Z. Wahrscheinlichkeitstheorie verw. Geb. 26, 33–41 (1973)

Some Results in the Probabilistic Theory of Asymptotic Uniform Distribution Modulo 1

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1. Introduction and Summary

The concept of the asymptotic uniform distribution (or equidistribution) modulo 1 of a sequence belongs, strictly, to the Theory of Numbers, but in recent years, for two rather different reasons, probabilistic ideas have been applied to it. On the one hand the basic idea of asymptotic distribution is a particular case of a standard probabilistic notion, that of weak convergence of distributions (see Billingsley [1], pp. 50–52), and on the other such a property is essentially an expression of some long-term regularity of the sequence, of just the kind that is of interest in probability theory. (See, for example, Robbins [7].) The description "probabilistic theory" in the title is intended to refer particularly to the second aspect.

In § 2 we give the continuous-time analogue of a result of Holewijn [4]: we shall find (sufficient) conditions under which

$$\hat{F}_T(x)=\frac{1}{T}\int_0^T I[\{X(t)\}\leqq x]\,dt\to x \text{ a.s.} \qquad \text{for all } x \text{ in } [0,1] \text{ as } T\to\infty,$$

where $X(t)$ is a stochastic process, $\{x\}$ denotes the fractional part of x, and $I[A]$ is the indicator function of the set A.

In §3 we prove a rather general version of a result that has been widely used, while §4 is concerned with the generalisations of equidistribution: A-equidistribution, in which

$$\hat{F}_{a,T}(x)=\int_0^\infty a(T,t)\,I[\{X(t)\leqq x]\,dt\to x \text{ a.s.} \qquad \text{for all } x \text{ in } [0,1] \text{ as } T\to\infty,$$

where $a(T,t)\geqq 0$ and $\int_0^\infty a(T,t)\,dt=1$ for all T; and B-equidistribution, the special case of A-equidistribution that arises by setting $a(T,t)=b(t)/B(T)$, where $b(t)\geqq 0$ and $B(T)=\int_0^T b(t)\,dt$. [The notation $\hat{F}_T$ and $\hat{F}_{a,T}$ is adopted, to underline the analogy with the empirical distribution function.]

2. Asymptotic Equidistribution in Continuous Time

We suppose that $\{X(t)\colon 0\leqq t<\infty\}$ is a stochastic process with the following property:

(I_c): the distribution of $X(t+h)-X(t)$, where $t, h\geqq 0$, depends on h but not on t.

It was observed in Loynes [6] § 4 that (I_c) implies that $\{\exp(i 2\pi\lambda X(t))\}$ is a second-order stationary process for every real λ, so that

$$e^{i 2\pi\lambda X(t)} = \int_{-\infty}^{\infty} e^{i\theta t}\, d Z_\lambda(\theta) \tag{1}$$

and

$$E[e^{i 2\pi\lambda (X(t+h)-X(t))}] = \int_{-\infty}^{\infty} e^{i\theta h}\, d F_\lambda(\theta) \tag{2}$$

for appropriate Z_λ and F_λ, Z_λ having orthogonal increments and F_λ being the spectral measure. Then we have the following result.

Theorem 1. *If the stochastic process $\{X(t)\}$ satisfies (I_c) and if*

(i) *$\Delta F_m(0)$, the jump in F_m at $\theta=0$, vanishes for every $m=1, 2, \ldots$,*

(ii) *$\int_{-1}^{+1} \log|\theta|\, dF_m(\theta) < \infty$ for every $m=1, 2, \ldots$,*

then a.s. as $T\to\infty$

$$\hat{F}_T(x) \to x \qquad \text{for all } x \text{ in } [0, 1];$$

that is, $\hat{F}_T$ converges weakly to the uniform distribution in [0, 1].

This is quite analogous to Theorem 2 of Holewijn [4], and his proof needs only minor changes. First we observe that, $\hat{F}_T$ being for every T a distribution function which concentrates all the probability on [0, 1], the claimed weak convergence occurs if and only if the characteristic functions

$$\phi_T(m) = \int_{[0,1]} e^{i 2\pi m x}\, d\hat{F}_T(x) = \frac{1}{T}\int_0^T e^{i 2\pi m X(t)}\, dt \tag{3}$$

converge to 0 a.s. for $m=1, 2, \ldots$. Next we use the obvious extension of Theorem 2.1 of Kuipers and van der Steen [5] (see also the next section) to the case in which a general probability space replaces the interval $[a, b]$ to see that it is sufficient to show that

$$\lim_{T\to\infty} \int_1^T \frac{1}{t} E|\phi_t(m)|^2\, dt < \infty, \qquad m=1, 2, \ldots. \tag{4}$$

Finally we use (2) to transform the expression in (4) into

$$\int_{-\infty}^{\infty} dF_m(\theta) \int_1^T \frac{|e^{i\theta t}-1|^2}{t^3\theta^2}\, dt \tag{5}$$

where the inner integral is to be understood as $\log T$ if $\theta=0$. Now

$$\int_1^T \frac{|e^{i\theta t}-1|^2}{t^3\theta^2}\, dt = \int_{|\theta|}^{|\theta| T} \frac{|e^{ix}-1|^2}{x^3}\, dx$$

which is $O(\log|\theta|)$ for small θ and bounded in any interval excluding the origin, so the result follows.

[There are various ways of showing that convergence of the functions $\phi_T(m)$ for integral m is sufficient for weak convergence. One possibility is to identify the half-open interval [0, 1) with the circumference of the unit circle via the correspondence $x \leftrightarrow e^{2\pi i x}$, for then the integrands $e^{2\pi i m x}$ in (3) are the characters of the circle group and the result, aside from one complication, follows at once, as in Billingsley (p. 51). The difficulty arises from the fact that the points 0 and 1 on the line are to be identified, but since points are closed sets, by Billingsley (Thm. 2.1)

$$\limsup F_n(\{0, 1\}) \leqq F(\{0, 1\})$$

when F is the weak limit of F_n: when, as in this case, the limit is continuous, $F_n(\{0\})$ and $F_n(\{1\}) \to 0$ so that the points 0 and 1 can merely be ignored in the limit – it is unimportant whether they are identified or not. The problem cannot be overcome in Proposition 1 below: using only integral m it is clearly impossible to distinguish between 0 and 1.]

Corollary. *If $\{X(t): t \geqq 0\}$ has stationary independent increments, and $X(0)=0$, then, provided there is no rational number r such that for all t the values of $X(t)$ are a.s. integral multiples of r, $\{X(t)\}$ is asymptotically uniformly distributed modulo* 1.

The proof is quite straightforward, and again is the natural analogue of Holewijn's, in his § 3. By the hypothesis that $X(t)$ has stationary and independent increments, the left hand side of (2) has the form $e^{-h\psi(\lambda)}$ for $h>0$ and $e^{+h\overline{\psi(\lambda)}}$ for $h<0$, for some $\psi(\lambda)$, where the real part of $\psi(\lambda)$ is non-negative for all λ and $\psi(m) \neq 0$ for $m=1, 2, \ldots$. [See Breiman [8], Proposition 14.18.] Suppose $\psi(m)=c+id$ where $c>0$: then it is easily seen that F_m has a density

$$f_m(\theta) = \frac{c}{\pi[c^2 + (d+\theta)^2]} \tag{6}$$

so that (i) and (ii) are satisfied. If on the other hand $c=0$ and $d \neq 0$, $F_m(\theta)$ places all its mass at $-d$, and again it is seen that (i) and (ii) follow.

Note that this in effect includes Theorem 3 of Robbins [7] and Satz 1 of Hlawka [3].

Remarks. (A) Whether in continuous time (as here) or discrete time the same techniques could be applied in cases where the values of $X(t)$ are not real but are elements of some other locally compact group.

(B) Condition (i) of Theorem 1 ensures that convergence is to the uniform distribution: without it but with condition (ii) (amended so that 0 is omitted from the range of integration) we should still have a.s. convergence to some distribution (perhaps random) on the circle, or equivalently on [0, 1] with 0 and 1 identified. Condition (ii) is rather different: if we drop this we have what might be called asymptotic equidistribution in L_2, but perhaps not a.s. equidistribution, and such a concept might be of interest in some applications. There is, incidentally, a possibility that condition (ii) could be weakened, for an inspection of the proof of the theorem in the next section shows that condition I_c is nowhere used.

It is of some interest to discuss the possible limit laws of $\hat{F}_T$ when condition (i) is not satisfied, and we have the following partial result.

Proposition 1. *If $\{X(t)\}$ satisfies (I_c) and if $X(0)=0$, then the L_2-limit of F_T (which always exists) is* a.s. *constant if and only if*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T E(e^{2\pi i m X(t)})\, dt$$

is 0 or 1 for $m=1,2,\ldots$. The possible non-random limits of $\hat{F}_T$ are the distribution functions H_0 and $H_k(p)$, for $k=1,2,\ldots$, and $0\leq p\leq\frac{1}{k}$, where H_0 corresponds to the uniform distribution on $[0,1]$ and $H_k(p)$ puts probability p at 0, $k^{-1}-p$ at 1, and k^{-1} at each of the points $\frac{1}{k},\frac{2}{k},\ldots,\frac{k-1}{k}$.

Mutatis mutandis this result is also valid in the discrete time situation, and in either case if there is an a.s. limit for $\hat{F}_T$ it concides with the L_2 limit. Whether, in either case, all the possible laws mentioned can actually arise does not seem obvious.

Proof. From the von Neumann ergodic theorem [Doob [9], Thm. X.6.1] the characteristic functions $\phi_T(m)$ of (3) L_2-converge to $\Delta Z_m(0)$. Now this is non-random if and only if for all m

$$E[|\Delta Z_m(0)-E\,\Delta Z_m(0)|^2]=0;$$

i.e. if

$$E\,|\Delta Z_m(0)|^2=|E\,\Delta Z_m(0)|^2.$$

Now the left hand side is $\Delta F_m(0)$, while from (1)

$$\begin{aligned} E[\Delta Z_m(0)] &= \lim_T \frac{1}{T}\int_0^T E(e^{2\pi i m X(t)})\,dt,\\ &= \lim_T \frac{1}{T}\int_0^T E(e^{2\pi i m (X(t)-X(0))})\,dt\\ &= \Delta F_m(0)\end{aligned}$$

from (2). The main part of the result is now proved, while if F_∞ is the non-random limit of $\hat{F}_T$, with characteristic function ϕ, we have $\phi^2=\phi$, so that F_∞ is an idempotent positive measure on the circle, and the result follows by using Theorem 3.2.1 of Grenander [10].

3. An Auxiliary Theorem

The following theorem contains the essence of Davenport, Erdös and LeVeque's theorem [2], and Theorem 2.1 of Kuipers and van der Steen. We suppose that B is a positive measure on the non-negative reals, and write $B(x)=B([0,x])$.

Theorem 2. *Let $S_T=\frac{1}{B(T)}\int_0^T u(t)\,B(dt)$ where*

(i) *$|u(t)|\leq K<\infty$ for almost all t (B-measure), and u is measurable;*

(ii) *$B(T)\to\infty$ as $T\to\infty$;*

(iii) *$\frac{\Delta B(T)}{B(T)}\to 0$ as $T\to\infty$, where $\Delta B(T)$ is the mass assigned by B to the point T.*

Then from $\int_a^\infty \psi(S_T)\frac{B(dT)}{B(T)}<\infty$, where ψ is non-negative and $\psi(x)\to 0$ implies $x\to 0$, follows $S_T\to 0$.

The proof depends on the following lemma.

Lemma. *If μ is a positive bounded measure on $[0, \infty)$, then there exists a positive function λ which is monotone increasing and tends to $+\infty$ as $x \to \infty$ such that $\int \lambda \, d\mu < \infty$.*

To show this we let, for $r = 1, 2, \ldots, x_r$ be the smallest number for which $\mu([x_r, \infty)) \leqq 2^{-r}$, or 0 if there is no such x_r. For x such that $x_r \leqq x < x_{r+1}$ write $\lambda(x) = 2^{\frac{1}{2}r}$: then

$$\int_{x_1}^{\infty} \lambda \, d\mu = \sum_{r=1}^{\infty} \int_{[x_r, x_{r+1})} \lambda \, d\mu = \sum_{r=1}^{\infty} 2^{\frac{1}{2}r} \mu([x_r, x_{r+1})) \leqq \sum 2^{\frac{1}{2}r} 2^{-r} < \infty .$$

Returning to the proof of the theorem, there exists λ such that

$$\int_a^{\infty} \lambda(T) \, \psi(S_T) \frac{B(dT)}{B(T)} < \infty .$$

Let $M_1 < \infty$ be arbitrarily chosen so that $\lambda(M_1) \geqq 2$, and then define M_r, $r = 2, 3, \ldots$, inductively by

$$M_{r+1} = \inf \left\{ x \colon B(x) \geqq \frac{\lambda(M_r)}{\lambda(M_r) - 1} B(M_r) + 1 \right\}.$$

All M_r are finite, and

$$B(M_{r+1}) \geqq \frac{\lambda(M_r)}{\lambda(M_r) - 1} B(M_r) + 1 \geqq B(M_{r+1} -),$$

from which it follows that M_r increases monotonically to ∞, and since

$$\frac{B(M_{r+1})}{B(M_r)} = \frac{B(M_{r+1} -) + \Delta B(M_{r+1})}{B(M_r)}$$

in which the first term tends to unity while

$$\frac{\Delta B(M_{r+1})}{B(M_r)} \leqq \frac{\Delta B(M_{r+1}) \, \lambda(M_r)}{B(M_{r+1} -) \left(\lambda(M_r) - 1\right)}$$

we find that $B(M_{r+1})/B(M_r) \to 1$.

Now let N_r be any number in $M_r < T \leqq M_{r+1}$ at which $\psi(S_T)$ is less than $1/r^2$ greater than the infimum of $\psi(S_T)$ in that range: then

$$\begin{aligned} \psi(S_{N_r}) &\leqq \frac{1}{B(M_{r+1}) - B(M_r)} \int_{(M_r, M_{r+1}]} \left(\psi(S_T) + \frac{1}{r^2} \right) B(dt) \\ &\leqq \frac{B(M_{r+1})}{B(M_{r+1}) - B(M_r)} \int_{(M_r, M_{r+1}]} \frac{\psi(S_T) \, B(dt)}{B(T)} + \frac{1}{r^2} \\ &\leqq \int_{(M_r, M_{r+1}]} \lambda(T) \, \psi(S_T) \frac{B(dt)}{B(T)} + \frac{1}{r^2}, \end{aligned}$$

and by adding these inequalities together for $r = 1, 2, \ldots$ we conclude that $\sum \psi(S_{N_r}) < \infty$: hence $\psi(S_{N_r}) \to 0$, so that $S_{N_r} \to 0$. If we now notice that, when $N_r < T \leqq N_{r+1}$ we have

$$|B(T) \, S_T - B(N_r) \, S_{N_r}| \leqq K\left(B(N_{r+1}) - B(N_r)\right),$$

we also see that $S_T \to 0$.

Corollary. *Suppose that u is a random function such that a.s.* $|u(t)| \leqq K < \infty$ *for almost all t (B-measure), where K may be a random variable, and that the other conditions of Theorem 2 are satisfied. Then if*

$$\int_a^\infty E\psi(S_T)\frac{B(dT)}{B(T)} < \infty$$

for some a, $S_T \to 0$ *a.s.*

The proof is immediate, since by Fubini's theorem

$$\int_a^\infty \psi(S_T)\frac{B(dT)}{B(T)} < \infty \qquad \text{a.s.}$$

Remark. In the situation treated in the corollary, it is possible to strengthen a detail in the proof. As the proof stands now, the sequence N_r is a *random* sequence, but it is easy to modify it so that N_r may be supposed deterministic. All that need be done is to replace $\psi(S_T)$ by $E(\psi(S_T))$ everywhere until the statement $\sum E\psi(S_{N_r}) < \infty$ is reached: then Fubini's theorem may be applied to show that $\sum \psi(S_{N_r}) < \infty$ a.s., and the proof continues as before.

4. A and B-Equidistribution

For a.s. A-equidistribution it is necessary and sufficient that the characteristic functions

$$\phi_{a,T}(m) = \int_{[0,1]} e^{i2\pi m x}\, d\hat{F}_{a,T}(x) = \int_0^\infty a(T,t)\, e^{i2\pi m X(t)}\, dt \tag{7}$$

converge to 0 a.s. for $m = 1, 2, \ldots$.

Now using (1) we have

$$\phi_{a,T}(m) = \int_{-\infty}^\infty \left(\int_0^\infty a(T,t)\, e^{i\theta t}\, dt\right) dZ_m(\theta) \tag{8}$$

and hence

$$E|\phi_{a,T}(m)|^2 = \int_{-\infty}^\infty \left|\int_0^\infty a(T,t)\, e^{i\theta t}\, dt\right|^2 dF_m(\theta). \tag{9}$$

In order that the same approach may be used conditions must be imposed to allow some estimate of this quantity to be made. As will be seen we shall put restrictions on F_m (i.e. on $\{X(t)\}$) and on $a(T,t)$; no doubt other combinations of conditions which would suffice could be found.

Theorem 3. *Suppose that* $\{X(t)\}$ *satisfies* (I_c) *and for each* $m = 1, 2, \ldots, F_m$ *is absolutely continuous with* $dF_m(\theta)/d\theta < K_m < \infty$, *where* K_m *is a constant. If* $\int_0^\infty \alpha(T)\, dT < \infty$, *where* $\alpha(T) = \int_0^\infty a(T,t)^2\, dt$, *and if, for each m,* $\int_0^\infty a(T,t)\, e^{i2\pi m X(t)}\, dt$ *is a.s. uniformly continuous for sufficiently large T, then a.s. as* $T \to \infty$

$$\hat{F}_{a,T}(x) \to x \qquad \textit{for all } x \textit{ in } [0,1].$$

The proof is straightforward, since by hypothesis, and (9),

$$E|\phi_{a,T}(m)|^2 \leqq K_m \int_{-\infty}^{\infty} d\theta \left|\int_0^{\infty} a(T,t)\, e^{i\theta t}\, dt\right|^2 = 2\pi K_m \int_0^{\infty} a(T,t)^2\, dt = 2\pi K_m \alpha(T) \quad (10)$$

by Parseval's theorem. Thus by Fubini

$$E \int_0^{\infty} |\phi_{a,T}(m)|^2\, dT = \int E|\phi_{a,T}(m)|^2\, dT < \infty \quad (11)$$

so that a.s. $\int_0^{\infty} |\phi_{a,T}(m)|^2\, dT < \infty$. This, together with the a.s. uniform continuity of $\phi_{a,T}(m)$ implies that $\phi_{a,T}(m) \to 0$ a.s.

Remark. The condition on F_m is satisfied by processes with stationary independent increments, provided that there is no real number d and integer p such that for all t the values of $X(t)$ have the form $(k+t\,d)/p$, with $k=0, \pm 1, \ldots$; see the discussion at and near (6). In particular, it is satisfied by the Wiener process, but in this case Hlawka [3] needs only to impose the condition $\int_0^{\infty} e^{-\delta^2/\alpha(T)}\, dT < \infty$ for every $\delta > 0$, in place of our much stronger $\int_0^{\infty} \alpha(T)\, dT < \infty$; his proof, however, uses the properties of the Wiener process in an essential way.

For the case of B-equidistribution it is possible to show that under relatively weak conditions on $b(t)$ the requirement that $\phi_{a,T}(m)$ be a.s. uniformly continuous follows automatically, but this is of no great interest since the application of Theorem 2 allows an appreciably stronger result.

Theorem 4. *Suppose that $\{X(t)\}$ satisfies (I_c) and for each $m=1,2,\ldots,F_m$ is absolutely continuous with $dF_m(\theta)/d\theta < K_m < \infty$, where K_m is constant. If $B(T)\to\infty$ as $T\to\infty$ and if*

$$\int_a^{\infty} \frac{\int_0^T b(t)^2\, dt}{B(T)^3}\, b(t)\, dt < \infty$$

then a.s. $\{X(t)\}$ is asymptotically B-equidistributed: a.s.

$$\hat{F}_{b,T}(x) \equiv \frac{1}{B(T)} \int_0^T I[\{X(t)\} \leqq x]\, b(t)\, dt \to x \qquad \text{for all } x \in [0,1].$$

The proof is trivial, since we need only use (10) and write $S_T = \phi_{b,T}(m)$, $\psi(x) = |x|^2$, in the corollary to Theorem 2.

There are results for discrete time which are quite analogous to Theorems 3 and 4. Both are concerned with a stochastic process $\{X(n)\colon n=0,1,2,\ldots\}$ which satisfies (I_D).

(I_D): the distribution of $X_{m+n} - X_n$, where $m, n \geqq 0$, depends on m but not on n.

Corresponding to (1) and (2) we have

$$e^{i2\pi\lambda X_n} = \int_{[0,1)} e^{i2\pi\theta n}\, dZ_\lambda(\theta) \quad (12)$$

and

$$E[e^{i2\pi\lambda(X_{m+n} - X_n)}] = \int_{[0,1)} e^{i2\pi\theta m}\, dF_\lambda(\theta) \quad (13)$$

for appropriate Z_λ and F_λ, Z_λ having orthogonal increments and F_λ being the spectral measure.

By a.s. asymptotic A-equidistribution we now understand that a.s. as $n\to\infty$

$$\hat{F}_{a,n}(x)=\sum_{j=0}^{\infty} a(n,j)\, I[\{X_j\}\leqq x]\to x \qquad \text{for all } x\in[0,1],$$

where $a(n,j)\geqq 0$, and $\sum_{j=0}^{\infty} a(n,j)=1$ for all n.

Similarly a.s. asymptotic B-equidistribution means that a.s. as $n\to\infty$

$$\hat{F}_{b,n}=\frac{1}{B_n}\sum_{j=0}^{n} b_j\, I[\{X_j\}\leqq x]\to x \qquad \text{for all } x\in[0,1],$$

where $b_n\geqq 0$ for all n, and $B_n=\sum_{j=0}^{n} b_j$.

It is again (necessary and) sufficient that we show that characteristic functions converge a.s.: that a.s.

$$\phi_{a,n}(m)=\sum_{j=0}^{\infty} a(n,j)\, e^{i2\pi m X_j}\to 0, \tag{14}$$

or a.s.

$$\phi_{b,n}(m)=\frac{1}{B_n}\sum_{j=0}^{n} b_j\, e^{i2\pi m X_j}\to 0, \tag{15}$$

in either case for $m=1,2,\ldots$.

We shall again wish to deal with $E|\phi_{a,n}(m)|^2$. From (12) and (14) we have

$$E|\phi_{a,n}(m)|^2=\int_{[0,1)}\left|\sum_{j=0}^{\infty} a(n,j)\, e^{i2\pi\theta j}\right|^2 dF_m(\theta),$$

and if we again suppose F_m absolutely continuous with $dF_m(\theta)/d\theta<K_m<\infty$, we reach

$$E|\phi_{a,n}(m)|^2<K_m\int_{[0,1)}|\sum a(n,j)\, e^{i2\pi\theta j}|^2\, d\theta=K_m\sum_{j=0}^{\infty} a(n,j)^2=K_m\,\alpha_n \tag{16}$$

again by Parseval.

Theorem 5. *Suppose that* $\{X_n\}$ *satisfies* (I_D) *and for each* $m=1,2,\ldots,F_m$ *is absolutely continuous with* $dF_m(\theta)/d\theta<K_m<\infty$, *where* K_m *is constant. If* $\sum_0^\infty \alpha_n<\infty$, *where* $\alpha_n=\sum_{j=0}^{\infty} a(n,j)^2$, *then a.s. as* $n\to\infty$

$$\hat{F}_{a,n}(x)\to x \qquad \textit{for all } x\in[0,1].$$

Theorem 6. *Suppose that* $\{X_n\}$ *satisfies* (I_D) *and for each* $m=1,2,\ldots$ F_m *is absolutely continuous with* $dF_m(\theta)/d\theta<K_m<\infty$, *where* K_m *is constant. If* $B_n\to\infty$ *and* $b_n/B_n\to 0$ *as* $n\to\infty$, *and if*

$$\sum_{n=0}^{\infty}\frac{\left(\sum_{j=0}^{n} b_j^2\right) b_n}{B_n^3}<\infty,$$

then a.s. $\{X(n)\}$ *is asymptotically B-equidistributed.*

For the proof of Theorem 6 we again use the corollary to Theorem 2; this time B puts mass b_n at n.

Remark. If $X_0=0$ and X_n is the n-th partial sum of a sequence of independent and identically distributed random variables, then (I_D) is satisfied. Provided that the values of the individual summands are not a.s. of the form $d+kp$, where d is real, p is rational, and $k=0, \pm 1, \pm 2, \ldots$, then the condition on F_m is satisfied: note however that this *excludes* the Weyl situation in which, in effect, our summands are equal to a constant irrational number with probability 1.

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(Received March 20, 1972)