}$$

and condition C10 implies

$$[M(x)-\alpha]/(x-\theta)\to\alpha_1 \quad \text{as } x\to\theta,$$

(3.1) and (3.2) imply that, if conditions C3–C10 are satisfied, then $n^{\frac{1}{2}}[M(x_n)-\alpha]$ is asymptotically normally distributed with mean zero and variance

$$\alpha_1^2 A^2 \sigma^2/(2A\alpha_1-1).$$

Thus, since conditions C2–C12 are sufficient for $t_n\to\alpha_1$ and $s_n^2\to\sigma^2$ as $n\to\infty$ with probability one, an intuitively appealing stopping time is $T_{d,\gamma}$ defined as the smallest positive integer n such that

$$K_\gamma[t_n^2 A^2 s_n^2/(2At_n-1)\,n]^{\frac{1}{2}}\leqq d$$

or, equivalently,

$$n\geqq K_\gamma^2 A^2 s_n^2/(2At_n-1)\,(d/t_n)^2. \tag{3.10}$$

In fact, under conditions C1–C12

$$\begin{aligned}\lim_{d\to 0} &P[|M(x_{T_{d,\gamma}+1})-\alpha|\leqq d]\\ &=\lim_{d\to 0} P\big[|\alpha_1(x_{T_{d,\gamma}+1}-\theta)+o(|x_{T_{d,\gamma}+1}-\theta|)|\leqq d\big]\\ &=\lim_{d\to 0} P\left(|x_{T_{d,\gamma}+1}-\theta|\leqq\frac{d}{\alpha_1}\right)=1-2\gamma\end{aligned} \tag{3.11}$$

and with probability one

$$\lim_{d\to 0} T_{d,\gamma}/[K_\gamma^2\alpha_1^2\sigma^2 A^2/(2A\alpha_1-1)\,d^2]=1. \tag{3.12}$$

The proofs of (3.11) and (3.12) are similar to the proofs of Theorem1 and Theorem 2.

4. Proofs of Theorems 1 and 2

The proof of Theorem 1 follows easily from the definition of $N_{d,\gamma}$ and the convergence of s_n^2 and t_n to σ^2 and α_1 respectively.

The key aspect of the martingale theory which is used in proving Theorem 2 is a generalization of Kolmogorov's inequality for martingales which was proved by Csőrgő [4]. Csőrgő's result is Lemma 1.

Lemma 1. *Let the sequence $\{W_k\}$ be a sequence of random variables such that $E(W_1)=0$ and, for $n\geqq 2$, $E(W_n|W_1,\ldots,W_{n-1})=0$. Let $S_n=W_1+W_2+\cdots+W_n$ for* *each* $n \geqq 1$. *If* $E(S_n^2) < \infty$ *for each* $n \geqq 1$ *and the sequence* $\{g_n\}$ *is a nonincreasing sequence of positive constants, then, for any positive integers m and n with* $m < n$ *and arbitrary* $\varepsilon > 0$,

$$P\left(\max_{m \leqq k \leqq n} g_k |S_k| \geqq \varepsilon\right) \leqq \varepsilon^{-2} \left[\sum_{k=m}^{n-1} (g_k^2 - g_{k+1}^2) E(S_k^2) + g_n^2 E(S_n^2)\right]$$

$$= \varepsilon^{-2} \left[g_m^2 \sum_{k=1}^{m} E(W_k^2) + \sum_{k=m+1}^{n} g_k^2 E(W_k^2)\right].$$

The proof of Theorem 2 is as follows: Let $N = N_{d,\gamma}$. Since the conditions of Theorems 1 and 2 are the same, it follows from Theorem 1 that

$$\lim_{d \to 0} d/[K_\gamma^2 A^2 \sigma^2/(2A\alpha_1 - 1) N]^{\frac{1}{2}} = 1 \quad \text{w.p.1}. \tag{4.1}$$

The definition of K_γ and (4.1) imply that to prove Theorem 2 it is sufficient to show that, for all x,

$$\lim_{d \to 0} P[N^{\frac{1}{2}} (2A\alpha_1 - 1)^{\frac{1}{2}} A^{-1} \sigma^{-1} (x_{N+1} - \theta) \leqq x] = \Phi(x) \tag{4.2}$$

where Φ is the distribution function of a standard normal random variable.

Since Sacks [7] has shown that, for all x,

$$\lim_{n \to \infty} P[n^{\frac{1}{2}} (2A\alpha_1 - 1)^{\frac{1}{2}} A^{-1} \sigma^{-1} (x_{n+1} - \theta) \leqq x] = \Phi(x), \tag{4.3}$$

it need only be shown that this conclusion is valid when n is replaced by N.

Let $M_n = M(x_n)$, $Z_n = Y(x_n) - M(x_n)$, and $\delta_n = M_n - \alpha - \alpha_1(x_n - \theta)$. Thus

$$\begin{aligned} x_{n+1} - \theta &= (x_n - \theta) - (A/n)(Z_n + M_n - \alpha) \\ &= [1 - (A\alpha_1/n)](x_n - \theta) - (A/n) Z_n - (A/n)\delta_n. \end{aligned} \tag{4.4}$$

If

$$\begin{aligned} B_{mn} &= \prod_{j=m+1}^{n} (1 - A\alpha_1 j^{-1}) \quad &&\text{for } 0 \leqq m < n \\ &= 1 &&\text{for } m = n, \end{aligned}$$

then the iteration of (4.4) back to $n = 1$ implies

$$x_{n+1} - \theta = B_{0n}(x_1 - \theta) - A \sum_{m=1}^{n} B_{mn} m^{-1} \delta_m - (\sigma/\alpha_1) S_n \tag{4.5}$$

where

$$S_n = (A\alpha_1/\sigma) \sum_{m=1}^{n} B_{mn} m^{-1} Z_m.$$

Let

$$h_n = \left[\sum_{m=1}^{n} (A\alpha_1)^2 m^{-2} B_{mn}^2\right]^{-\frac{1}{2}}$$

It is easily shown that there exists a sequence of constants $\{\varepsilon_m\}$ such that $\varepsilon_m \to 0$ as $m \to \infty$ and, for all $n \geqq m$,

$$(1 - \varepsilon_m) m^{A\alpha_1} n^{-A\alpha_1} \leqq B_{mn} \leqq (1 + \varepsilon_m) m^{A\alpha_1} n^{-A\alpha_1}. \tag{4.6}$$

Thus

$$\lim_{n\to\infty} h_n/n^{\frac{1}{2}}(2A\alpha_1-1)^{\frac{1}{2}}(A\alpha_1)^{-1}=1. \tag{4.7}$$

Sacks establishes (4.3) by showing that

$$h_n B_{0n}(x_1-\theta)\to 0 \quad \text{as } n\to\infty, \tag{4.8}$$

$$h_n \sum_{m=1}^{n} B_{mn} m^{-1}|\delta_m|\to 0 \quad \text{in probability as } n\to\infty, \tag{4.9}$$

and

$$P(h_n S_n\leqq x)\to\Phi(x) \quad \text{as } n\to\infty \text{ for all } x. \tag{4.10}$$

Since (3.7) implies

$$N\to\infty \quad \text{as} \quad d\to 0 \text{ w.p.1}, \tag{4.11}$$

(4.5) and (4.8) imply that (4.2) will hold if

$$h_N \sum_{m=1}^{N} B_{mN} m^{-1}|\delta_m|\to 0 \quad \text{in probability as } d\to 0 \tag{4.12}$$

and, for each x,

$$P(h_N S_N\leqq x)\to\Phi(x) \quad \text{as } d\to 0. \tag{4.13}$$

If $n_d=K_\gamma^2 A^2\sigma^2/[(2A\alpha_1-1)d^2]$, then (3.7) implies

$$\lim_{d\to 0} N/n_d=1 \text{ w.p.1}. \tag{4.14}$$

Thus, (4.6) and (4.7) imply (4.12) will hold if

$$n_d^{\frac{1}{2}-A\alpha_1}\sum_{m=1}^{N} m^{A\alpha_1-1}|\delta_m|\to 0 \quad \text{in probability as } d\to 0. \tag{4.15}$$

Now (4.6), (4.7), and (4.9) imply

$$n^{\frac{1}{2}-A\alpha_1}\sum_{m=1}^{N} m^{A\alpha_1-1}|\delta_m|\to 0 \quad \text{in probability as } n\to\infty. \tag{4.16}$$

Thus, since for any $\varepsilon>0$

$$\begin{aligned} &P\left(n_d^{\frac{1}{2}-A\alpha_1}\sum_{m=1}^{N} m^{A\alpha_1-1}|\delta_m|>\varepsilon\right)\\ &\leqq P\left(n_d^{\frac{1}{2}-A\alpha_1}\sum_{m=1}^{2n_d} m^{A\alpha_1-1}|\delta_m|>\varepsilon \text{ and } N/n_d\leqq 2\right)+P(N/n_d>2)\\ &\leqq P\left[(2n_d)^{\frac{1}{2}-A\alpha_1}\sum_{m=1}^{2n_d} m^{A\alpha_1-1}|\delta_m|>\varepsilon\, 2^{\frac{1}{2}-A\alpha_1}\right]+P(N/n_d>2), \end{aligned}$$

(4.14) and (4.16) imply (4.15) and consequently (4.12).

Anscombe's central limit theorem [1] will imply (4.13) if the sequence $\{S_n\}$ is uniformly continuous in probability; that is, given any small positive ε and η, there are positive constants v and ρ such that $n>v$ implies

$$P\Big(\max_{i:\ |i-n|\leqq n\rho} |S_i - S_n| \geqq \varepsilon h_n^{-1}\Big) < \eta. \tag{4.17}$$

If $\varepsilon' = \varepsilon\sigma/h_n A\alpha_1$ and m_1 and m_2 are integers such that $n(1-\rho)\leqq m_1 < n(1-\rho)+1$ and $n(1+\rho)-1 < m_2 \leqq n(1+\rho)$, then

$$\begin{aligned}
P\Big(\max_{i:\ |i-n|\leqq n\rho} |S_i - S_n| \geqq \varepsilon h_n^{-1}\Big) &\leqq P\left(\max_{m_1\leqq i\leqq n-1}\left|\sum_{m=1}^{i} m^{-1} B_{mi} Z_m - \sum_{m=1}^{n} m^{-1} B_{mn} Z_m\right| \geqq \varepsilon'\right)\\
&\quad + P\left(\max_{n+1\leqq i\leqq m_2}\left|\sum_{m=1}^{i} m^{-1} B_{mi} Z_m - \sum_{m=1}^{n} m^{-1} B_{mn} Z_m\right| \geqq \varepsilon'\right)\\
&\leqq P\left[\max_{m_1\leqq i\leqq n-1}\left|\sum_{m=1}^{i} (B_{mi}-B_{mn})\, m^{-1} Z_m\right| \geqq \varepsilon'/2\right]\\
&\quad + P\left(\max_{m_1\leqq i\leqq n-1}\left|\sum_{m=i+1}^{n} m^{-1} B_{mn} Z_m\right| \geqq \varepsilon'/2\right)\\
&\quad + P\left[\max_{n+1\leqq i\leqq m_2}\left|\sum_{m=1}^{n} (B_{mi}-B_{mn})\, m^{-1} Z_m\right| \geqq \varepsilon'/2\right]\\
&\quad + P\left(\max_{n+1\leqq i\leqq m_2}\left|\sum_{m=n+1}^{i} m^{-1} B_{mi} Z_m\right| \geqq \varepsilon'/2\right).
\end{aligned} \tag{4.18}$$

The establishement of (4.17), which will complete the proof of Theorem 2, is accomplished by proving that, for n sufficiently large, each of the last four probabilities in (4.18) converge to zero as $\rho\to 0$. For example, in the first of these last four probabilities, if

$$\begin{aligned}
D_{mm_1} &= B_{mm_1} && \text{for } m\leqq m_1\\
&= B_{m_1 m}^{-1} && \text{for } m > m_1,
\end{aligned}$$

then it can be readily verified that

$$B_{mi} - B_{mn} = (1-B_{in})\, B_{m_1 i}\, D_{mm_1},$$

so that

$$\begin{aligned}
&P\left[\max_{m_1\leqq i\leqq n-1}\left|\sum_{m=1}^{i} (B_{mi}-B_{mn})\, m^{-1} Z_m\right| \geqq \varepsilon'/2\right]\\
&\quad = P\left[\max_{m_1\leqq i\leqq n-1} (1-B_{in})\, B_{m_1 i}\left|\sum_{m=1}^{i} D_{mm_1} m^{-1} Z_m\right| \geqq \varepsilon'/2\right]\\
&\quad \leqq (\varepsilon'/2)^{-2}\left[(1-B_{m_1 n})^2 B_{m_1 m_1}^2 \sum_{m=1}^{m_1} m^{-2} D_{mm_1}^2 E(Z_m^2)\right.\\
&\qquad \left. + \sum_{m=m_1+1}^{n-1} (1-B_{mn})^2 B_{m_1 m}^2 D_{mm_1}^2 m^{-2} E(Z_m^2)\right]
\end{aligned} \tag{4.19}$$

where the inequality in (4.19) follows from Lemma 1. The convergence of the upper bound in (4.19) to zero as $\rho\to 0$ is a straightforward consequence of condition C7, (4.6), and (4.7).

5. Two Modifications of RM

An unfortunate characteristic of the use of $N_{d,\gamma}$ as a stopping time for RM is that, although two observations must be taken at each step in order to obtain convenient information of α_1, only one of those two observations is actually used to generate the next estimate of θ. One way to avoid this shortcoming is to generate the sequence $\{x_n)$ by the following procedure which is herein called the modified Robbins-Monro procedure: Let x_1 be any constant, and let

$$x_{n+1}=x_n+a_n[\alpha-(\tfrac{1}{2})(y'_n+y''_n)]$$

in which y'_n and y''_n are random variables whose conditional distributions for given x_n are independent, coincide with the distributions of $Y(x_n-c_n)$ and $Y(x_n+c_n)$ respectively, and are independent of $x_1,x_2,\ldots,x_{n-1},y'_1,y'_2,\ldots,y'_{n-1},y''_1,y''_2,\ldots,y'_{n-1}$.

Venter [9] actually went one step further than the modified Robbins-Monro procedure. He proposed a successive approximation procedure which at the n-th step incorporates the cumulative information on α_1 as well as the observations on $Y(x_n+c_n)$ and $Y(x_n-c_n)$ into the recurrence relation which generates x_{n+1}. If the conditions C3–C10 are satisfied, the asymptotic variance of the error in RM, $A^2\sigma^2/(2A\alpha_1-1)n$, is minimized with respect to A if $A=\alpha_1^{-1}$. Unfortunately, α_1 will usually be unknown. Venter recognized these facts and proposed that the sequence $\{x_n\}$ be generated according to the recurrence relation

$$x_{n+1}=x_n+(1/n\,t_n^*)\cdot[\alpha-(\tfrac{1}{2})(y''_n+y'_n)]$$

where t_n^* is an estimator of α_1, x_1 is any constant, and the form of the sequences $\{y''_n\}$ and $\{y'_n\}$ is the same as in the modified Robbins-Monro procedure.

Stopping times analogous to $N_{d,\gamma}$ and $T_{d,\gamma}$ have been proposed in [8] for both the modified Robbins-Monro procedure and Venter's procedure. Statements analogous to Theorems 1 and 2, (3.11), and (3.12) are given there for the stopping times of these two additional procedures. Also, in [8] are the results of a fairly extensive Monte Carlo study of the empirical behaviors of the stopping times for all three procedures.

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