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Mixtures of Infinitely Divisible Distributions as Limit Laws for Sums of Dependent Random Variables

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1. Introduction

Let there be given a double sequence $\mathfrak{X}$ of random variables (rv's)

$$
\begin{array}{l}
X_{11}, X_{12}, \ldots, X_{1,k_1}, \\
X_{21}, X_{22}, \ldots, X_{2,k_2}, \\
\ldots\ldots\ldots\ldots\ldots\ldots \\
X_{n1}, X_{n2}, \ldots, X_{n,k_n}, \\
\ldots\ldots\ldots\ldots\ldots\ldots
\end{array}
$$

and a sequence of their row sums

$$S_n = \sum_{k=1}^{k_n} X_{nk}, \qquad n \in \mathbb{N},$$

where all these rv's are defined on a common probability space $(\Omega, \mathscr{F}, \mathbf{P})$.

The problem of the asymptotic behaviour of the probability distributions of S_n, $n \in \mathbb{N}$, is mainly contained in two questions:

1° which measures can appear as weak limit laws?

2° which properties of $\mathfrak{X}$ imply the weak convergence of S_n, $n \in \mathbb{N}$, to the specified probability measure?

Bavly's idea of accompanying laws allows us to find the complete solution to this problem for row-wise independent arrays $\mathfrak{X}$. It is based on the fact that the characteristic function (chf) of the sum S_n can be approximated by a suitably constructed, infinitely divisible chf. If we assume some "smallness" conditions on $\mathfrak{X}$ then these two sequences of chf's have the same common limit. Of course, in this case the limit law of S_n, $n \in \mathbb{N}$, must be infinitely divisible. Moreover, the method of the above construction produces necessary and sufficient conditions for the weak convergence of the sum distributions given in terms of $\mathfrak{X}$. If we omit the postulate of the independence of the rv's in the same rows, then the situation is more complicated (e.g. every probability measure can appear as a

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limit law). One way to look for limit theorems for dependent rv's is to generalize the classical situation i.e. to give such conditions which imply weak convergence and which turn out to be known from the classical theory if we apply them to independent rv's. An essential step in this direction was made by Brown and Eagleson ([1]) by putting to a good use the idea of accompanying laws. Observe that with $\mathfrak{X}$ we can associate (non-uniquely) a double array $\mathfrak{F}$ of σ-fields:

$$\begin{array}{l}\mathscr{F}_{1,0}\subset\mathscr{F}_{1,1}\subset\ldots\subset\mathscr{F}_{1,k_1}\subset\mathscr{F}\\ \mathscr{F}_{2,0}\subset\mathscr{F}_{2,1}\subset\ldots\subset\mathscr{F}_{2,k_2}\subset\mathscr{F},\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ \mathscr{F}_{n,0}\subset\mathscr{F}_{n,1}\subset\ldots\subset\mathscr{F}_{n,k_n}\subset\mathscr{F},\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{array}$$

such that every rv X_{nk} is $\mathscr{F}_{n,k}$-measurable. The pair $(\mathfrak{X}, \mathfrak{F})$ will be called a system. Because of a certain approximation lemma (see [1] and Lemma 3.2 of [3]) we can imitate the classical case by the construction of some "conditional chf's", which are defined in terms of conditional quantities of the rv's from $\mathfrak{X}$ with respect to the σ-fields from $\mathfrak{F}$ and approximate the chf's of S_n, $n\in\mathbb{N}$. If in the known necessary and sufficient conditions all mean values are replaced by conditional mean values with respect to σ-fields from $\mathfrak{F}$ and such obtained sequences of rv's are convergent in probability, then these new conditions guarantee the convergence of the "conditional chf's". The possibility of such an approximation is given by the analogous conditional "smallness" properties. The above procedure was first applied for infinitely divisible laws with finite variance by Brown and Eagleson in [1]; then it was extended by Kłopotowski for an arbitrary infinitely divisible law in $\mathbb{R}^d$ ([3]). Now we extend the class of possible weak limits taking into account mixtures of infinitely divisible laws. This extension is maximal; every probability distribution in $\mathbb{R}^1$, induced by some rv on Ω, can be trivially decomposed as the mixture of infinitely divisible laws. Eagleson in [2] has proved a limit theorem for martingale difference sequences with finite variances which generalizes the preceeding situation, giving sufficient conditions for weak convergence to mixtures of laws with finite variances. The proof of his theorem is based on a very artificial construction involving some regular conditional probabilities on $\mathbb{R}^\infty$. A purpose of this note is to show that in the case of mixtures the idea of the accompanying laws can also be applied. Proofs thereof will be given in the most general case without any assumptions about the existence of moments of the rv's and of the mixed limit laws.

2. Mixtures

For every $t\in\mathbb{R}^1$ let us define a function g_t: $\mathbb{R}^1\to\mathbb{C}$ as follows

$$g_t(x)=\begin{cases}\left(e^{itx}-1-\dfrac{itx}{1+x^2}\right)\dfrac{1+x^2}{x^2} & \text{for } x\neq 0\\ -t^2/2 & \text{for } x=0.\end{cases}$$

The function g_t is continuous and bounded on $\mathbb{R}^1$ i.e. there exists a constant $M = M_t > 0$ such that $|g_t(x)| \leqq M$ for $x \in \mathbb{R}^1$.

For the distribution function K of some finite measure on $\mathbb{R}^1$ and some $a \in \mathbb{R}^1$ the function

$$(1) \quad \varphi(t) = \exp\left\{ita + \int_{-\infty}^{+\infty} \left(e^{itx} - 1 - \frac{itx}{1+x^2}\right) \frac{1+x^2}{x^2} dK(x)\right\}, \qquad t \in \mathbb{R}^1,$$

is the chf of some infinitely divisible law on $\mathbb{R}^1$. Conversely, every chf of an infinitely divisible law on $\mathbb{R}^1$ can be uniquely decomposed in the form (1).

Now let us assume that both the parameters a and K in (1) are random i.e. $a(\cdot)$ and $K(x, \cdot)$, $x \in \mathbb{R}^1$, are rv's and for a.e. $\omega \in \Omega$ $K(\cdot, \omega)$ is bounded, nondecreasing, left continuous, $\lim_{x \to -\infty} K(x, \omega) = 0$. Thus we have the family of chf's

$$(2) \quad \varphi(t, \omega) = \exp\left\{ita(\omega) + \int_{-\infty}^{+\infty} g_t(x)\, dK(x, \omega)\right\}, \qquad t \in \mathbb{R}^1,$$

defined for a.e. $\omega \in \Omega$. Integrating both sides of (2) with respect to $\mathbf{P}$ we obtain the chf

$$(3) \quad \psi(t) = \int_{\Omega} \varphi(t, \omega)\, d\mathbf{P}(\omega), \qquad t \in \mathbb{R}^1.$$

Its corresponding probability measure is called the mixture of laws given by (2) and will be denoted by $\mathcal{M}ix(a, K)$. If, instead of g_t, we use the functions h_t, $t \in \mathbb{R}^1$, defined by

$$h_t(x) = \begin{cases} (e^{itx} - 1 - itx)/x^2 & \text{for } x \neq 0 \\ -t^2/2 & \text{for } x = 0 \end{cases}$$

then

$$\rho(t) = \int_{\Omega} \exp\left\{ita(\omega) + \int_{-\infty}^{+\infty} h_t(x)\, dK(x, \omega)\right\} d\mathbf{P}(\omega), \qquad t \in \mathbb{R}^1,$$

is the chf of the mixture of some infinitely divisible laws with finite variances; it will be denoted by $mix(a, K)$.

3. Accompanying Conditional Laws

For given $\mathfrak{F}$ let us define the σ-field $\mathscr{F}_0 = \bigcap_{n=1}^{\infty} \mathscr{F}_{n,0}$. Our fundamental assumption about $\mathcal{M}ix(a, K)$ and $\mathfrak{F}$ is:

(C.0) 1° $a(\cdot)$ is $\mathscr{F}_0$-measurable,
2° for every fixed $x \in \mathbb{R}^1$ $K(x, \cdot)$ is $\mathscr{F}_0$-measurable,
3° $K(+\infty, \cdot) = \lim_{x \to +\infty} K(x, \cdot)$ is finite a.e.

Because of 3° all $K(x,\cdot)$, $-\infty \leqq x \leqq +\infty$, are finite a.e. Of course $K(-\infty,\cdot) = \lim_{x\to-\infty} K(x,\cdot)=0$. 2° implies the $\mathscr{F}_0$-measurability of $K(+\infty,\cdot)$.

In this part we shall consider only systems $(\mathfrak{X},\mathfrak{F})$ having the following properties:

$$\text{(C.1)}\quad \sum_{k=1}^{k_n} \left\{A_{nk}+E_{n,k-1}\left(\frac{Y_{nk}}{1+Y_{nk}^2}\right)\right\} \xrightarrow{\text{a.e.}} a(\cdot);$$

$$\text{(C.2)}\quad \sum_{k=1}^{k_n} E_{n,k-1}\left(\frac{Y_{nk}^2}{1+Y_{nk}^2} I(Y_{nk}<x)\right) \xrightarrow{\text{a.e.}} K(x,\cdot)$$

for every x belonging to some countable dense set $D\subset\mathbb{R}^1$;

$$\text{(C.3)}\quad \sum_{k=1}^{k_n} E_{n,k-1}\left(\frac{Y_{nk}^2}{1+Y_{nk}^2}\right) \xrightarrow{\text{a.e.}} K(+\infty,\cdot);$$

where

$$A_{nk}=E_{n,k-1}(X_{nk}I(|X_{nk}|\leqq\tau)),\quad 1\leqq k\leqq k_n,\quad n\in\mathbb{N},$$

$$Y_{nk}=X_{nk}-A_{nk},\quad 1\leqq k\leqq k_n,\quad n\in\mathbb{N},$$

for arbitrarily fixed $\tau>0$.

(Here and in the sequel we use the notation

$$E_{n,k-1}(X)=E(X|\mathscr{F}_{n,k-1}),\quad P_{n,k-1}(A)=P(A|\mathscr{F}_{n,k-1}).$$

All equalities and inequalities between rv's are considered in the sense "with probability one", $\xrightarrow{\text{a.e.}}$ denotes the convergence almost sure, $\xrightarrow{\mathbf{P}}$ denotes the convergence in probability.)

Observe that (C.2) implies

$$\text{(4)}\quad \sum_{k=1}^{k_n} E_{n,k-1}\left(\frac{Y_{nk}^2}{1+Y_{nk}^2} I(a\leqq Y_{nk}<b)\right) \xrightarrow{\text{a.e.}} K(b,\cdot)-K(a,\cdot)$$

for every a, $b\in D$.

We are going to prove that if, in addition, the system $(\mathfrak{X},\mathfrak{F})$ satisfies

$$\text{(C.4)}\quad \sum_{k=1}^{k_n} E_{n,k-1}\left(\frac{Y_{nk}^2}{1+Y_{nk}^2}\right)\leqq C,\quad n\in\mathbb{N},$$

for some $\mathscr{F}_0$-measurable rv $0\leqq C<+\infty$, then for every fixed $t\in\mathbb{R}^1$

$$\text{(5)}\quad \sum_{k=1}^{k_n} E_{n,k-1}\left(g_t(Y_{nk})\frac{Y_{nk}^2}{1+Y_{nk}^2}\right) \xrightarrow{\text{a.e.}} \int_{-\infty}^{+\infty} g_t(x)\,dK(x,\cdot).$$

Let us fix a sequence of real numbers ε_s, $s\in\mathbb{N}$, strictly decreasing to zero. For every $s\in\mathbb{N}$ we choose a sufficiently large number of points on the real line

$$x_0^{(s)}<x_1^{(s)}<\ldots<x_{m_s}^{(s)}$$

from the dense set D such that

a) $x_0^{(s)} \searrow -\infty$, $x_{m_s}^{(s)} \nearrow +\infty$,

b) $\max_{1 \leqq j \leqq m_s} |x_j^{(s)} - x_{j-1}^{(s)}| \searrow 0$, $s \to +\infty$,

c) $\max_{1 \leqq j \leqq m_s} |g_t(x_j^{(s)}) - g_t(x_{j-1}^{(s)})| < \varepsilon_s$, $s \in \mathbb{N}$.

Since the improper Lebesgue-Stieltjes stochastic integral $\int_{-\infty}^{+\infty} g_t(x)\, dK(x, \cdot)$ is well defined for a.e. $\omega \in \Omega$, then by the continuity and the boundedness of g_t this integral is equal to the improper Riemann-Stieltjes one. Therefore (for fixed t)

$$(6) \quad \sum_{j=1}^{m_s} g_t(x_{j-1}^{(s)})[K(x_j^{(s)}, \cdot) - K(x_{j-1}^{(s)}, \cdot)] \xrightarrow{\text{a.e.}} \int_{-\infty}^{+\infty} g_t(x)\, dK(x, \cdot)$$

and the integral is an $\mathscr{F}_0$-measurable, a.e. finite rv. For the moment let us fix $s \in \mathbb{N}$. From (4)

$$(7) \quad \sum_{k=1}^{k_n} \sum_{j=1}^{m_s} g_t(x_{j-1}^{(s)})\, E_{n,k-1}\left(\frac{Y_{nk}^2}{1+Y_{nk}^2} I(x_{j-1}^{(s)} \leqq Y_{nk} < x_j^{(s)})\right)$$
$$\xrightarrow{\text{a.e.}} \sum_{j=1}^{m_s} g_t(x_{j-1}^{(s)})[K(x_j^{(s)}, \cdot) - K(x_{j-1}^{(s)}, \cdot)].$$

Next, observe that for this fixed s and every $n \in \mathbb{N}$

$$(8) \quad \Bigg| \sum_{k=1}^{k_n} \sum_{j=1}^{m_s} E_{n,k-1}\left(g_t(x_{j-1}^{(s)}) \frac{Y_{nk}^2}{1+Y_{nk}^2} I(x_{j-1}^{(s)} \leqq Y_{nk} < x_j^{(s)})\right)$$
$$- \sum_{k=1}^{k_n} \sum_{j=1}^{m_s} E_{n,k-1}\left(g_t(Y_{nk}) \frac{Y_{nk}^2}{1+Y_{nk}^2} I(x_{j-1}^{(s)} \leqq Y_{nk} < x_j^{(s)})\right)\Bigg| \leqq \varepsilon_s \cdot C.$$

Finally

$$(9) \quad \sum_{k=1}^{k_n} E_{n,k-1}\left(g_t(Y_{nk}) \frac{Y_{nk}^2}{1+Y_{nk}^2}\right)$$
$$- \sum_{k=1}^{k_n} E_{n,k-1}\left(g_t(Y_{nk}) \frac{Y_{nk}^2}{1+Y_{nk}^2} I(x_0^{(s)} \leqq Y_{nk} < x_{m_s}^{(s)})\right)$$
$$\xrightarrow{\text{a.e.}} K(+\infty, \cdot) - K(x_{m_s}^{(s)}, \cdot) + K(x_0^{(s)}, \cdot).$$

For every natural number s, each of conditions (6)–(9) determines a **P**-null set on which this condition is not fulfilled; the same is true of (C.4). The sum N_0 of these sets is a **P**-null set. On the set $\Omega \setminus N_0$ all sequences of rv's in (6)–(9) are pointwise convergent. (Moreover, we can assume that the rv C is finite on $\Omega \setminus N_0$.) It remains to prove that this is also true for (5) i.e. for every $\omega \in \Omega \setminus N_0$

$$(10) \quad \sum_{k=1}^{k_n} E_{n,k-1}\left(g_t(Y_{nk}) \frac{Y_{nk}^2}{1+Y_{nk}^2}\right)(\omega) \xrightarrow[n \to \infty]{} \int_{-\infty}^{+\infty} g_t(x)\, dK(x, \omega).$$

Let us fix $\varepsilon>0$. Choose s so large that

1° $\varepsilon_s \cdot C<\varepsilon/8$,

2° $K(+\infty,\omega)-K(x_{m_s}^{(s)},\omega)+K(x_0^{(s)},\omega)<\varepsilon/8$,

3° the absolute value of the difference between both sides in (6) at ω is less than $\varepsilon/2$.

For this s there exists n_ε so large that for $n>n_\varepsilon$ the absolute value of the difference between both sides in (7) and (9) at ω is less than $\varepsilon/8$. This gives (10). From (C.1) and (5), by the continuity of exp, we obtain the final conclusion of this part:

$$(11)\quad \exp\left\{it\sum_{k=1}^{k_n}\left[A_{nk}+E_{n,k-1}\left(\frac{Y_{nk}}{1+Y_{nk}^2}\right)\right] + \sum_{k=1}^{k_n}E_{n,k-1}\left(g_t(Y_{nk})\frac{Y_{nk}^2}{1+Y_{nk}^2}\right)\right\} \xrightarrow{\text{a.e.}} \exp\left\{ita(\cdot)+\int_{-\infty}^{+\infty}g_t(x)\,dK(x,\cdot)\right\},\qquad t\in\mathbb{R}^1.$$

The idea of the accompanying laws is contained in this fact. To see this, it suffices to take into account a row-wise independent system $\mathfrak{X}$ and non-random a, K.

For technical reasons we shall use a property which is very similar to (11) and follows from an identical argument:

Lemma 1. *Assumptions* (C.0)–(C.4) *imply*

$$(12)\quad \exp\left\{-it\sum_{k=1}^{k_n}\left[A_{nk}+E_{n,k-1}\left(\frac{Y_{nk}}{1+Y_{nk}^2}\right)\right] - \sum_{k=1}^{k_n}E_{n,k-1}\left(g_t(Y_{nk})\frac{Y_{nk}^2}{1+Y_{nk}^2}\right)\right\} \xrightarrow{\text{a.e.}} \exp\left\{-ita(\cdot)-\int_{-\infty}^{+\infty}g_t(x)\,dK(x,\cdot)\right\},\qquad t\in\mathbb{R}^1.$$

4. Comments about (C.4)

Putting

$$V_{nk}=\sum_{j=1}^{k}E_{n,j-1}\left(\frac{Y_{nj}^2}{1+Y_{nj}^2}\right),\qquad 1\leqq k\leqq k_n,\quad n\in\mathbb{N},$$

$$C=K(+\infty,\cdot)+1,$$

we obtain from (C.3) that

$$(13)\quad P(\limsup_{n\to+\infty}[V_{n,k_n}>C])=0.$$

If we truncate the rv's x_{nk} in the following manner

$$X^*_{nk} = X_{nk} I(V_{nk} \leqq C), \quad 1 \leqq k \leqq k_n, \quad n \in \mathbb{N},$$

then by the $\mathscr{F}_{n,k-1}$-measurability of $V_{nk} - C$ and $I(V_{nk} \leqq C)$ we have

$$A^*_{nk} = E_{n,k-1}(X^*_{nk} I(|X^*_{nk}| \leqq \tau)) = A_{nk} I(V_{nk} \leqq C),$$

$$Y^*_{nk} = X^*_{nk} - A^*_{nk} = Y_{nk} I(V_{nk} \leqq C),$$

$$V^*_{nk} = \sum_{j=1}^{k} E_{n,j-1}\left(\frac{Y^{*2}_{nj}}{1+Y^{*2}_{nj}}\right) = \sum_{j=1}^{k} E_{n,j-1}\left(\frac{Y^2_{nj}}{1+Y^2_{nj}}\right) I(V_{nj} \leqq C),$$

for $1 \leqq k \leqq k_n$, $n \in \mathbb{N}$.

Thus the last equality gives the property (C.4) for $(\mathfrak{X}, \mathfrak{F})$. Using (13), one can easily prove that $(\mathfrak{X}, \mathfrak{F})$ satisfies (C.1)–(C.3) and that the limit distributions of S_n, $n \in \mathbb{N}$, and $S^*_n = \sum_{k=1}^{k_n} X^*_{nk}$, $n \in \mathbb{N}$, are equal. Moreover

$$\exp(itS_n) - \exp(itS^*_n) \xrightarrow{\text{a.e.}} 0$$

and then by the bounded convergence theorem for a conditional expectation we have

$$(14) \quad E(\exp(itS_n)|\mathscr{F}_0) - E(\exp(itS^*_n)|\mathscr{F}_0) \xrightarrow{\text{a.e.}} 0$$

5. Approximation

The following version of a lemma of Brown and Eagleson gives us the possibility to approximate the conditional chf's of the accompanying laws.

Lemma 2. *Let there be given a σ-field $\mathscr{F}_0 \subset \mathscr{F}$ and a function $f: \mathbb{R}^1 \times \Omega \to \mathbb{C}$ such that $f(t, \cdot)$ is $\mathscr{F}_0$-measurable for every $t \in \mathbb{R}^1$, $|f(t,\omega)| \leqq 1$, $f(t,\omega) \neq 0$ for all $t \in \mathbb{R}^1$ and a.e. $\omega \in \Omega$. For fixed $t \in \mathbb{R}^1$ let U_n, W_n be sequences of rv's such that*

1° *for some $\mathscr{F}_0$-measurable a.e. finite rv C*

$$|W_n^{-1}| \leqq C, \quad n \in \mathbb{N},$$

2° $f(t, \cdot) W_n^{-1} \xrightarrow{\text{a.e.}} 1.$

Then

3° $E(\exp(itU_n)|\mathscr{F}_0) \xrightarrow{\text{a.e.}} f(t, \cdot)$

if and only if

4° $E(W_n^{-1} \exp(itU_n)|\mathscr{F}_0) \xrightarrow{\text{a.e.}} 1.$

Proof. $4° \Rightarrow 3°$. Since

$$\begin{aligned}
&|E(\exp(itU_n)|\mathscr{F}_0)-f(t,\cdot)| \\
&\quad \leqq |E(\exp(itU_n)[1-W_n^{-1}f(t,\cdot)]|\mathscr{F}_0)| \\
&\quad +|E(f(t,\cdot)[W_n^{-1}\exp(itU_n)-1]|\mathscr{F}_0)| \\
&\quad \leqq E(|1-W_n^{-1}f(t,\cdot)||\mathscr{F}_0)+|E(W_n^{-1}\exp(itU_n)|\mathscr{F}_0)-1|
\end{aligned}$$

we obtain the result by the following elementary sublemma:

If $X_n \xrightarrow{\text{a.e.}} 0$, $|X_n| \leqq G$ a.e. for an $\mathscr{F}_0$-measurable, a.e. finite rv G, then $E(|X_n||\mathscr{F}_0) \xrightarrow{\text{a.e.}} 0$.

Similarly we prove $3° \Rightarrow 4°$.

Applying Lemmas 1 and 2 to

$$U_n = S_n, \quad n \in \mathbb{N},$$

$$\begin{aligned}
W_n = \exp\Bigg\{ & it \sum_{k=1}^{k_n} \left[A_{nk} + E_{n,k-1}\left(\frac{Y_{nk}}{1+Y_{nk}^2}\right)\right] \\
& + \sum_{k=1}^{k_n} E_{n,k-1}\left(g_t(Y_{nk})\frac{Y_{nk}^2}{1+Y_{nk}^2}\right)\Bigg\}, \quad n \in \mathbb{N},
\end{aligned}$$

we obtain:

Theorem 1. *If the system* $(\mathfrak{X}, \mathfrak{F})$ *satisfies* (C.0)–(C.4), *then for every* $t \in \mathbb{R}^1$

$$(15) \quad E(\exp(itS_n)|\mathscr{F}_0) \xrightarrow{\text{a.e.}} \exp\left[ita(\cdot) + \int_{-\infty}^{+\infty} g_t(x)\, dK(x,\cdot)\right]$$

if and only if

$$(16) \quad E\left(\exp\left[it\sum_{k=1}^{k_n} Y_{nk} - \sum_{k=1}^{k_n} E_{n,k-1}(\exp(itY_{nk})-1)\right]\Bigg|\mathscr{F}_0\right) \xrightarrow{\text{a.e.}} 1.$$

Corollary 1. *If the system* $(\mathfrak{X}, \mathfrak{F})$ *satisfies* (C.0)–(C.3) *then it fulfils* (15) *if and only if each of the equivalent conditions is satisfied:*

1° (16) *holds for every* $\mathfrak{X}^*$ *given (in the manner described in Sect.* 4) *by all* $\mathscr{F}_0$-*measurable rv* C, $K(+\infty) < C < +\infty$;

2° (16) *holds for* $\mathfrak{X}^*$ *given by some* $\mathscr{F}_0$-*measurable rv* C, $K(+\infty) < C < +\infty$.

Observe that if $(\mathfrak{X}, \mathfrak{F})$ satisfies (C.0)–(C.4), then the sequence integrated in (16) is bounded and therefore the property

$$(17) \quad \exp\left[it\sum_{k=1}^{k_n} Y_{nk} - \sum_{k=1}^{k_n} E_{n,k-1}(\exp(itY_{nk})-1)\right] \xrightarrow{\text{a.e.}} 1$$

is sufficient for (15) to be satisfied. Then using the $*$-procedure one can prove:

Corollary 2. *If* $(\mathfrak{X}, \mathfrak{F})$ *satisfies* (C.0)–(C.3) *and* (17), *then it fulfils* (15).

It seems that conditions (16) and (17) have no intuitive meaning (thanks to their generality). Now we are going to show that they are implied by the more restrictive requirement that the rv's are uniformly asymptotically negligible in some sense. In the first place such a property is contained in the following

Theorem 2. *If the system* $(\mathfrak{X}, \mathfrak{F})$ *satisfies* (C.0)–(C.3) *and*

$$\text{(C.5)}\quad \sum_{k=1}^{k_n} |E_{n,k-1}(\exp(itY_{nk})-1)|^2 \xrightarrow{\text{a.e.}} 0, \qquad t\in\mathbb{R}^1,$$

then it satisfies (15).

Proof. Applying the above definition of $\mathfrak{X}^*$ we obtain

$$\begin{aligned}|E_{n,k-1}(\exp(itY_{nk}^*)-1)|^2 &= I(V_{nk}\leqq C)\,|E_{n,k-1}(\exp(itY_{nk})-1)|^2\\ &\leqq |E_{n,k-1}(\exp(itY_{nk})-1)|^2\end{aligned}$$

and thus we may assume (C.4) for $(\mathfrak{X}, \mathfrak{F})$.

Repeating the arguments used in the proof of Theorem 3.4 of [3] we can prove that

a) the sequence in (C.5) is bounded from above by $2(M^2+t)\cdot C$ and then we have

$$\text{(18)}\quad \sum_{k=1}^{k_n} E(|E_{n,k-1}(\exp(itY_{nk})-1)|^2 | \mathscr{F}_0) \xrightarrow{\text{a.e.}} 0,$$

b) the following inequality holds

$$\begin{aligned}&\left|E\left(\exp\left[it\sum_{k=1}^{k_n} Y_{nk} - \sum_{k=1}^{k_n} E_{n,k-1}(\exp(itY_{nk})-1)\right]-1\,\middle|\,\mathscr{F}_0\right)\right|\\ &\leqq \tfrac{1}{2}\exp(2+M\cdot C)\sum_{k=1}^{k_n} E(|E_{n,k-1}(\exp(itY_{nk})-1)|^2|\mathscr{F}_0), \qquad n\in\mathbb{N}.\end{aligned}$$

Then (18) implies the required result.

Now let us assume that $(\mathfrak{X}, \mathfrak{F})$ is strongly conditionally infinitesimal i.e. for every $\varepsilon>0$

$$\text{(C.6)}\quad \max_{1\leqq k\leqq k_n} P_{n,k-1}(|X_{nk}|>\varepsilon) \xrightarrow{\text{a.e.}} 0.$$

Observe that if (C.6) is satisfied for every element of some sequence ε_s, $s\in\mathbb{N}$, strictly decreasing to zero, then $(\mathfrak{X}, \mathfrak{F})$ is strongly conditionally infinitesimal.

This condition implies

$$\begin{aligned}\text{(19)}\quad &\max_{1\leqq k\leqq k_n} |A_{nk}| \xrightarrow{\text{a.e.}} 0, \qquad \tau>0;\\ &\max_{1\leqq k\leqq k_n} P_{n,k-1}(|Y_{nk}|>\varepsilon) \xrightarrow{\text{a.e.}} 0, \qquad \varepsilon>0.\end{aligned}$$

The last limit is equivalent to

$$\max_{1\leqq k\leqq k_n} E_{n,k-1}\left(\frac{Y_{nk}^2}{1+Y_{nk}^2}\right) \xrightarrow{\text{a.e.}} 0,$$

which, with (C.3), gives

$$\sum_{k=1}^{k_n}\left[E_{n,k-1}\left(\frac{Y_{nk}^2}{1+Y_{nk}^2}\right)\right]^2 \xrightarrow{\text{a.e.}} 0.$$

In (3) we have proved that if

$$(20)\qquad \max_{1\leq k\leq k_n}|A_{nk}|\leq 1/2,\qquad n\in\mathbb{N},$$

then for every $t\in\mathbb{R}^1$ there exists a constant $\eta(t)>0$, independent of k and n, such that

$$|E_{n,k-1}(\exp(itY_{nk})-1)|<\eta(t)\,E_{n,k-1}\left(\frac{Y_{nk}^2}{1+Y_{nk}^2}\right).$$

Because of (19) we can assume (20), eventually applying a construction similar to the $*$-procedure. Therefore we have obtained:

Theorem 3. *If the system* $(\mathfrak{X},\mathfrak{F})$ *satisfies* (C.0)–(C.3) *and* (C.6), *then* (15) *holds.*

Integrating both sides in (15) we obtain the final result of this part:

Theorem 4. *If the system* $(\mathfrak{X},\mathfrak{F})$ *satisfies* (C.0)–(C.3) *and at least one of conditions* (17), (C.5) *or* (C.6)., *then its row sums* S_n, $n\in\mathbb{N}$, *converge in distribution to* $\mathcal{M}ix(a,K)$.

6. Weaker Assumptions

It would be unfortunate if, when proving limit theorems for mixtures, we had to assume almost sure convergence in our conditions, while for pure infinitely divisible laws we have criteria involving the weaker convergence in probability of relative sequences. Eagleson in [2] has shown how we can weaken the assumptions of Theorem 4. Using his method we can prove:

Theorem 5. *If the system* $(\mathfrak{X},\mathfrak{F})$ *satisfies* (C.0) *and*

$$(\text{C.7})\qquad \max_{1\leq k\leq k_n} P_{n,k-1}(|X_{nk}|>\varepsilon)\xrightarrow{\mathbf{P}}0,\qquad \varepsilon>0,$$

$$(\text{C.8})\qquad \sum_{k=1}^{k_n}\left\{A_{nk}+E_{n,k-1}\left(\frac{Y_{nk}}{1+Y_{nk}^2}\right)\right\}\xrightarrow{\mathbf{P}}a(\cdot),$$

$$(\text{C.9})\qquad \sum_{k=1}^{k_n}E_{n,k-1}\left(\frac{Y_{nk}^2}{1+Y_{nk}^2}I(Y_{nk}<x)\right)\xrightarrow{\mathbf{P}}K(x,\cdot)$$

for every x belonging to some countable dense set $D\subset\mathbb{R}^1$,

$$(\text{C.10})\qquad \sum_{k=1}^{k_n}E_{n,k-1}\left(\frac{Y_{nk}^2}{1+Y_{nk}^2}\right)\xrightarrow{\mathbf{P}}K(+\infty,\cdot),$$

then its row sums S_n, $n\in\mathbb{N}$, *converge in law to* $\mathcal{M}ix(a,K)$.

Proof. It suffices to show that every subsequence $\{S_{n_k}\}\subset\{S_n\}$ contains a further subsequence $\{S_{n_{k_1}}\}$ which is convergent in law to $\mathcal{M}ix(a,K)$. Of course we may consider only the case of $\{S_{n_k}\}=\{S_n\}$. Observe that in (C.8)-(C.10) we have only countably many conditions; the same is true for (C.7) as we have remarked above. Then using the diagonal method we can choose a subsequence $\{S_{n_k}\}\subset\{S_n\}$ such that all these conditions are fulfilled with the convergence almost sure and Theorem 4 implies the conclusion.

7. Case of Finite Variances

In this part we shall consider only systems $(\mathfrak{X},\mathfrak{F})$ with all rv's having finite variance.

Theorem 6. *If the system* $(\mathfrak{X},\mathfrak{F})$ *satisfies* (C.0) *and*

(C.11) $$\sum_{k=1}^{k_n} B_{nk}\xrightarrow{\text{a.e.}} a(\cdot),$$

(C.12) $$\sum_{k=1}^{k_n} E_{n,k-1}(Z_{nk}^2 I(Z_{nk}<x))\xrightarrow{\text{a.e.}} K(x,\cdot)$$

for every x belonging to some countable dense subset $D\subset\mathbb{R}^1$,

(C.13) *there exists an* $\mathscr{F}_0$*-measurable rv* $0<C<+\infty$ *such that*

$$\mathbf{P}\left(\limsup_{n\to\infty}\left[\sum_{k=1}^{k_n} E_{n,k-1}(Z_{nk}^2)>C\right]\right)=0,$$

where

$$B_{nk}=E_{n,k-1}(X_{nk}),\qquad 1\leqq k\leqq k_n,\quad n\in\mathbb{N},$$

$$Z_{nk}=X_{nk}-B_{nk},\qquad 1\leqq k\leqq k_n,\quad n\in\mathbb{N},$$

then each of the conditions

(C.14) $$\exp\left[it\sum_{k=1}^{k_n} Z_{nk}-\sum_{k=1}^{k_n} E_{n,k-1}(\exp(itZ_{nk})-1)\right]\xrightarrow{\text{a.e.}} 1,\qquad t\in\mathbb{R}^1,$$

(C.15) $$\sum_{k=1}^{k_n}|E_{n,k-1}(\exp(itZ_{nk})-1)|^2\xrightarrow{\text{a.e.}} 0,\qquad t\in\mathbb{R}^1,$$

(C.16) $$\max_{1\leqq k\leqq k_n} P_{n,k-1}(|Z_{nk}|>\varepsilon)\xrightarrow{\text{a.e.}} 0,\qquad \varepsilon>0,$$

(C.17) $$\max_{1\leqq k\leqq k_n} E_{n,k-1}(Z_{nk}^2)\xrightarrow{\text{a.e.}} 0,$$

is sufficient for

$$E(\exp(itS_n)|\mathscr{F}_0)\xrightarrow{\text{a.e.}}\exp\left[ita(\cdot)+\int_{-\infty}^{+\infty} h_t(x)\,dK(x,\cdot)\right],\qquad t\in\mathbb{R}^1$$

and hence for the convergence in law of S_n, $n\in\mathbb{N}$, *to* $\mathcal{M}ix(a,K)$.

Theorem 7. *If the system* $(\mathfrak{X}, \mathfrak{F})$ *satisfies* (C.0) *and*

(C.18) $\sum_{k=1}^{k_n} B_{nk} \xrightarrow{\mathbf{P}} a(\cdot)$,

(C.19) $\sum_{k=1}^{k_n} E_{n,k-1}(Z_{nk}^2 I(Z_{nk}<x)) \xrightarrow{\mathbf{P}} K(x,\cdot)$

for every x *belonging to some countable dense set* $D \subset \mathbb{R}^1$,

(C.20) *there exists an* $\mathscr{F}_0$*-measurable a.e. finite rv* $C>0$ *such that*

$$\lim_{n\to\infty} \mathbf{P}\left(\left[\sum_{k=1}^{k_n} E_{n,k-1}(Z_{nk}^2) > C\right]\right) = 0,$$

then each of the conditions

(C.21) $\max_{1\leqq k\leqq k_n} P_{n,k-1}(|Z_{nk}|>\varepsilon) \xrightarrow{\mathbf{P}} 0, \quad \varepsilon>0$,

(C.22) $\max_{1\leqq k\leqq k_n} E_{n,k-1}(Z_{nk}^2) \xrightarrow{\mathbf{P}} 0$,

is sufficient for the convergence in law of S_n, $n\in\mathbb{N}$, *to* $\mathscr{mix}(a,K)$.

We omit the proofs of these theorems because they are very similar to the previous ones.

After these general considerations we shall give some comments about a special case of mixtures.

Let us assume that for almost every

$$K(t,\omega) = \begin{cases} 0 & \text{for } t\leqq 0 \\ \eta(\omega)\geqq 0 & \text{for } t>0. \end{cases}$$

Then $\mathscr{N}(a,\eta) = \mathscr{mix}(a,K) = \mathscr{Mix}(a,K)$ is a mixture of normal distributions on $\mathbb{R}^1$ with the chf

$$\varphi(t) = E\exp[ita(\cdot) - \tfrac{1}{2}t^2\eta(\cdot)], \quad t\in\mathbb{R}^1.$$

Conditions (C.1)–(C.3) in this case are equivalent to

(CN.1) $\sum_{k=1}^{k_n}\left\{A_{nk} + E_{n,k-1}\left(\frac{Y_{nk}}{1+Y_{nk}^2}\right)\right\} \xrightarrow{\text{a.e.}} a(\cdot)$,

(CN.2) $\sum_{k=1}^{k_n} E_{n,k-1}\left(\frac{Y_{nk}^2}{1+Y_{nk}^2} I(|Y_{nk}|>\varepsilon)\right) \xrightarrow{\text{a.e.}} 0, \quad \varepsilon>0$,

(CN.3) $\sum_{k=1}^{k_n} E_{n,k-1}\left(\frac{Y_{nk}^2}{1+Y_{nk}^2}\right) \xrightarrow{\text{a.e.}} \eta(\cdot)$.

Condition (CN.2) is equivalent to

$$\text{(CN.4)} \quad \sum_{k=1}^{k_n} P_{n,k-1}(|Y_{nk}|>\varepsilon) \xrightarrow{\text{a.e.}} 0, \qquad \varepsilon>0.$$

One can easily prove that

Theorem 8. *If the system* $(\mathfrak{X}, \mathfrak{F})$ *satisfies* (C.0) *and*

$$\text{(CN.5)} \quad \sum_{k=1}^{k_n} P_{n,k-1}(|X_{nk}|>\varepsilon) \xrightarrow{\text{a.e.}} 0, \qquad \varepsilon>0,$$

$$\text{(CN.6)} \quad \sum_{k=1}^{k_n} E_{n,k-1}(X_{nk} I(|X_{nk}|\leqq\varepsilon)) \xrightarrow{\text{a.e.}} a(\cdot), \qquad \varepsilon>0,$$

$$\text{(CN.7)} \quad \sum_{k=1}^{k_n} E_{n,k-1}(X_{nk}^2 I(|X_{nk}|\leqq\varepsilon) - \sum_{k=1}^{k_n} [E_{n,k-1}(X_{nk} I(|X_{nk}|\leqq\varepsilon))]^2 \xrightarrow{\text{a.e.}} \eta(\cdot), \qquad \varepsilon>0,$$

then S_n, $n\in\mathbb{N}$, *converges in law to* $\mathcal{N}(a,\eta)$.

We can prove more, namely, that under the assumption (C.6) conditions (CN.5)–(CN.7) are equivalent to (CN.1)–(CN.3). If we put the convergence in probability into (CN.5)–(CN.7), then the conclusion also holds.

If all rv's of the system $(\mathfrak{X}, \mathfrak{F})$ have finite variances then the straight reformulation of Theorems 6 and 7 gives sufficient conditions, which improve the results obtained in [2].

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