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On Palm Probabilities

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1. Introduction

Consider a point process on some space X. If it is known that a certain element $x \in X$ happens to be one of the points of occurrence of the process, what can be said about the rest of the process? This question about conditional probabilities, so easily stated in common-sense terms, contains the essence of the problem of Palm probabilities. It was first considered by Palm himself [18] for stationary point processes on the real line. His reflections were made rigorous by Hinčin [11] and generalized, using analytic tools, by Slivnyak [23] and by Fieger [4] to not necessarily stationary processes, still on the line. More measure theoretic approaches have been made, for stationary processes, by Matthes [5], Mecke [16], Neveu [17] and Leadbetter [13]; for more general processes on the line by Ryll-Nardzewski [21], Papangelou [19] and Belyaev [2], and on general spaces by Kummer and Matthes [12].[1]

Here Ryll-Nardzewski's definition and Mecke's results for stationary random measures will be taken as starting point for a discussion of Palm measures of not necessarily stationary point processes on locally compact second countable Hausdorff spaces (though many results are nontopological). Stationary processes, infinitely divisible and Poisson processes will be characterized in terms of their Palm measures. Some words will be said about limits of Palm measures. The method we use applies to general random measures, whatever the value of such generalizations be.

In parts the paper is expository. The characterization given of the Poisson process is close to the one obtained by Papangelou and also a consequence of Mecke [16, Satz 3.1]. The characterization of infinitely divisible processes is due to Kummer and Matthes [12].[1]

2. Fundamentals

Let X be a locally compact, second countable Hausdorff space. Denote by $\mathscr{C}_K$ the class of continuous functions $X \to R$ with compact support, by $\mathscr{B}(X)$ the Borel algebra on X and by $\mathscr{N}(X)$ the class of integer- or infinite-valued nonnegative Radon measures (i.e. finite on compact sets) defined on $\mathscr{B}(X)$. Endow $\mathscr{N}(X)$ with the vague topology, (the coarsest topology rendering all maps $\mathscr{N}(X) \ni \mu \mapsto \mu f = \int f(x)\,\mu(dx)$ for $f \in \mathscr{C}_K$ continuous), and denote by $\mathscr{B}(\mathscr{N})$ the corresponding Borel algebra on $\mathscr{N}(X)$. A point process is a measurable map from some probability space $(\Omega, \mathscr{S}, \mathbb{P})$ into $(\mathscr{N}(X), \mathscr{B}(\mathscr{N}))$.

[1] I am grateful to the referee for bringing that paper to my cognizance.

Now consider some point process ξ. For any A we define

$$\xi^* A = \sum_{x \in A} \min(\xi\{x\}, 1).$$

ξ^* and ξ coincide if and only if ξ has no multiple points, i.e. $\mathbb{P}\{\forall x \in X \;\; \xi\{x\} \leqq 1\} = 1$. Throughout the paper we assume that $\lambda A = E\,\xi^* A$, $A \in \mathscr{B}(X)$, is a Radon measure.

We have to check that the map $T\colon \mu \mapsto \mu^*$ is measurable. Put $\psi_A(\mu) = \min(\mu A, 1)$, $A \in \mathscr{B}(X)$. Since $\mathscr{B}(\mathscr{N})$ coincides with the σ-algebra generated by sets of the form $\{\mu \in \mathscr{N}(X);\; \mu A_j = r_j,\; 1 \leqq j \leqq n\}$, $n = 1, 2, \ldots, A_j \in \mathscr{B}(X)$, [9], each ψ_A is measurable. Let $A \in \mathscr{B}(X)$ be bounded (=contained in some compact set) and led d be some metric metrizing the topology (any locally compact, second countable space is metrizable with a complete metric, [3, p. 294]). Suppose that $A_{n1}, \ldots, A_{nr_n}$ is a sequence of finer and finer partitions of A into sets with diameter less than $1/n$. The sequence

$$\psi_n = \sum_{j=1}^{r_n} \psi_{A_{nj}}$$

increases and $\lim_{n \to \infty} \psi_n(\mu) = \mu^* A$ is measurable. Hence

$$T^{-1}\{\mu \in \mathscr{N}(X);\; \mu A_j = r_j,\; 1 \leqq j \leqq n\} = \{\mu \in \mathscr{N}(X);\; \mu^* A_j = r_j,\; 1 \leqq j \leqq n\} \in \mathscr{B}(\mathscr{N}).$$

If φ is a suitable function, say measurable $X \times \mathscr{N}(X) \to R_+$, such that

$$\eta A = \int_A \varphi(x, \xi)\, \xi^*(dx)$$

has a finite expectation for bounded A, then the Radon measure $E\eta$ is absolutely continuous with respect to λ. (We have that $\lambda A = 0$, implies that $\mathbb{P}\{\xi^* A = 0\} = 1$. Therefore $\mathbb{P}\{\eta A = 0\} = 1$ and $E\eta A = 0$.) By the Radon-Nikodym theorem there is a function $x \mapsto E_x \varphi$ such that

$$E[\varphi(x, \xi)\, \xi^*(dx)] = E[\eta(dx)] = E_x \varphi\, \lambda(dx).$$

Note that if $\mathbb{P}\{\xi\{x\} > 0\}$ is positive, then

$$\mathbb{P}\{\xi\{x\} > 0\} = \mathbb{P}\{\xi^*\{x\} = 1\} = E[\xi^*\{x\}] = \lambda\{x\},$$
$$E[\varphi(x, \xi);\; \xi\{x\} > 0] = E[\varphi(x, \xi)\, \xi^*\{x\}] = E[\eta\{x\}] = E_x \varphi\, \lambda\{x\}.$$

Hence, using the elementary definition of the right hand side,

$$E_x \varphi = E[\varphi(x, \xi) \mid \xi\{x\} > 0].$$

More generally, consider an $x \in X$ such that $\mathbb{P}\{\xi V_x > 0\} > 0$ for all neighbourhoods V_x of x. Obviously the class S of such x's is the only relevant part of X in the sense that $\mathbb{P}\{\xi S' = 0\} = 1$ – prime for complement: If $x \in S'$, there is a neighbourhood V_x of x such that $\mathbb{P}\{\xi V_x = 0\} = 1$. Obviously S' is open: if $x \in S'$, then $V_x \subset S'$. By Lindelöf's theorem [3, p. 174] S' is covered by a countable number of V_x-sets, say

$$S' \subset \bigcup_{n=1}^{\infty} V_n.$$

Hence

$$\mathbb{P}\{\xi S' = 0\} \geqq \mathbb{P}\{\xi \cup_n V_n = 0\} = \mathbb{P}\big(\bigcap_n \{\xi V_n = 0\}\big) = 1.$$

Returning to S we have

Proposition 1. *If φ is bounded and $V_n(x)$ is a sequence of neighbourhoods of x such that $V_n(x)\downarrow\{x\}$ and $\mathbb{P}\{\xi V_n(x)>0\}>0$ (such a sequence exists if and only if $x\in S$), then*

$$\lim_{n\to\infty} E[\varphi(x,\xi)\,|\,\xi V_n(x)>0]=E_x\varphi \qquad \text{a.s. } \lambda$$

provided $E[\xi^* A;\ \xi^* A\geqq 2]=o(\lambda A)$ as $\lambda A\downarrow 0$ and the family $\{\varphi(\cdot,\mu)\colon \mu\in\mathcal{N}(X)\}$ is equicontinuous.

Proof.

$$\begin{aligned}
&E[\varphi(x,\xi);\ \xi V_n(x)>0]\\
&\quad=E\Big[\varphi(x,\xi)-\int_{V_n(x)}\varphi(y,\xi)\,\xi^*(dy);\ \xi V_n(x)>0\Big]\\
&\qquad+E\Big[\int_{V_n(x)}\varphi(y,\xi)\,\xi^*(dy);\ \xi V_n(x)>0\Big]\\
&\quad=E\Big[\int_{V_n(x)}\{\varphi(x,\xi)-\varphi(y,\xi)\}\,\xi^*(dy);\ \xi^* V_n(x)=1\Big]\\
&\qquad+E\Big[\varphi(x,\xi)-\int_{V_n(x)}\varphi(y,\xi)\,\xi^*(dy);\ \xi^* V_n(x)\geqq 2\Big]+E\Big[\int_{V_n(x)}\varphi(y,\xi)\,\xi^*(dy)\Big].
\end{aligned}$$

The last expectation obviously equals

$$\int_{V_n(x)} E_y\varphi\,\lambda(dy).$$

The Lebesgue-Vitali theorem [22, p. 209] yields

$$\lim_{n\to\infty}\frac{E\Big[\int_{V_n(x)}\varphi(y,\xi)\,\xi^*(dy)\Big]}{\lambda V_n(x)}=E_x\varphi \qquad \text{a.s. } \lambda$$

for $x\in S$. The second expression gives (taking $0\leqq\varphi\leqq 1$ for simplicity)

$$\Big|E\Big[\varphi(x,\xi)-\int_{V_n(x)}\varphi(y,\xi)\,\xi^*(dy);\ \xi^* V_n(x)\geqq 2\Big]\Big|\leqq E[\xi^* V_n(x);\ \xi^* V_n(x)\geqq 2]=o(\lambda V_n(x)).$$

Next

$$\begin{aligned}
&\Big|E\Big[\int_{V_n(x)}\{\varphi(x,\xi)-\varphi(y,\xi)\}\,\xi^*(dy);\ \xi^* V_n(x)=1\Big|\\
&\quad\leqq \sup_{y\in V_n(x)}\ \sup_{\mu\in\mathcal{N}(X)}|\varphi(x,\mu)-\varphi(y,\mu)|\,\lambda V_n(x).
\end{aligned}$$

Also

$$\begin{aligned}
\mathbb{P}\{\xi V_n(x)>0\}&=\lambda V_n(x)-E[(\xi^* V_n(x)-1);\ \xi^* V_n(x)\geqq 2]\\
&=\lambda V_n(x)+o(\lambda V_n(x)).
\end{aligned}$$

Note. Already the first proviso renders this proposition superficial. For a thorough analysis of similar convergences on the line see [3].

The second condition is satisfied if φ is independent of x, like $\varphi(x,\mu)=1_B(\mu)$ for fixed $B\in\mathscr{B}(\mathcal{N})$. In this case $E_x\varphi$ exists and if $P=\mathbb{P}\,\xi^{-1}$ is the distribution of ξ we write $E_x\varphi=P_x B$. Since all measures on $\mathscr{B}(\mathcal{N})$ are regular and tight and $\mathcal{N}(X)$ is separable, the usual proof for Polish spaces [1, p. 258] applies to show that the family $\{P_x B,\ x\in X,\ B\in\mathscr{B}(\mathcal{N})\}$ can be so chosen that P_x is for all x a probability measure on $\mathscr{B}(\mathscr{N})$ and for each $B \in \mathscr{B}(\mathscr{N})$ the function $x \mapsto P_x B$ is measurable. In the sequel we assume this to be done.

For ξ with distribution P we also let ξ_x denote a point process with distribution P_x, "ξ conditioned to have an occurrence at x" and write $\mathbb{P}\{\xi_x \in B\}$, $\mathbb{P}\{\xi \in B | \xi\{x\} > 0\}$ or $P_x\{\mu \in B\}$ for $P_x B$.

In the case of general random measures ξ there is of course no ξ^*. But then ξ it self could be used to define E_x. This approach would even have some advantages for the development of the present theory. The link to elementary concepts of conditioning would be lost, though.

Proposition 2. *For any φ: $X \times \mathscr{N}(X) \to R_+$ which is measurable with respect to the product algebra $\mathscr{B}(X) \times \mathscr{B}(\mathscr{N})$ and such that $E_x \varphi$ exists*

$$E_x \varphi = \int_{\mathscr{N}(X)} \varphi(x, \mu) P_x(d\mu).$$

Note. All relations like the one above should be interpreted as a.s. equalities with respect to λ. Proposition 2 shows that conditional probabilities of events related to the place x of the conditioning occurrence can also be calculated from P_x. For example it follows that

$$\mathbb{P}\{\xi\{x\} > 0 | \xi\{x\} > 0\} = 1 \quad \text{or if} \quad \delta_x(\mu) = \inf\{d(y, x);\ y \neq x, \mu(y) > 0\}$$

for some metric d on X, $\varphi_s(x, \mu) = e^{-s\delta_x(\mu)}$, $s \geqq 0$, then

$$\int_0^\infty e^{-st} P_x\{\delta_x(\mu) \in dt\} = E_x \varphi_s.$$

Proof. If $\varphi = 1_B$, $B \in \mathscr{B}(\mathscr{N})$, the proposition holds by definition and by the usual linearity and limit arguments, we can prove it for any measurable function of μ, $\varphi(x, \mu) = \vartheta(\mu)$. Now let $\varphi(x, \mu) = f(x)\,\vartheta(\mu)$. Then since

$$E[\vartheta(\xi)\,\xi^*(dx)] = \int_{\mathscr{N}(X)} \vartheta(\mu) P_x(d\mu)\,\lambda(dx),$$

$$E_x \varphi\,\lambda(dx) = E[f(x)\,\vartheta(\xi)\,\xi^*(dx)] = \int f(x)\,\vartheta(\mu) P_x(d\mu)\,\lambda(dx),$$

and again linearity and monotonicity completes the proof.

Proposition 3. *Any probability measure P on $\mathscr{N}(X)$ is uniquely determined by λ and its Palm probabilities.*

Proof. Using an idea of Mecke we shall derive an explicit formula for

$$E[\vartheta(\xi)] = \int \vartheta(\mu) P(d\mu)$$

where $P = \mathbb{P}\,\xi^{-1}$ and ϑ is any non-negative measurable function on $\mathscr{N}(X)$ such that $\vartheta(0) = 0$, $E[\vartheta(\xi)] < \infty$, the first zero denoting the zero measure. Let $\{A_n\}$ be a countable partition of X in bounded disjoint measurable sets and define

$$a(x, \mu) = \begin{cases} 2^{-n}(\mu^* A_n)^{-1} & \text{if } x \in A_n,\ \mu A_n > 0 \\ 0 & \text{if } x \in A_n,\ \mu A_n = 0. \end{cases}$$

Then, if $\mu X > 0$,

$$0 < \int a(x, \mu)\,\mu^*(dx) \leqq 1$$

and

$$\psi(x,\mu)=\begin{cases}\dfrac{a(x,\mu)}{\int a(x,\mu)\,\mu^*(dx)} & \mu \neq 0\\ 0 & \mu=0\end{cases}$$

satisfies $\int \psi(x,\mu)\,\mu^*(dx)=1$. Hence, with $\varphi(x,\mu)=\vartheta(\mu)\,\psi(x,\mu)$,

$$E[\vartheta(\xi)]=E[\int \vartheta(\xi)\,\psi(x,\xi)\,\xi^*(dx)]=\int E_x\varphi\,\lambda(dx)=\iint \vartheta(\mu)\,\psi(x,\mu)\,P_x(d\mu)\,\lambda(dx),$$

by Proposition 2.

Palm-Hinčin equations can of course be expected only for point processes on the line. However, if ξ has no multiple points a simple formula for $\mathbb{P}\{\xi A=k\}$ can be deduced:

Let φ be the indicator function of the set $\{\mu A=k\}$ for $k\geqq 1$. It holds that

$$\begin{aligned}\mathbb{P}\{\xi A=k\}&=E[\xi A;\,\xi A=k]/k=E\Big[\int_A \varphi(\xi)\,\xi(dx)\Big]/k=E\Big[\int_A \varphi(\xi)\,\xi^*(dx)\Big]/k\\&=\int_A E_x\varphi(dx)/k=\int_A \mathbb{P}\{\xi_x A=k\}\,\lambda(dx)/k.\end{aligned}$$

Thus

Proposition 4. *If ξ has no multiple points and $A\in\mathscr{B}(X)$*

$$\mathbb{P}\{\xi A=k\}=\int_A P_x\{\mu A=k\}\,\lambda(dx)/k \qquad \text{for } k=1,2,\ldots.$$

3. The Poisson and Infinitely Divisible Cases

The Poisson process ξ with intensity λ is by definition a completely random point process (i.e. such that $\xi A_1,\ldots,\xi A_n$ are independent as soon as the A_j are disjoint) satisfying

$$\mathbb{P}\{\xi A=k\}=\frac{(\lambda A)^k}{k!}\,e^{-\lambda A}, \qquad k=0,1,2,\ldots$$

for A bounded and λ a non-atomic Radon measure on X. We denote the distribution of ξ by $\Pi_\lambda=\mathbb{P}\,\xi^{-1}$, Poisson measure on $\mathscr{N}(X)$, and by Δ_x, $x\in X$, the measure on $\mathscr{N}(X)$ that gives only mass one to the measure e_x, $e_x A=1_A(x)$.

In the stationary case, the following elegant characterization of the Poisson process is due to Slivnyak [23], Kersten and Matthes [10] and Mecke [16, Satz 4.1]. With our definition of the Palm probability it is also a consequence of Mecke's Satz 3.1 [16].

Proposition 5. *Let λ be a non-atomic Radon measure and ξ a point process with $E\,\xi^*=\lambda$. Then ξ is Poisson with intensity λ if and only if ξ_x and $\xi+e_x$ have the same distribution.*

Note. In convolution notation (formally, if P and Q are two measures on $\mathscr{N}(X)$ then

$$P*Q(B)=\iint 1_B(\mu+\nu)\,P(d\mu)\,Q(d\nu),\quad B\in\mathscr{B}(\mathscr{N}))$$

we can write Proposition 5: $P=\Pi_\lambda$ if and only if $P_x=P*\Delta_x$ and $\lambda A=\int \mu^* A\,P(d\mu)$ is non-atomic.

That this last requirement is essential follows from considering two processes which place mass k only at some fixed point x, one of them with Poisson probability $a^k e^{-a}/k!$, $a>0$, the other with geometric probability $(1-b)\,b^k$, $0<b<1$.

Proof. I. Let $A, B \in \mathscr{B}(X)$ and assume that ξ is Poisson λ.

$$\begin{aligned}\int_A \Pi_\lambda * \Delta_x\{\mu;\,\mu B=k\}\,\lambda(dx) &= \int_{A\setminus B} \mathbb{P}\{\xi B=k\}\,\lambda(dx) + \int_{A\cap B} \mathbb{P}\{\xi B=k-1\}\,\lambda(dx)\\ &= \mathbb{P}\{\xi B=k\}\,E[\xi(A\setminus B)] + \mathbb{P}\{\xi B=k-1\}\,E[\xi(A\cap B)]\\ &= E[\xi(A\setminus B);\,\xi B=k] + \mathbb{P}\{\xi B=k-1\}\,E[\xi(A\cap B)],\end{aligned}$$

since $\xi(A\setminus B)$ and ξB are independent and $E\xi=\lambda$ as Poisson processes have no multiple points.

Writing C for $A\cap B$ we obtain

$$\begin{aligned}E[\xi C;\,\xi B=k] &= \sum_{j=1}^{k} j\,\frac{(\lambda C)^j}{j!}\,e^{-\lambda C}\,\frac{(\lambda(B\setminus C))^{k-j}}{(k-j)!}\,e^{-\lambda(B\setminus C)}\\ &= e^{-\lambda B}\,\frac{\lambda C}{(k-1)!}\sum_{j=1}^{k}\binom{k-1}{j-1}(\lambda C)^{j-1}(\lambda(B\setminus C))^{k-1-(j-1)}\\ &= \mathbb{P}\{\xi B=k-1\}\,E[\xi(A\cap B)].\end{aligned}$$

Hence

$$\int_A \Pi_\lambda * \Delta_x\{\mu;\,\mu B=k\}\,\lambda(dx) = E[\xi A;\,\xi B=k],$$

proving that

$$\Pi_{\lambda x}\{\mu;\,\mu B=k\} = \Pi_\lambda * \Delta_x\{\mu B=k\}.$$

The same equality holds for sets $\{\mu;\,\mu B_i=r_i,\ 1\leqq i\leqq k\}$, as follows by similar arguments, and since the class of such sets is a π-system generating $\mathscr{B}(\mathcal{N})$ it follows that $\Pi_{\lambda x}=\Pi_\lambda * \Delta_x$.

II. Now suppose that $P_x = P * \Delta_x$. If ξ is a point process with distribution P and $A\in\mathscr{B}(X)$

$$\begin{aligned}\mathbb{P}\{\xi A\geqq 2\} &\leqq E\int_A 1_{\{\mu A\geqq 2\}}(\xi)\,\xi^*(dx)\\ &= \int_A P_x\{\mu A\geqq 2\}\,\lambda(dx) = \mathbb{P}\{\xi A\geqq 1\}\,\lambda A \leqq (\lambda A)^2.\end{aligned}$$

Hence for any $\varepsilon>0$ and compact set $K\subset X$ there are disjoint sets $A_1,\dots,A_n$, $\lambda A_j<\varepsilon$, $K=\bigcup_{j=1}^{n} A_j$ and

$$\mathbb{P}\{\exists x\in K;\,\xi\{x\}\geqq 2\} \leqq \mathbb{P}\left(\bigcup_{j=1}^{n}\{\xi A_j\geqq 2\}\right) \leqq \sum_{j=1}^{n}\mathbb{P}\{\xi A_j\geqq 2\} < \varepsilon\lambda K.$$

By σ-compactness it follows that ξ can have no multiple points. But then Proposition 4 yields that for any $A\in\mathscr{B}(X)$, $k=1, 2\dots$

$$\mathbb{P}\{\xi A=k\} = \mathbb{P}\{\xi A=k-1\}\,\lambda A/k.$$

By induction

$$\mathbb{P}\{\xi A=k\}=\mathbb{P}\{\xi A=0\}(\lambda A)^k/k!.$$

Summing over $k=0,1\ldots$ we obtain

$$\mathbb{P}\{\xi A=0\}=e^{-\lambda A}$$

since $\mathbb{P}\{\xi A<\infty\}=1$ for bounded A. By Renyi's theorem [20] it follows already from this that ξ is Poisson. However the complete randomness of ξ, can also be shown directly:

Let $A_1,\ldots,A_n$ be disjoint and bounded, $s_j\geqq 0$, $1\leqq j\leqq n$,

$$\eta=\exp\left\{-\sum_{j=1}^{n-1} s_j\,\xi A_j\right\},$$

$$a=E[\eta],\quad b=E[\xi A_n],\quad f_0(x)=1,\quad x\geqq 0.$$

Then

$$E[\eta(\xi A_n)^0]=a f_0(b)$$

and we assume that

$$E[\eta(\xi A_n)^j]=a f_j(b),\qquad 1\leqq j\leqq k$$

for some functions $f_j\colon R_+\to R_+$ independent of A_n.

Then

$$\begin{aligned} E[\eta(\xi A_n)^{k+1}]&=E\Big[\int_{A_n}\eta(\xi A_n)^k\,\xi(dx)\Big]\\ &=\int_{A_n}E_x\,\eta(\xi A_n)^k\,\lambda(dx)=\int_{A_n}E[\eta(\xi A_n+1)^k]\,\lambda(dx)\\ &=b\sum_{j=0}^{k}a\binom{k}{j}f_j(b)=a f_{k+1}(b),\end{aligned}$$

if we define

$$f_{k+1}(x)=x\sum_{j=0}^{k}\binom{k}{j}f_j(x),\qquad x\geqq 0.$$

Hence

$$\begin{aligned} E\big[e^{-\sum_{j=1}^{n}s_j\xi A_j}\big]&=E\big[e^{-\sum_{j=1}^{n-1}s_j\xi A_j}\big]\cdot\sum_{j=0}^{\infty}\frac{(-s_n)^j}{j!}f_j(b)\\ &=E\big[e^{-\sum_{j=1}^{n-1}s_j\xi A_j}\big]\,E\big[e^{-s_n\xi A_n}\big].\end{aligned}$$

The last equality follows from the special case $s_1=\cdots=s_{n-1}=0$. Repeating the argument concludes the proof.

It is interesting to observe that a convolution characterization of Palm measure applies not only to the Poisson process, but to all non-atomic infinitely divisible processes without multiple points [12]. A point process with distribution P is said to be infinitely divisible if for all $n=1,2\ldots$ there exist probability measures P_n on $\mathcal{N}(X)$ such that $P=P_n^{*n}$. It is known [13, 15] that ξ is infinitely divisible if and only if there is a uniquely determined Borel measure Λ on $\mathcal{N}(X)\setminus\{0\}$, called the canonic measure, satisfying

$$\Lambda\{\mu;\,\mu K\geqq 1\}<\infty$$

for all compact K and

$$-\log E[e^{-\xi f}] = \int_{\mathcal{N}(X)} (1-e^{-\mu f})\, \Lambda(d\mu)$$

for $0 \leqq f \in \mathscr{C}_K$, where Λ can be extended to all of $\mathcal{N}(X)$ in an arbitrary way. Λ must be non-atomic: if for some $\mu_0 \in \mathcal{N}(X) \setminus \{0\}$, $\Lambda\{\mu_0\}$ were positive there would exist an $a \in X$ such that $\mu_0\{a\} > 0$. Replacing ξf by $\xi\{a\}$ and μf by $\mu\{a\}$ in the formula above would contradict the fact that $\mathbb{P}\{\xi\{a\}=0\}=1$ (O. Kallenberg pointed this out to me). Also Λ is Radon on $\mathcal{N}(X) \setminus \{0\}$, given the relative topology. Though this space is not generally locally compact, we can define a Poisson process Ξ on $\mathcal{N}(X) \setminus \{0\}$ with intensity Λ and extend it to all of $\mathcal{N}(X)$ by setting for any outcome $\Xi\{0\} = \Lambda\{0\} = 0$. Then

$$\eta A = \int_{\mathcal{N}(X)} \mu A\, \Xi(d\mu), \qquad A \in \mathscr{B}(X),$$

has the same characteristic functional as ξ, $E[e^{i\xi f}] = E[e^{i\eta f}]$ and hence the same distribution [8]. If $P = \mathbb{P}\xi^{-1}$ we wish to determine P_x: Take $A \in \mathscr{B}(X)$, $B \in \mathscr{B}(\mathcal{N})$ and note that

$$\lambda A = \int_{\mathcal{N}(X)} \mu A\, \Lambda(d\mu) = E\xi A$$

assumed, as usual, to be Radon. Since ξ was assumed without multiple points it follows, in an obvious notation, that

$$\begin{aligned}
\int_A P_x B\, \lambda(dx) &= E\Big[\int_A 1_B(\eta)\, \eta(dx)\Big] \\
&= E\Big[\int_{\mathcal{N}(X)} \Big\{\int_A 1_B(\eta)\, \mu(dx)\Big\}\, \Xi(d\mu)\Big] = \int_{\mathcal{N}(X)} E[\mu A\, 1_B(\eta) | \Xi\{\mu\} > 0]\, \Lambda(d\mu) \\
&= \int_{\mathcal{N}(X)} E[1_B(\eta+\mu)]\, \mu A\, \Lambda(d\mu) = \int_{\mathcal{N}(X)} P * \Delta_\mu(B)\, \mu A\, \Lambda(d\mu),
\end{aligned}$$

where we have used Proposition 5 on Ξ, and Δ_μ is the distribution of a process ζ such that $\mathbb{P}\{\zeta = \mu\} = 1$.

However for any B

$$\int \Delta_\mu(B)\, \mu A\, \Lambda(d\mu) = \int_B \mu A\, \Lambda(d\mu) \leqq \lambda A$$

and the term on the left is a measure in A. Hence for some probability measure $P(x)$ on $\{\mu^*;\ \mu \neq 0,\ \mu \in \mathcal{N}(X)\}$ such that $P(x)(B)$ is measurable in x it holds that

$$\int_B \mu A\, \Lambda(d\mu) = \int_A P(x)(B)\, \lambda(dx)$$

and

$$P_x = P * P(x).$$

A more suggestive notation might be $P(x) = \Lambda_x$, since $P(x)$ is obtained from Λ as P_x from P. Only Λ need not be probability measure. Still, for the converse, Proposition 3 applies, and if $\{P(x), x \in X, \lambda\}$ is given we can define Λ by

$$\Lambda(B) = \int_X \int_B \psi(x,\mu)\, P(x)(d\mu)\, \lambda(dx),$$

$B \subset \mathscr{N}(X) \setminus \{0\}$, $\Lambda\{0\}=0$. Then, by an obvious extension of Proposition 4,

$$\begin{aligned}\Lambda\{\mu K \geqq 1\} &= \sum_{n=1}^{\infty} \int_K P(x)\{\mu K = n\}\, \lambda(dx)/n \\ &\leqq \int_K P(x)\{\mu K \geqq 1\}\, \lambda(dx) \leqq \lambda K < \infty\end{aligned}$$

if $K \subset X$ is compact. This means that Λ could serve as a canonic measure. However writing for $0 \leqq s$, $0 \leqq f \in \mathscr{C}_K$

$$\varphi(s) = E[e^{-s\xi f}]$$

we see by the definition of Palm measure that

$$\begin{aligned}\varphi'(s) &= -E[\int e^{-s\xi f} f(x)\, \xi(dx)] \\ &= -\int_X [\int_{\mathscr{N}(X)} e^{-s\mu f}\, P * P(x)(d\mu)]\, f(x)\, \lambda(dx) \\ &= -\varphi(s) \int_X \int_{\mathscr{N}(X)} e^{-s\mu f} f(x)\, P(x)(d\mu)\, \lambda(dx) \\ &= -\varphi(s) \int_{\mathscr{N}(X)} e^{-s\mu f}\, \mu f\, \Lambda(d\mu).\end{aligned}$$

Since $\varphi(0)=1$

$$\begin{aligned}-\log E[e^{-\xi f}] = -\log \varphi(1) &= -\int_0^1 \frac{\varphi'(s)}{\varphi(s)}\, ds \\ &= \int_{\mathscr{N}(X)} (1-e^{-\mu f})\, \Lambda(d\mu),\end{aligned}$$

completing the proof of

Proposition 6. *Assume P to be the distribution of an atomless point process without multiple points. Then P is infinitely divisible if and only if there is a family $\{P(x);\, x \in X\}$ of probability measures on $\{\mu^*;\, \mu \in \mathscr{N}(X) \setminus \{0\}\}$ such that for each $B \in \mathscr{B}(\mathscr{N})$ the function $x \to P(x)(B)$ is measurable and*

$$P_x = P * P(x).$$

4. The Stationary Case

Suppose, in this section only, that X is an Abelian topological group with operation $+$ and unit 0. Define for $x \in X$, $T_x\colon \mathscr{N}(X) \to \mathscr{N}(X)$ by $T_x \mu A = \mu(A+x)$. Then $T_x^{-1} = T_{-x}$ and T_0 is the identity operator. A process ξ is said to be stationary if $T_x \xi$ has the same distribution for all x.

Proposition 7. *If P is the distribution of a stationary point process ξ then there is a probability measure P^0 (usually called the Palm measure of the process) such that $P_x T_{-x} = P^0$ almost everywhere (λ).*

Proof. Since $T_x \xi^* = (T_x \xi)^*$ stationarity of ξ implies the same for ξ^*. Thus λ must be Haar measure and we write $\lambda(dx) = dx$. Let $B \in \mathscr{B}(\mathscr{N})$. If we can prove that $\int f(x)\, P_x T_{-x} B\, dx$ has the same value for all $f \geqq 0$ such that $\int f(x)\, dx = 1$, the proposition follows. Therefore, let f and g be two such functions. Exactly like Mecke [16, Satz 2.1] we obtain a chain of equalities using in each step the definition of Palm measure, the stationarity of ξ, substitutions like $z=x-y$ in Haar integrals or the relation $\int f(y)\, T_x\mu(dy)=\int f(y-x)\,\mu(dy)$:

$$\begin{aligned}\int f(x)\, P_x T_{-x} B\, dx &= E[\int f(x)\, 1_B(T_x\xi)\, \xi^*(dx)]\\ &= E[\iint f(x)\, g(y)\, 1_B(T_x\xi)\, \xi^*(dx)\, dy]\\ &= E[\iint f(x)\, g(y)\, 1_B(T_{x+y}\xi)\, T_y\xi^*(dx)\, dy]\\ &= E[\iint f(x-y)\, g(y)\, 1_B(T_x\xi)\, \xi^*(dx)\, dy]\\ &= E[\iint f(z)\, g(x-z)\, 1_B(T_x\xi)\, \xi^*(dx)\, dz]\\ &= E[\iint f(z)\, g(x)\, 1_B(T_{x+z}\xi)\, T_z\xi^*(dx)\, dz]\\ &= E[\iint f(z)\, g(x)\, 1_B(T_x\xi)\, \xi^*(dx)\, dz]\\ &= E[\int g(x)\, 1_B(T_x\xi)\, \xi^*(dx)] = \int g(x)\, P_x\, T_{-x} B\, dx.\end{aligned}$$

Note. The proof also shows that P^0 has the form

$$P^0(B)=E[\int f(x)\, 1_B(T_x\xi)\, \xi^*(dx)]$$

where f is any non-negative function with Haar integral one. In particular if $K\in\mathscr{B}(X)$ is some set such that $\lambda K<\infty$ and $B=\{\mu;\ \mu A=k\}$, then

$$P^0\{\mu A=k\}=E[\xi^*\{y\in K;\ \xi(A+y)=k\}]/\lambda K.$$

This gives the usual frequency interpretation – on the line we can choose $K=[0,1]$.

It is interesting that a converse holds, as in the Poisson case.

Proposition 8. *If λ is translation invariant and $P_x T_{-x}$ is the same for almost all x, then ξ is stationary.*

Note. Propositions 7 and 8 in process formulation have the following attractive form:

Let $\lambda=E\,\xi^*$. Then ξ is stationary if and only if almost all $T_x\,\xi_x$ (with respect to λ) have the same distribution and λ is Haar measure.

Proof. Write $P_x T_{-x}=P^0$ and $\lambda(dx)=dx$. First note that, $\varphi(x,\mu)\geqq 0$,

$$\int \varphi(x,\mu)\, P^0(d\mu)\, dx=\int \varphi(x, T_x\mu)\, P_x(d\mu)\, dx=E[\int \varphi(x, T_x\xi)\, \xi^*(dx)]$$

by Proposition 2. Also observe that if $\psi(x,\mu)$ is the function defined in the proof of Proposition 3, then $\Theta(\mu)=\psi(x, T_{-x}\mu)$ is independent of x. Now let $\vartheta\colon \mathscr{N}(X)\to R_+$ satisfy $\vartheta(0)=0$ and take $y\in X$. Then,

$$\begin{aligned}E[\vartheta(T_y\xi)] &= E[\int \vartheta(T_x T_{y-x}\xi)\, \psi(x, T_x T_{-x}\xi)\, \xi^*(dx)]\\ &= \int \vartheta(T_{y-x}\mu)\, \psi(x, T_{-x}\mu)\, P^0(d\mu)\, dx\\ &= \int \vartheta(T_{y-x}\mu)\, \Theta(\mu)\, P^0(d\mu)\, dx=\int \vartheta(T_z\mu)\, \Theta(\mu)\, P^0(d\mu)\, dz,\end{aligned}$$

independently of y after the substitution $z=y-x$. Since $E[\vartheta(T_y\xi)]$ is independent of y for arbitrary ϑ such that $\vartheta(0)>0$ stationarity follows.

May be it is worthwhile to point out again that the condition on λ is necessary.

5. Superpositions of Point Processes

If $\xi_1, \xi_2, \ldots, \xi_n$ are point processes, their superposition is by definition the sum $\xi_1+\xi_2+\cdots+\xi_n$. We shall study Palm measures of superpositions of independent process. Since addition of independent point processes corresponds to convolution of their distributions, this amounts to an analysis of the relation between the convolution operation and Palm conditioning. The basic result is the "differentiation rule" of Proposition 9, due in the stationary case to Mecke [16].

Let ξ and η be two independent point processes on x with distributions P and Q. As always we assume that $\lambda=E\xi^*$ and $\kappa=E\eta^*$ are Radon measures. The study of $\xi+\eta$ given $\xi\{x\}+\eta\{x\}>0$ is somewhat complicated by the fact that $(\xi+\eta)^*$ may differ from $\xi^*+\eta^*$. However the following holds:

Lemma. *For ξ and η as above $\mathbb{P}\{\xi+\eta)^*=\xi^*+\eta^*\}=1$ if and only if ξ and η have no joint atoms.*

Note. ξ has an atom at x if $\mathbb{P}\{\xi\{x\}>0\}>0$.

Proof. The necessity of the condition is obvious. For the sufficiency assume first that ξ has no atoms. Let $\varepsilon>0$ be given. Each $x\in X$ has a neighbourhood V_x such that $\mathbb{P}\{\xi^*\, V_x\geqq 1\}<\varepsilon$. By Lindelöf's theorem a countable number $\{V_n\}_1^\infty$ of these cover X. Defining

$$A_1=V_1, \quad A_k=V_k\setminus\bigcup_{n<k} V_n, \quad k=1,2,\ldots$$

we obtain a measurable partition of X into disjoint sets and for B bounded

$$\begin{aligned}\mathbb{P}\{\exists x\in B;\ \xi^*\{x\}+\eta^*\{x\}\geqq 2\}&=\mathbb{P}\{\exists x\in B;\ \xi^*\{x\}=\eta^*\{x\}=1\}\\ &\leqq\mathbb{P}\big(\bigcup_k\{\xi^*A_k\geqq 1,\eta^*A_k\cap B\geqq 1\}\big)\leqq\sum_k\mathbb{P}\{\xi^*A_k\geqq 1\}\,P\{\eta^*A_k\cap B\geqq 1\}\\ &<\varepsilon\sum_k\mathbb{P}\{\eta^*A_k\cap B\geqq 1\}\leqq\varepsilon\kappa B<\infty.\end{aligned}$$

It follows that $\xi^*+\eta^*$ can have no multiple points in B and by σ-compactness none at all.

Now to the general case. Let S be the set of atoms of ξ and B again bounded

$$\begin{aligned}&\mathbb{P}\{\exists x\in B;\ \xi^*\{x\}+\eta^*\{x\}\geqq 2\}\\ &\quad\leqq\mathbb{P}\{\exists x\in B\setminus S;\ \xi^*\{x\}+\eta^*\{x\}\geqq 2\}+\mathbb{P}\{\exists x\in B\cap S;\ \xi^*\{x\}+\eta^*\{x\}\geqq 2\}.\end{aligned}$$

To both these entities the preceding argument is applicable, ξ has no atom in $B\setminus S$, η none in $B\cap S$.

Proposition 9. *Let ξ and η with distributions P and Q have no joint atoms and $\lambda=E\xi^*$, $\kappa=E\eta^*$, $\nu=E(\xi+\eta)^*$. Then*

$$(P*Q)_x\,\nu(dx)=P_x*Q\,\lambda(dx)+P*Q_x\,\kappa(dx).$$

Note. This has the following interpretation: $(\xi+\eta)_x$ is distributed like $\xi_x+\eta$ with probability $\lambda(dx)/\nu(dx)$ and like $\xi+\eta_x$ with probability $\kappa(dx)/\nu(dx)$. Observe that $\lambda+\kappa=\nu$.

Proof. Let $B\in\mathscr{B}(\mathscr{N})$ and assume ξ and η independent

$$\begin{aligned}(P*Q)_x(B)\,\nu(dx) &= E[1_B(\xi+\eta)\,(\xi+\eta)^*(dx)]\\ &= E\left[1_B(\xi+\eta)\left(\xi^*(dx)+\eta^*(dx)\right)\right] = E[1_B(\xi+\eta)\,\xi^*(dx)] + E[1_B(\xi+\eta)\,\eta^*(dx)]\\ &= \int_{\mathscr{N}(X)} E[1_B(\xi+\mu)\,\xi^*(dx)]\,Q(d\mu) + \int_{\mathscr{N}(X)} E[1_B(\mu+\eta)\,\eta^*(dx)]\,P(d\mu)\\ &= P_x*Q(B)\,\lambda(dx) + P*Q_x(B)\,\kappa(dx).\end{aligned}$$

By induction we get

Proposition 10. *Let $P_1, P_2, \ldots, P_n$ be probability measures on $\mathscr{N}(X)$, such that if ξ_j and ξ_k have distributions P_j and P_k they have no joint atoms, $1\leqq j<k\leqq n$. Set $\lambda_k = E\xi_k^*$, $1\leqq k\leqq n$, $\lambda = E(\xi_1+\cdots+\xi_n)^*$, $f_k = d\lambda_k/d\lambda$. Then*

$$\left(\prod_{k=1}^{n}{}^{*} P_k\right)_x = \sum_{k=1}^{n} P_{kx} * \prod_{j\neq k}{}^{*} P_j \cdot f_k(x)$$

almost everywhere (λ).

Here $\prod_{k=1}^{n}{}^{*} P_k$ is the convolution product $P_1 * P_2 * \cdots * P_n$. Note that each λ_k is absolutely continuous with respect to λ and that the f_k can be so chosen that the differentiation rule holds for all x. In the next section we assume this to be done.

6. Limit Results

If P_n, P are probability measures on $\mathscr{N}(X)$ endowed with its vague topology we write $P_n \xrightarrow{w} P$ for the weak convergence of P_n towards P ($\int \varphi\, dP_n \to \int \varphi\, dP$ for all continuous bounded $\varphi\colon \mathscr{N}(X)\to R$). Similarly, if ξ_n has distribution P_n and ξ distribution P, $\xi_n \xrightarrow{w} \xi$ means that $P_n \xrightarrow{w} P$. By $\chi(P)$ or $\chi(\xi)$ we denote the characteristic functional defined on $\mathscr{C}_K$:

$$\chi(P)(f) = \int e^{i\mu f} P(d\mu) = E[e^{i\xi f}] = \chi(\xi)(f)$$

if $P = \mathbb{P}\,\xi^{-1}$, $f\in\mathscr{C}_K$. A continuity theorem holds: $P_n \xrightarrow{w} P$ if and only if $\chi(P_n)\to\chi(P)$ pointwise [8]. Of course Laplace transforms can be used as well [9].

In applications, like in queuing theory, it is often taken for granted that the convergence $\xi_n \xrightarrow{w} \xi$ enhances a similar convergence of conditioned processes: ξ_{nx} tends to ξ_x. Simple examples show that this cannot hold generally – for any sensible definition of the conditioned process. We shall use the result of the preceding section to make some remarks about conditioned convergence in the classical situation of superpositions of processes in a triangular array $\{\xi_{n1}, \xi_{n2}, \ldots, \xi_{nr_n}\}$ $n=1,2,\ldots$ of independent point processes.

Proposition 11. *Assume that ξ_{nk} is distributed according to P_{nk}, for each n no two ξ_{nj}, ξ_{nk} have joint atoms and $\xi_{n1}, \ldots, \xi_{nr_n}$ are independent,*

$$\xi_n = \sum_{k=1}^{r_n} \xi_{nk}$$

has distribution

$$P_n = \prod_{k=1}^{r_n}{}^* P_{nk},$$

$$\lambda_{nk} = E\,\xi^*_{nk}, \qquad \lambda_n = E\,\xi^*_n, \qquad f_{nk} = d\lambda_{nk}/d\lambda_n, \qquad P_n \xrightarrow{w} P.$$

Then, if for $x \in S \in \mathscr{B}(X)$ $P(x)$ is a probability measure on $\mathscr{N}(X)$ and

$$\lim_{n\to\infty} \sum_{k=1}^{r_n} f_{nk}(x)\,\chi(P_{nkx})/\chi(P_{nk}) = \chi(P(x)),$$

it holds that

$$P_{nx} \xrightarrow{w} P * P(x)$$

for all $x \in S$, provided f_{nk} has been chosen as required in the note after Proposition 10.

The proof is a direct consequence of Proposition 10 and the continuity theorem:

$$\begin{aligned}\chi(P_{nx}) &= \sum_{k=1}^{r_n} \chi(P_{nkx}) \cdot \prod_{j \neq k} \chi(P_{nk}) f_{nk}(x)\\ &= \chi(P_n) \sum_{k=1}^{r_n} f_{nk}(x)\,\chi(P_{nkx})/\chi(P_{nk}) \to \chi(P)\,\chi(P(x)).\end{aligned}$$

If the triangular array is infinitesimal, that is

$$\lim_{n\to\infty} \max_{1 \leqq k \leqq r_n} \mathbb{P}\{\xi_{nk} K \geqq 1\} = 0$$

for all compact sets K, then the limit law P must be infinitely divisible. Moreover, if its process satisfies the conditions of Proposition 6, $P_x = P * P(x)$ for some family $P(x)$. Thus in this case we find that under the conditions of Proposition 11 $P_{nx} \xrightarrow{w} P_x$, $x \in S$, provided both the $P(x)$ are the same.

Let us now assume that the infinitesimal triangular array is also conditionally infinitesimal: for all compact $K \subset X$ and $x \in S$

$$\lim_{n\to\infty} \max_{1 \leqq k \leqq r_n} \mathbb{P}\{\xi_{nk}(K \setminus \{x\}) \geqq 1 \mid \xi_{nk}\{x\} > 0\} = 0,$$

and assume that, $x \in S$,

$$\lim_{n\to\infty} \mathbb{P}\{\xi_{nk}\{x\} = j \mid \xi_{nk}\{x\} > 0\} = p_j(x),$$

$j = 1, 2, \ldots$ uniformly in k.

Then for any $f \in \mathscr{C}_K$

$$\lim_{n\to\infty} \chi(P_{nk})(f) = 1,$$

$$\lim_{n\to\infty} \chi(P_{nkx})(f) = \sum_{j=1}^{\infty} P_j(x)\, e^{if(x)j} = \chi(P(x))(f),$$

if $P(x)\{\mu \in \mathscr{N}(X);\ \mu = j\,e_x\} = p_j(x)$, both limits holding uniformly in k, $1 \leqq k \leqq r_n$.

Since

$$\sum_{k=1}^{r_n} f_{nk}(x) = 1 \qquad \text{a.s. } \lambda_n,$$

say for $x \in S_n$, we have for $x \in S \cap \limsup S_n$ that

$$P_{nx} \xrightarrow{w} P * P(x),$$

if $P_n \xrightarrow{w} P$ and the condition of no joint atoms is satisfied. In particular, if the array has asymptotically no multiple points in the sense that $P_j(x) = 0$ for $j > 0$, i.e.

$$\lim_{n\to\infty} \max_{1 \leqq k \leqq r_n} \mathbb{P}\{\xi_{nk}\{x\} \geqq 2 \mid \xi_{nk}\{x\} > 0\} = 0$$

for $x \in S$, then it follows that

$$P_{nx} \to P * \Delta_x$$

for $x \in S \cap \limsup S_n$ as above. Thus when P is Poisson measure, and only then, $P_{nx} \to P_x$. However P is Π_λ if and only if for all compact $K \subset X$ and bounded $A \in \mathscr{B}(X)$ such that $\lambda(\partial A) = 0$, ∂ for "boundary of",

$$\sum_{k=1}^{r_n} \mathbb{P}\{\xi_{nk} K \geqq 2\} \to 0, \tag{1}$$

$$\sum_{k=1}^{r_n} \mathbb{P}\{\xi_{nk} A = 1\} \to \lambda A \tag{2}$$

[8, 9]. Combining these facts we obtain

Proposition 12. *Let $\{\xi_{n1}, \xi_{n2}, \ldots, \xi_{nr_n}\}$ be an infinitesimal array of independent point processes such that, for fixed n, no two ξ_{nj}, ξ_{nk} have joint atoms,* (1) *and* (2) *are satisfied, and for any compact set K*

$$\lim_{n\to\infty} \max_{1 \leqq k \leqq r_n} \mathbb{P}\{\xi_{nk} K \geqq 2 \mid \xi_{nk}\{x\} > 0\} = 0, \qquad x \in S.$$

Then

$$\xi_n = \sum_{k=1}^{r_n} \xi_{kn} \xrightarrow{w} \xi$$

where ξ is Poisson with intensity λ, and for $x \in S \cap \limsup S_n$

$$\xi_{nx} \xrightarrow{w} \xi_x.$$

If $\lim_{n\to\infty} E\xi_n^* A = 0$ implies that $\lambda A = 0$, the convergence holds almost everywhere (λ) on S.

Proof. The last assertion is easily checked,

$$\lambda(\limsup S_n)' = \lim_{k\to\infty} \lambda\left(\bigcap_{m=k}^{\infty} S'_m\right)$$

and, $\lambda_n = E\xi_n^*$,

$$\lambda_n\left(\bigcap_{m=k}^{\infty} S'_m\right) \leqq \lambda_n(S'_n) = 0$$

for $n \geqq k$. The rest follows from

$$\mathbb{P}\{\xi_{nk}\{x\} \geqq 2 \mid \xi_{nk}\{x\} > 0\} \leqq \mathbb{P}\{\xi_{nk} K \geqq 2 \mid \xi_{nk}\{x\} > 0\}$$

and

$$\mathbb{P}\{\xi_{nk}(K \setminus \{x\}) \geqq 1 \mid \xi_{nk}\{x\} > 0\} \leqq \mathbb{P}\{\xi_{nk} K \geqq 2 \mid \xi_{nk}\{x\} > 0\}.$$

The explicit reference to Palm distributions in the conditions for Proposition 12 can of course be suppressed, for example using the following argument:

For any compact $K \subset X$ assume that

$$\lim_{n\to\infty} \max_{1\leqq k\leqq r_n} \sup \{E[\xi_{nk} A; \xi_{nk} K \geqq 2]/\lambda_{nk} A; A \in \mathscr{B}(X), \text{ bounded}\} = 0.$$

There is an S_{nk} with full λ_{nk} measure such that

$$\mathbb{P}\{\xi_{nk} K \geqq 2 \mid \xi\{x\} > 0\} \leqq \sup_A E[\xi_{nk} A; \xi_{nk} K \geqq 2]/\lambda_{nk} A$$

for $x \in S_{nk}$. Obviously

$$\lim_{n\to\infty} \max_{1\leqq k\leqq r_n} \mathbb{P}\{\xi_{nk} K \geqq 2 \mid \xi\{x\} > 0\} = 0$$

for $x \in \limsup \bigcap_{k=1}^{r_n} S_{nk}$.

7. Conditioning upon more Knowledge

Let us denote by ξ^{*k} the random product measure $\xi^* \times \cdots \times \xi^*$ k times on X^k and define the higher moment measure $\lambda_k = E\,\xi^{*k}$. For $\varphi\colon X^k \times \mathcal{N}(X) \to R_+$ $E_{x_1 x_2, \ldots, x_k} \varphi$ can be defined as before:

$$E_{x_1 x_2, \ldots, x_k} \varphi\, \lambda_k(dx_1\, dx_2, \ldots, dx_k) = E[\varphi(x_1, x_2, \ldots, x_k, \xi)\, \xi^{*k}(dx_1\, dx_2, \ldots, dx_k)].$$

$P_{x_1 x_2, \ldots, x_k}$ is defined exactly as with $k=1$ and it is easy to check that

$$(P_{x_1, \ldots, x_k})_{x_{k+1}} = P_{x_1, \ldots, x_k x_{k+1}}$$

and since

$$\int \mu A \Delta_x(d\mu) = 0$$

if $x \notin A$ repeated use of Proposition 9 yields

$$\Pi_{\lambda x_1 x_2, \ldots, x_k} = \Pi_\lambda * \Delta_{x_1} * \Delta_{x_2} * \cdots * \Delta_{x_k},$$

provided $x_1, x_2, \ldots, x_k$ are all different.

Using

$$P_{x_1 x_2, \ldots, x_k}(r_1, r_2, \ldots, r_k) = \mathbb{P}\{\xi\{x_j\} = r_j,\ 1 \leqq j \leqq k \mid \xi\{x_j\} > 0,\ 1 \leqq j \leqq k\},$$

we can define the conditional distribution, given complete knowledge about the number of points in $x_1, x_2, \ldots, x_k$:

$$\begin{aligned} &\mathbb{P}\{\xi \in B \mid \xi\{x_j\} = r_j,\ 1 \leqq j \leqq k\}\, p_{x_1 x_2 \ldots x_k}(r_1, r_2 \ldots r_k)\, \lambda_k(dx_1\, dx_2 \ldots dx_k) \\ &\quad = E\left[1_B(\xi)\, 1_{\{\nu;\, \nu\{x_j\} = r_j,\ 1 \leqq j \leqq k\}}(\xi)\, \xi^{*k}(dx_1\, dx_2, \ldots, dx_k)\right]. \end{aligned}$$

And it is not difficult to verify that the distribution of the places of occurrences of a point process ξ, that is ξ^*'s distribution, together with the mass distributions on places of occurrences, that is the $P_{x_1 x_2, \ldots, x_k}(r_1, r_2, \ldots, r_k)$, determine the distribution of the whole process.

Let us end by noting that $E_{x_1 x_2 \ldots x_k}$ could also be defined via a process ξ_k^* which assigns to sets $A \subset X_k$ the number

$$\xi^{*k}\{(x_1, x_2, \ldots, x_k) \in A;\ \text{all } x_j \text{ are different}\}.$$

From some points of view such an approach would be attractive.

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