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On the Functional Central Limit Theorem for Martingales, II

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1. Introduction

The problem of finding necessary and sufficient conditions for the convergence to a Brownian motion of summation processes based on an array $\{X_{n,i}\}$ of random variables which are "close" to a martingale difference array was studied in the paper [16], "On the functional central limit theorem for martingales". The present paper is mainly concerned with some questions left open in [16] concerning the case when $\{X_{n,i}\}$ itself is a martingale difference array, and for that case we obtain conditions which are necessary and sufficient for the functional central limit theorem in rather general circumstances, the main restriction being that the sequence of row-maxima is assumed to be uniformly integrable. In particular the sufficient conditions obtained are somewhat weaker than the best previously published results, which require that the sequence of row-maxima should have uniformly bounded second moments, see [12]. In the important special cases when normalization is by means of variances or conditional variances our results lead to a complete solution to the martingale functional central limit problem, and thus in particular to a (partial) generalization of the Lindeberg-Feller theorem. Furthermore, weak conditions for convergence to mixtures are obtained.

As in [16] the proofs are made in two steps. The first one is to reduce the problem to the case of bounded martingale differences. In the present paper an efficient and rather simple truncation procedure is used to this end. The next step is to compare various normalizations (or "time-scales") with a natural time-scale given by sums of squares. Other time-scales can then be seen as "random changes of time" from the natural time-scale. This point of view is used in [16, 17] and more fully in [5], and it leads to natural and simple proofs. In fact, if one wants to use this approach to prove central limit theorems "starting from scratch", the only ingredient which has to be added to this paper is a proof of convergence to a Brownian motion for the case when the martingale differences are uniformly bounded by constants tending to zero and when the time-scale is given by the sum of squares. In this case the customary proofs of finite-

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dimensional convergence simplify considerably and moreover, using a Burkholder-inequality, it is possible to give a very short proof of tightness.

For further discussion of the central limit problem for martingales we refer to the recent papers [9–11, 14] and to the references listed in [16].

In the present paper, Section 2 contains most of the necessary notation, and the main lemma, Lemma 1, which gives the truncation procedure. Section 3 contains some sets of necessary and sufficient conditions for convergence to a Brownian motion. Finally, the results on convergence to mixtures are contained in Section 4.

2. A Truncation Lemma

We briefly recall the notation of [16], which we are going to use also in this paper. For $n=1,2,\ldots$, $\{X_{n,i}\}_{i=1}^{\infty}$ is a sequence of random variables on a probability space $(\Omega_n, \mathscr{B}_n, P_n)$, the sub-sigmaalgebra $\mathscr{B}_{n,i}$ of $\mathscr{B}_n$ is generated by $X_{n,1},\ldots,X_{n,i}$, and $S_n(k)=\sum_{i=1}^{k} X_{n,i}$. Furthermore, $\tau_n(t)$; $t\in[0,1]$ are stopping times of $\{\mathscr{B}_{n,i}\}_{i=1}^{\infty}$ which are increasing and right continuous in t a.s., and for simplicity it is throughout assumed that

$$\tau_n(1)<\infty \qquad \text{a.s. } n\geqq 1.$$

Thus $\{\tau_n\}$ and $\{S_n\circ\tau_n(t);\ t\in[0,1]\}$ are sequences of random variables in $D(0,1)$, which we take to be endowed with the Skorokhod topology. For brevity we will write P instead of P_n, $S\circ\tau_n$ for $S_n\circ\tau_n$, etc., and $E_i(\cdot)$ for $E\{\cdot\,\|\mathscr{B}_{n,i}\}$ when taking expectation of variables from the n-th row ($E_0(\cdot)=E(\cdot)$). If $E_{i-1}(X_{n,i})=0$, $i\geqq 2$, $n\geqq 1$ then $\{X_{n,i}\}$ is a martingale difference array (m.d.a.).

Let $M_n=\max_{1\leqq i\leqq \tau_n(1)}|X_{n,i}|$. In the situations we are interested in $M_n\xrightarrow{P}0$, i.e., $P(M_n>\varepsilon)\to 0$, $\forall\varepsilon>0$. From this follows the existence of a sequence $\{\varepsilon_n\}$ of constants satisfying

$$P(M_n>\varepsilon_n)\to 0 \quad \text{and} \quad \varepsilon_n\to 0 \qquad \text{as } n\to\infty. \tag{1}$$

The results of this paper will rest on the following simple truncation lemma.

Lemma 1. *Let $\{X_{n,i}\}$ be a m.d.a., let the constants $\{\varepsilon_n\}$ satisfy* (1), *put $\nu_n=\inf\{i\geqq 1;\ |X_{n,i}|>\varepsilon_n\}\wedge\tau_n(1)$, and suppose that*

$$E|X_{n,\nu_n}|\to 0 \qquad \textit{as } n\to\infty. \tag{2}$$

Then there exists a m.d.a. $\{\xi_{n,i}\}$ with $|\xi_{n,i}|<2\varepsilon_n$ a.s. such that, writing $S_n'(k)=\sum_{i=1}^{k}\xi_{n,i}$,

$$\sup_{0\leqq t\leqq 1}|S\circ\tau_n(t)-S'\circ\tau_n(t)|\xrightarrow{P}0 \qquad \textit{as } n\to\infty, \tag{3}$$

and

$$\sup_{0\leq t\leq 1}\left|\sum_{i=1}^{\tau_n(t)} X_{n,i}^2-\sum_{i=1}^{\tau_n(t)}\xi_{n,i}^2\right|\xrightarrow{P}0 \quad \text{as } n\to\infty. \tag{4}$$

Further, writing J_n *for the (random) set of integers between* 1 *and* $\tau_n(1)$ *such that* $X_{n,i}=0$,

$$\sum_{i\in J_n}|\xi_{n,i}|\xrightarrow{P}0 \quad \text{as } n\to\infty. \tag{5}$$

The conditions (1) *and* (2) *are satisfied for some suitably choosen sequence* $\{\varepsilon_n\}$ *if* $\{M_n\}_{n=1}^{\infty}$ *is uniformly integrable and* $M_n\xrightarrow{P}0$.

Proof. Define

$$\xi_{n,i}=X_{n,i}I(v_n>i)-E_{i-1}(X_{n,i}I(v_n>i)),$$

where I is the indicator function, i.e. $I(A)$ is one on A and zero on A^c.

Clearly $\{\xi_{n,i}\}$ is a m.d.a., and since $|X_{n,i}I(v_n>i)|\leqq\varepsilon_n$ by the definition of v_n we have $|\xi_{n,i}|\leqq 2\varepsilon_n$ a.s. Further

$$|S\circ\tau_n(t)-S'\circ\tau_n(t)|\leqq\sum_{i=1}^{\tau_n(1)}|X_{n,i}-\xi_{n,i}|. \tag{6}$$

and by elementary algebra

$$\begin{aligned}\left|\sum_{i=1}^{\tau_n(t)}X_{n,i}^2-\sum_{i=1}^{\tau_n(t)}\xi_{n,i}^2\right|&\leqq 2\sum_{i=1}^{\tau_n(t)}|\xi_{n,i}|\,|X_{n,i}-\xi_{n,i}|+\sum_{i=1}^{\tau_n(t)}(X_{n,i}-\xi_{n,i})^2\\&\leqq 4\varepsilon_n\sum_{i=1}^{\tau_n(1)}|X_{n,i}-\xi_{n,i}|+\sum_{i=1}^{\tau_n(1)}(X_{n,i}-\xi_{n,i})^2,\end{aligned} \tag{7}$$

for $t\in[0,1]$. Now, $\left\{\sum_{i=1}^{\tau_n(1)}(X_{n,i}-\xi_{n,i})^2>\varepsilon\right\}\subset\left\{\sum_{i=1}^{\tau_n(1)}|X_{n,i}-\xi_{n,i}|>\varepsilon\right\}$ for $0<\varepsilon<1$, and hence $\sum_{i=1}^{\tau_n(1)}(X_{n,i}-\xi_{n,i})^2\xrightarrow{P}0$ if

$$\sum_{i=1}^{\tau_n(1)}|X_{n,i}-\xi_{n,i}|\xrightarrow{P}0 \quad \text{as } n\to\infty. \tag{8}$$

Thus, by (6) and (7), the relations (3) and (4) follow if we prove (8). However,

$$\sum_{i=1}^{\tau_n(1)}|X_{n,i}-\xi_{n,i}|\leqq\sum_{i=1}^{\tau_n(1)}|X_{n,i}-X_{n,i}I(v_n>i)|+\sum_{i=1}^{\tau_n(1)}|E_{i-1}(X_{n,i}I(v_n>i))|,$$

and the first sum on the right equals $\sum_{i=v_n}^{\tau_n(1)}|X_{n,i}|$, which tends to zero in probability by (1) and the definition of v_n. Moreover, using that $\{v_n>i-1\}$

$\in \mathscr{B}_{n,i-1}$ and that $\{X_{n,i}\}$ is a m.d.a., we have that

$$E_{i-1}(X_{n,i} I(v_n > i)) = E_{i-1}(X_{n,i} I(v_n > i-1)) - E_{i-1}(X_{n,i} I(v_n = i))$$
$$= -E_{i-1}(X_{n,v_n} I(v_n = i)).$$

Hence,

$$E \sum_{i=1}^{\tau_n(1)} |E_{i-1}(X_{n,i} I(v_n > i))| = E \sum_{i=1}^{\tau_n(1)} |E_{i-1}(X_{n,i} I(v_n = i))|$$
$$\leqq E \sum_{i=1}^{\tau_n(1)} |X_{n,v_n}| I(v_n = i)$$
$$= E|X_{n,v_n}| \to 0 \quad \text{as } n \to \infty \tag{9}$$

by (2), which proves (8), and thus (3) and (4). Since $X_{n,i}=0$ implies that $\xi_{n,i} = -E_{i-1}(X_{n,i} I(v_n > i))$ the relation (5) follows at once from (9). This concludes the verification that $\{\xi_{n,i}\}$ has the properties stated in the lemma.

Finally, as noted above, $M_n \xrightarrow{P} 0$ implies that there exists a sequence $\{\varepsilon_n\}$ satisfying (1), and if furthermore $\{M_n\}$ is uniformly integrable,

$$E|X_{n,v_n}| \leqq EM_n \to 0 \quad \text{as } n \to \infty,$$

i.e. (2) holds, which proves the last part of the lemma. □

It may be noted that Condition (1) and Condition (2) put different restrictions on the ε_n-sequence. Of course the first part of (1) says that the ε_n's should not be too small. On the other hand, (2) is easier to satisfy the smaller the ε_n's. In fact, suppose $\{\varepsilon'_n\}$ is another sequence of constants, with $\varepsilon'_n \leqq \varepsilon_n$ and let v'_n be defined from ε'_n in the same way as v_n is defined from ε_n. Then $|X_{n,v'_n}| \leqq |X_{n,v_n}|$, and hence $E|X_{n,v_n}| \to 0$ implies $E|X_{n,v'_n}| \to 0$.

Condition (2) is rather obscure, and for that reason we will use the more transparent conditions that $\{M_n\}$ is uniformly integrable or that $EM_n \to 0$ in the statements of the theorems in the following sections. However, the theorems of course hold also under the somewhat weaker condition (2).

3. Convergence to a Brownian Motion

The following concept is important for the first theorem. We will say that the time-scale τ_n "takes all relevant values" with respect to an array $\{X_{n,i}\}$ if the event $X_{n,i} \neq 0$ and $1 \leqq i \leqq \tau_n(1)$ implies the event that $\tau_n(t) = i$ for some $t \in [0, 1]$.

Theorem 2. *Let $\{X_{n,i}\}$ be a m.d.a., let $\tau_n(t)$; $t \in [0, 1]$ be stopping times of $\{\mathscr{B}_{n,i}\}$ which are increasing and right continuous in t a.s. and suppose that $\{M_n\}_{n=1}^{\infty}$ is uniformly integrable. Then*

$$\sum_{i=1}^{\tau_n(t)} X_{n,i}^2 \xrightarrow{P} t \quad \text{as } n \to \infty, \tag{10}$$

for $t \in [0,1]$, *implies that*

$$S \circ \tau_n \xrightarrow{d} B \quad \text{as } n \to \infty, \text{ in } D(0,1), \tag{11}$$

where B is a standard Brownian motion. Conversely, if in addition τ_n *takes all relevant values with respect to* $\{X_{n,i}\}$, *then* (11) *implies* (10).

Proof. It is easy to see that if (10) holds then $M_n \xrightarrow{P} 0$, see [13], and hence the hypothesis of Lemma 1 is satisfied. Letting $\{\xi_{n,i}\}$ be the m.d.a. defined in the lemma, (10) and (4) give that $\sum_{i=1}^{\tau_n(t)} \xi_{n,i}^2 \xrightarrow{P} t$, $t \in [0,1]$. Since furthermore $|\xi_{n,i}| \leqq 2\varepsilon_n \to 0$ it follows from the first part of Lemma 3 of [16] that $S' \circ \tau_n \xrightarrow{d} B$ (where as above $S'_n(k) = \sum_{i=1}^{k} \xi_{n,i}$). By (3) this proves $S \circ \tau_n \xrightarrow{d} B$.

Now to the converse. By assumption $\{M_n\}$ is uniformly integrable and, since τ_n is assumed to take all relevant values with respect to $\{X_{n,i}\}$, (11) implies that $M_n \xrightarrow{P} 0$. Hence the conditions of Lemma 1 are satisfied, and we can again approximate with the array $\{\xi_{n,i}\}$, which by (3) and (11) satisfies $S' \circ \tau_n \xrightarrow{d} B$. Moreover, according to (4), to prove (10) we only have to show that

$$\sum_{i=1}^{\tau_n(t)} \xi_{n,i}^2 \xrightarrow{P} t, \quad \text{as } n \to \infty, \tag{12}$$

for $t \in [0,1]$. For this we will modify the proof of the second part of Lemma 3 of [16] slightly. No changes are needed in that proof up to the point where stopping times $\tau(i) = \tau_n(i/n')$, $0 \leqq i \leqq n'$, satisfying

$$\sum_{i=1}^{n'} \{S' \circ \tau(i) - S' \circ \tau(i-1)\}^2 \xrightarrow{P} 1 \quad \text{as } n \to \infty, \tag{13}$$

are introduced and where $\tau'(i)$ is defined as the minimum of $\tau(i)$ and of

$$\inf\left\{k > \tau(i-1); \left|\sum_{i=\tau(i-1)+1}^{k} \xi_{n,i}\right| > 1\right\}.$$

Further, A_n is the event that $\tau(i) = \tau'(i)$, $i = 1, \ldots, n'$. It is straight-forward to check that since $\tau_n(t)$; $t \in [0,1]$ takes all relevant values with respect to $\{X_{n,i}\}$,

$$P(A_n^c) \leqq P(Y_n > 1/2) + P\Big(\sum_{J_n} |\xi_{n,i}| > 1/2\Big),$$

where Y_n is defined in [16], p. 203, l. 16, and J_n is the set of integers between 1 and $\tau_n(1)$ such that $X_{n,i} = 0$. As in [16] $Y_n \xrightarrow{P} 0$ and by (5) above $\sum_{J_n} |\xi_{n,i}| \xrightarrow{P} 0$ and hence it follows that $P(A_n^c) \to 0$ as $n \to \infty$. (At this point there is a gap in the proof of Lemma 3 of [16]; however, this can be filled, cf. the acknowledgement.)

Next, with $\zeta_{n,i} = \sum_{j=\tau(i-1)+1}^{\tau(i)} \xi_{n,j}$ set $\theta_n = \min\left\{k; \sum_{i=1}^{k} \zeta_{n,i}^2 > 2\right\} \wedge n'$ and set $\tilde{\theta}_n = \tau(\theta_n)$. Since

$$\{\tilde{\theta}_n = k\} = \bigcup_{i=1}^{n'} \{\theta_n = i, \tau(i) = k\}$$

and since $\{\theta_n=i, \tau(i)=k\}\in\mathscr{B}_{n,k}$ we have that $\tilde{\theta}_n$ is a stopping time of $\{\mathscr{B}_{n,k}\}_{k=1}^{\infty}$. Let $\tilde{\xi}_{n,i}=\xi_{n,i}I(\theta_n\geqq i)$ so that $\{\tilde{\xi}_{n,i}\}$ is a m.d.a., put $\tilde{S}_n(k)=\sum_{i=1}^{k}\tilde{\xi}_{n,i}$, and put $\tilde{\zeta}_{n,i}=\sum_{j=\tau(i-1)+1}^{\tau'(i)}\tilde{\xi}_{n,j}$. From (13) and $P(A_n)\to 1$ follows that

$$P(\xi_{n,i}\neq\tilde{\xi}_{n,i} \text{ for some } i\leqq\tau_n(1))\to 0 \quad \text{as } n\to\infty. \tag{14}$$

Furthermore, $\max_{1\leqq i\leqq n'}|\tilde{\zeta}_{n,i}|\leqq Y_n+\sum_{J_n}|\xi_{n,i}|$, and as above the two last terms tend to zero in probability, and hence $\max_{1\leqq i\leqq n'}|\tilde{\zeta}_{n,i}|\xrightarrow{P}0$. By definition, $\sum_{i=1}^{n'}\tilde{\zeta}_{n,i}^2\leqq 2+\max_{1\leqq i\leqq n'}\zeta_{n,i}^2\leqq 2+(1+2\varepsilon_n)^2$ so $\left\{\sum_{i=1}^{n'}\tilde{\zeta}_{n,i}^2\right\}$ is uniformly integrable and since $\tilde{\xi}_{n,i}$ is a m.d.a. $E\sum_{i=1}^{n'}\sum_{j=\tau(i-1)+1}^{\tau'(i)}\tilde{\xi}_{n,i}^2=E\sum_{i=1}^{n'}\tilde{\zeta}_{n,i}^2\leqq 2+(1+2\varepsilon_n)^2$. Now, using the proof in [16], p. 204, 1.4 to p. 205, 1.4, with $\xi_{n,i}$, replaced by $\tilde{\xi}_{n,i}$, S'_n replaced by $\tilde{S}_n$ and $\zeta_{n,i}$ replaced by $\tilde{\zeta}_{n,i}$, we have that

$$\sum_{i=1}^{n'}\sum_{j=\tau(i-1)+1}^{\tau'(i)}\tilde{\xi}_{n,j}^2\xrightarrow{P}1 \quad \text{as } n\to\infty.$$

Together with $P(A_n)\to 1$ and (14) this proves that (12) holds for $t=1$. The case $0<t<1$ follows by dividing by t, and then (12) must hold also for $t=0$. □

It should be noted that also when τ_n does not take all relevant values, Theorem 2 gives necessary and sufficient conditions for convergence to a Brownian motion, expressed in terms of sums of squares of the jumps of the summation processes $S\circ\tau_n$ (or equivalently, in terms of the square variation of $S\circ\tau_n$). Of course the jumps can – in contrast to the $X_{n,i}$'s – be retrieved directly from the summation process and they seem to be intrinsically more important when convergence is considered.

To make this precise, let $t_{n,1}=\inf\{t\in[0,1];\ \tau_n(t)\neq\tau_n(t-)\}$ and define recursively $t_{n,i+1}=\inf\{t\in(t_{n,i},1];\ \tau_n(t)\neq\tau_n(t-)\}$, defining the infimum to be ∞ if the set within brackets is empty. The jump of $S\circ\tau_n$ at $t_{n,i}$ is then $\tilde{X}_{n,i}=S\circ\tau_n(t_{n,i})-S\circ\tau_n(t_{n,i}-)$ for $t_{n,i}<\infty$ and $\tilde{X}_{n,i}=0$ for $t_{n,i}=\infty$. If $\{\tilde{X}_{n,i}\}$ is a m.d.a. and $\{\max|\tilde{X}_{n,i}|\}_{n=1}^{\infty}$ is uniformly integrable, then by Theorem 2 the convergence of $S\circ\tau_n$ to a Brownian motion is equivalent to

$$\sum_{\{i;\, t_{n,i}\leqq t\}}\tilde{X}_{n,i}^2\xrightarrow{P}t \quad \text{as } n\to\infty, \tag{15}$$

for $t\in[0,1]$. That $\{\tilde{X}_{n,i}\}$ is a m.d.a. follows e.g. if for all n the stopping-time $\tau_n(1)$ is regular for $\{X_{n,i}\}_{i=1}^{\infty}$ (or equivalently, if $\{X_{n,\tau_n(1)\wedge i}\}_{i=1}^{\infty}$ is L_1-convergent).

One, more restrictive, condition which ensures this is

$$\sup_n E\sum_{i=1}^{\tau_n(1)}X_{n,i}^2<\infty, \tag{16}$$

since it implies that $\{X_{n,\tau_n(1)\wedge i}\}_{i=1}^{\infty}$ is L_2-convergent. Furthermore, it follows that $E\sum_{i=1}^{\tau_n(1)} X_{n,i}^2 = E\sum_i \tilde{X}_{n,i}^2$ and thus that $\{\max_i |\tilde{X}_{n,i}|\}_{n=1}^{\infty}$ is uniformly integrable. Thus, assuming (16), the condition (15) is necessary and sufficient for $S\circ\tau_n \xrightarrow{d} B$, and in fact it can be shown that the conditions together imply that the difference between $\sum_{i=1}^{\tau_n(1)} X_{n,i}^2$ and the sum of squares in (15) tend to zero in probability and that hence also $\sum_{i=1}^{\tau(t)} X_{n,i}^2 \xrightarrow{P} t$, $t\in[0,1]$. The latter result is proved by Gänssler and Häusler in the paper mentioned in the acknowledgement, so we will not discuss it further here but will instead give a direct derivation of necessary and sufficient conditions for $S\circ\tau_n \xrightarrow{d} B$ for the cases when normalization is made by variances or by conditional variances.

Theorem 3. *Let $\{X_{n,i}\}$ be a m.d.a. and let $\tau_n(t)$; $t\in[0,1]$ be stopping times of $\{\mathscr{B}_{n,i}\}_{i=1}^{\infty}$ which are increasing and right continuous in t a.s. and which satisfy*

$$\sum_{i=1}^{\tau_n(t)} E_{i-1}(X_{n,i}^2) \xrightarrow{P} t \qquad \text{as } n\to\infty,\ t\in[0,1]. \tag{17}$$

Then $S\circ\tau_n \xrightarrow{d} B$ as $n\to\infty$, in $D(0,1)$, if and only if both $M_n \xrightarrow{P} 0$ and

$$\sum_{i=1}^{\tau_n(1)} E_{i-1}(X_{n,i}^2 I(|X_{n,i}|>d)) \xrightarrow{P} 0 \qquad \text{as } n\to\infty, \tag{18}$$

for some $d>0$.

Remark 4. The proof below shows that if $S\circ\tau_n \xrightarrow{d} B$ then

$$\sum_{i=1}^{\tau_n(1)} E_{i-1}(X_{n,i}^2 I(|X_{n,i}|>d)) \xrightarrow{P} 0 \qquad \text{as } n\to\infty,\ \forall d>0, \tag{19}$$

and it is easy to show that if (19) holds then $S\circ\tau_n \xrightarrow{d} B$ (see e.g. [5]). Hence, assuming (17), also "the conditional Lindeberg condition" (19) is necessary and sufficient for $S\circ\tau_n \xrightarrow{d} B$.

Proof. We first prove that the conditions are sufficient to ensure $S\circ\tau_n \xrightarrow{d} B$. Put $X'_{n,i} = X_{n,i} I(|X_{n,i}|\leqq d) - E_{i-1}(X_{n,i} I(|X_{n,i}|\leqq d))$ and let $S'_n(k) = \sum_{i=1}^{k} X'_{n,i}$, Straightforward calculations show that $\sup_{0\leqq t\leqq 1} |S\circ\tau_n(t) - S'\circ\tau_n(t)| \xrightarrow{P} 0$ and that $\max_{1\leqq i\leqq \tau_n(1)} |X'_{n,i}| \xrightarrow{P} 0$. Now,

$$E_{i-1}(X'^2_{n,i}) = E_{i-1}(X_{n,i}^2 I(|X_{n,i}|\leqq d)) - E_{i-1}^2(X_{n,i} I(|X_{n,i}|\leqq d))$$

and

$$E_{i-1}^2(X_{n,i} I(|X_{n,i}|\leqq d)) = E_{i-1}^2(X_{n,i} I(|X_{n,i}|>d)) \leqq E_{i-1}(X_{n,i}^2 I(|X_{n,i}|>d)$$

and hence, by (17) and (18), $\sum_{i=1}^{\tau_n(t)} E_{i-1}(X'^2_{n,i}) \xrightarrow{P} t$ for $t \in [0,1]$. It follows that $\sum_{i=1}^{\tau_n(t)} X'^2_{n,i} \xrightarrow{P} t$ for $t \in [0,1]$ (cf. [5, 12]) and then by Theorem 2 that $S' \circ \tau_n \xrightarrow{d} B$, and hence $S \circ \tau_n \xrightarrow{d} B$.

The proof of necessity is somewhat more complicated. We first consider the particular timescale $\tau'_n(t) = \inf\left\{k;\ \sum_{i=1}^{k+1} E_{i-1}(X^2_{n,i}) > t\right\}$ and prove that

$$\text{if } S \circ \tau'_n \xrightarrow{d} B \quad \text{then } \sum_{i=1}^{\tau'_n(1)} E_{i-1}(X^2_{n,i} I(|X_{n,i}| > d)) \xrightarrow{P} 0. \tag{20}$$

Clearly τ'_n takes all relevant values, and since $\tau'_n(1)$ is a stopping time

$$E \max_{1 \leq i \leq \tau'_n(1)} X^2_{n,i} \leq E \sum_{i=1}^{\tau'_n(1)} X^2_{n,i} = E \sum_{i=1}^{\tau'_n(1)} E_{i-1}(X^2_{n,i}) \leq 1,$$

so $\{\max_{1 \leq i \leq \tau'_n(1)} |X_{n,i}|\}_{n=1}^{\infty}$ is uniformly integrable. By Theorem 2 it follows that $\sum_{i=1}^{\tau'_n(1)} X^2_{n,i} \xrightarrow{P} 1$, and thus, since as above $E \sum_{i=1}^{\tau'_n(1)} X^2_{n,i} \leq 1$ and since the sum is positive, that $\left\{\sum_{i=1}^{\tau'_n(1)} X^2_{n,i}\right\}_{n=1}^{\infty}$ is uniformly integrable. Further, $\max_{1 \leq i \leq \tau'_n(1)} |X_{n,i}| \xrightarrow{P} 0$ and thus $\sum_{i=1}^{\tau'_n(1)} X^2_{n,i} I(|X_{n,i}| > d) \xrightarrow{P} 0$. Since $0 \leq \sum_{i=1}^{\tau'_n(1)} X^2_{n,i} I(|X_{n,i}| > d) \leq \sum_{i=1}^{\tau'_n(1)} X^2_{n,i}$,

$$E \sum_{i=1}^{\tau'_n(1)} E_{i-1}(X^2_{n,i} I(|X_{n,i}| > d)) = E \sum_{i=1}^{\tau'_n(1)} X^2_{n,i} I(|X_{n,i}| > d) \to 0 \quad \text{as } n \to \infty,$$

which proves (20).

Next, from (17) follows that $\max_{1 \leq i \leq \tau_n(1)} E_{i-1}(X^2_{n,i}) \xrightarrow{P} 0$ and that $P(\tau_n(1) > \tau'_n(1-\varepsilon)) \to 1$ for $\varepsilon > 0$, and hence $\sum_{i=1}^{\tau'_n(t)} E_{i-1}(X^2_{n,i}) \xrightarrow{P} t$ for $0 \leq t < 1$. Thus, since $\sum_{i=1}^{\tau'_n(t)} E_{i-1}(X^2_{n,i})$ is increasing in t we have $\sup_{0 \leq t \leq 1-\varepsilon} \left|\sum_{i=1}^{\tau'_n(t)} E_{i-1}(X^2_{n,i}) - t\right| \xrightarrow{P} 0$ for $\varepsilon > 0$ and similarly it follows from (17) that $\sup_{0 \leq t \leq 1} \left|\sum_{i=1}^{\tau_n(t)} E_{i-1}(X^2_{n,i}) - t\right| \xrightarrow{P} 0$. Hence, writing $A_n(\varepsilon) = \{\tau_n(t-\varepsilon) < \tau'_n(t) < \tau_n(t+\varepsilon);\ t \in [\varepsilon, 1-\varepsilon]\}$,

$$P(A_n(\varepsilon)) \to 1 \quad \text{as } n \to \infty,\ \forall \varepsilon > 0, \tag{21}$$

and thus there exists a sequence $\{\varepsilon_n\}$ of constants such that $\varepsilon_n \downarrow 0$ and $P(A_n(\varepsilon_n)) \to 1$. For $\varepsilon > 0$ and N with $\varepsilon/2 > 1/N$ and n large enough to make $2\varepsilon_n < 1/N$ we have the inclusion

$$\{\sup_{0\le t\le 1-\varepsilon} |S\circ\tau_n(t)-S\circ\tau_n'(t)|>\delta\}$$
$$\subset A_n^c(\varepsilon_n)\cup\bigcup_{i=0}^{N-3}\left\{\sup\left\{\left|\sum_{j=\tau_n(i/N)+1}^{k} X_{n,j}\right|;\ \tau_n(i/N)<k\right.\right.$$
$$\left.\left.\le\tau_n((i+2)/N)\wedge\tau_n'(1)\right\}>\delta/2\right\}. \tag{22}$$

(Here we have assumed $\tau_n(0)=0$; a further small argument shows that this can be done without loss of generality.) Now, using Lemma 4 of [3] and Cauchy's inequality on the m.d.a. $\{X_{n,j}I(\tau_n(i/N)<j)\}$ gives that for $0\le i\le N-3$

$$P\left(\sup\left\{\left|\sum_{j=\tau_n(i/N)+1}^{k} X_{n,j}\right|;\ \tau_n(i/N)<k\le\tau_n((i+2)/N)\wedge\tau_n'(1)\right\}>\delta\right)$$
$$\le 4\delta^{-1}\left(E\sum_{j=\tau_n(i/N)+1}^{\tau_n'(1)} X_{n,j}^2\right)^{1/2} P(|S(\tau_n((i+2)/N)\wedge\tau'(1))$$
$$-S(\tau_n(i/N))|>\delta/4)^{1/2}$$
$$\le 4\delta^{-1}P(|S(\tau_n((i+2)/N)\wedge\tau_n'(1))-S(\tau_n(i/N))|>\delta/4)^{1/2}. \tag{23}$$

By (17) and the assumption that $S\circ\tau_n\xrightarrow{d}B$

$$P(|S(\tau_n((i+2)/N)\wedge\tau_n'(1))-S(\tau_n(i/N))|>\delta/4)$$
$$\to 2(1-\Phi(\delta\sqrt{N/32})\le 4\phi(\delta\sqrt{N/32}),\qquad 0\le i\le N-3, \tag{24}$$

where Φ and ϕ are the standard normal distribution and density functions, respectively. Since $P(A_n^c(\varepsilon))\to 0$ it follows from (22)-(24) and Boole's inequality that

$$\limsup_{n\to\infty} P(\sup_{0\le t\le 1-\varepsilon}|S\circ\tau_n(t)-S\circ\tau_n'(t)|>\delta)$$
$$\le 4\delta^{-1}N\cdot\{4\phi(\delta\sqrt{N/32})\}^{1/2}\to 0\qquad\text{as } N\to\infty,$$

and hence $S\circ\tau_n\xrightarrow{d}B$ implies that $S\circ\tau_n'\xrightarrow{d}B$ as $n\to\infty$, in $D(0,1-\varepsilon)$.

Clearly (20) holds also if $D(0,1)$ is replaced by $D(0,1-\varepsilon)$ and thus

$$\sum_{i=1}^{\tau_n'(1-\varepsilon)} E_{i-1}(X_{n,i}^2 I(|X_{n,i}|>d)\xrightarrow{P}0\qquad\text{as } n\to\infty,$$

for all $\varepsilon>0$. Now

$$\sum_{i=1}^{\tau_n(1)} E_{i-1}(X_{n,i}^2 I(|X_{n,i}|>d))\le\sum_{i=1}^{\tau_n'(1-\varepsilon)} E_{i-1}(X_{n,i}^2 I(|X_{n,i}|>d))$$
$$+\sum_{i=\tau_n'(1-\varepsilon)+1}^{\tau_n(1)} E_{i-1}(X_{n,i}^2)\xrightarrow{P}0+1-(1-\varepsilon)=\varepsilon,$$

and since $\varepsilon>0$ is arbitrary, (18) follows. Finally, using (23) and (24) it is not difficult to show that there is a sequence $\varepsilon_n\downarrow 0$ of constants, with

$\max_{1 \leq i \leq \tau_n(1-\varepsilon_n)} |X_{n,i}| \xrightarrow{P} 0$, and since $\sum_{i=\tau_n(1-\varepsilon_n)+1}^{\tau_n(1)} E_{i-1}(X_{n,i}^2) \xrightarrow{P} 0$ it follows from Remark 4 that $\sup\left\{\left|\sum_{i=\tau_n(1-\varepsilon_n)+1}^{k} X_{n,i}\right|; \tau_n(1-\varepsilon_n)<k \leq \tau_n(1)\right\} \xrightarrow{P} 0$, and hence $M_n \xrightarrow{P} 0$ as $n \to \infty$. □

In the case of independence and non-random time-scales the condition (19) reduces to the Lindeberg condition and hence Theorem 3 is an extension of the Lindeberg-Feller Theorem from independence to dependence. In the general case normalization is made with conditional means and variances, which of course reduce to ordinary means and variances in the classical case. One change is however made from the classical setup; the u.a.n.-condition $\max_i P(|X_{n,i}|>\varepsilon) \to 0$, for all $\varepsilon>0$, which does not seem to be useful in the dependent case is replaced by the requirement that the entire summation process $S \circ \tau_n(t)$; $t \in [0,1]$ converges.

Sometimes it might be desirable to replace the random normalizations of Theorems 2 and 3 by non-random normalizations. The next theorem gives conditions for this when normalization is made by variances. We omit its proof, since it is analogous to the proof of Theorem 4, but simpler.

Theorem 5. *Let $\{X_{n,i}\}$ be a m.d.a. and let $\tau_n(t)$; $t \in [0,1]$ be deterministic functions which are increasing and right continuous and which satisfy*

$$\sum_{i=1}^{\tau_n(t)} EX_{n,i}^2 \to t \qquad \text{as } n \to \infty,\ t \in [0,1]. \tag{25}$$

Then $S \circ \tau_n \xrightarrow{d} B$ as $n \to \infty$, in $D(0,1)$, if and only if

$$\sum_{i=1}^{\tau_n(t)} X_{n,i}^2 \xrightarrow{P} t \qquad \text{as } n \to \infty,\ t \in [0,1]. \tag{26}$$

Remark 6. The theorem still holds if (26) is replaced by (17) and (19) together; in fact assuming (25) these conditions are equivalent. Furthermore in this situation also the ordinary Lindeberg condition holds (cf. [12, 18]). □

That (25) and (26) are sufficient to ensure convergence to a Brownian motion is of course well known, see e.g. [12] or [18]. Also, Theorem 5 generalizes known results from the central limit theory for independent summands, cf. [8], p. 143.

4. Convergence to Mixtures

It has recently been noted by several authors, e.g. [5, 6, 9, 10, 17] that theorems on convergence to a Brownian motion often can be extended to give conditions for convergence to mixtures of Brownian motions. The most direct and elegant way to do this seems to be via "a random change of time". This method is used to some extent in [17] and is more fully developed in [5, 10]. It has also been much used to prove "random indices limit theorems", see e.g. [1, 2, 15]. Here we will use it together with the truncation procedure given in Lemma 1 to prove a theorem which generalizes many of the results from both these areas. It should be noted that in the theorem it is not assumed that $\eta_n(t)$ is a stopping time, and that all random variables are supposed to be defined on the same probability space.

Theorem 6. *Let* $\{X_{n,i}\}$ *be a m.d.a., let the random processes* $\eta_n(t)$; $t\in[0,1]$ *be integer valued, increasing, and right continuous in t a.s., and assume that the* $\mathscr{B}_{n,i}$*'s are nested, i.e.* $\mathscr{B}_{n,i}\subset\mathscr{B}_{n+1,i}$, $n\geqq 1$, $i\geqq 1$. *If there exists a sequence* $\{\zeta_n\}$ *of stopping times with* $\lim_{n\to\infty} P(\zeta_n\geqq\eta_n(1))=1$ *and with*

$$E\max_{1\leqq i\leqq\zeta_n}|X_{n,i}|\to 0 \quad \text{as } n\to\infty \tag{27}$$

and if

$$\sum_{i=1}^{\eta_n(\cdot)} X_{n,i}^2 \xrightarrow{P} \eta \quad \text{as } n\to\infty, \text{ in } D(0,1), \tag{28}$$

for some random process $\eta\in D(0,1)$, *then*

$$\left(S\circ\eta_n, \sum_{i=1}^{\eta_n(\cdot)} X_{n,i}^2\right) \xrightarrow{d} (B\circ\eta, \eta) \quad \text{as } n\to\infty, \tag{29}$$

in $D(0,1)\times D(0,1)$, *where B is a Brownian motion independent of* η. *Further, if* η *is non-random then the requirement that the sigma-algebras are nested can be deleted.*

Remark 7. The condition (27) can be stated in the following equivalent way, which may be easier to check; "for each $\varepsilon>0$ there exists a sequence $\{\zeta_n\}$ of stopping times with $\limsup_{n\to\infty} P(\zeta_n\geqq\eta_n(1))\geqq 1-\varepsilon$ and with $E\max_{1\leqq i\leqq\zeta_n}|X_{n,i}|\to 0$". Further, since $\sum_{i=1}^{\eta_n(t)} X_{n,i}^2$ is increasing in t also $\eta(t)$ is increasing, and hence if $\eta\in C(0,1)$ a.s. then (28) is equivalent to

$$\sum_{i=1}^{\eta_n(t)} X_{n,i}^2 \xrightarrow{P} \eta(t) \quad \text{as } n\to\infty, \text{ for } t\in[0,1]. \tag{28$'$}$$

Corollary 8. *Let* $\{X_{n,i}\}$ *be a m.d.a. with the sigma-algebras* $\mathscr{B}_{n,i}$ *nested and let* $\{\eta_n\}$ *be a sequence of integer valued random variables. Further assume that there is a sequence* $\{\zeta_n\}$ *of stopping times with* $\lim_{n\to\infty} P(\zeta_n\geqq\eta_n)=1$ *and with* $E\max_{1\leqq i\leqq\zeta_n}|X_{n,i}|\to 0$, *and that* $\sum_{i=1}^{\eta_n} X_{n,i}^2 \xrightarrow{P} \eta$, *for some random variable* η. *Then*

$$P\left(\sum_{i=1}^{\eta_n} X_{n,i}\leqq x\right)\to\int\Phi(x/\sqrt{t})\,dF_\eta(t) \quad \text{as } n\to\infty,$$

where F_η is the distribution function of η, and

$$P\left(\sum_{i=1}^{\eta_n} X_{n,i} \Big/ \left\{\sum_{i=1}^{\eta_n} X_{n,i}^2\right\}^{1/2} \leqq X \,\middle|\, \eta>0\right) \to \Phi(x) \quad as \ n\to\infty.$$

If η is a constant, the condition that the sigma-algebras are nested can be deleted.

Proof. Since $\lim_{n\to\infty} P(\zeta_n \geqq \eta_n(1)) = 1$ it is enough to prove that (29) holds with $\eta_n(\cdot)$ replaced by $\eta_n(\cdot)\wedge\zeta_n$, and since (28) holds also after this replacement we may during the proof assume that $\eta_n(1)\leqq\zeta_n$. Thus, by switching over to a suitable sequence of martingale differences for $i>\zeta_n$, we without loss of generality assume that, writing $M_n = \max_{1\leqq i} |X_{n,i}|$, $EM_n\to 0$, that $\sum_{i=1}^{k} X_{n,i}^2 \xrightarrow{\text{a.s.}} \infty$ as $k\to\infty$, for each n, and that still the sigma-algebras are nested.

Let $\tau_n(t) = \inf\left\{k;\ \sum_{i=1}^{k} X_{n,i}^2 > t\right\}$ be the natural time-scale of $\{X_{n,i}\}$ and define $\tau_n^{-1}(k) = \inf\{t\geqq 0;\ \tau_n(t)\geqq k\}$. It can be seen that $|S(k) - S\circ\tau_n\circ\tau_n^{-1}(k)| \leqq M_n$ and that

$$\sum_{i=1}^{\eta_n(t)} X_{n,i}^2 - M_n^2 \leqq \tau_n^{-1}\circ\eta_n(t) \leqq \sum_{i=1}^{\eta_n(t)} X_{n,i}^2, \tag{30}$$

for $t\in[0,1]$. Hence (29) follows if we prove

$$(S\circ\tau_n\circ\tau_n^{-1}\circ\eta_n,\ \tau_n^{-1}\circ\eta) \xrightarrow{d} (B\circ\eta, \eta) \tag{31}$$

in $D(0,1)\times D(0,1)$.

To do this we first show that the sequence $\{S\circ\tau_n(t);\ t\geqq 0\}_{n=1}^{\infty}$ of random variables in $D(0,\infty)$ given the topology of [19] is Rényi-mixing with limiting distribution that of a Brownian motion. This follows if we show that

$$S\circ\tau_n \xrightarrow{d} B \quad \text{in } D(0,\infty), \tag{32}$$

with respect to $P(\cdot\,|A)$ if $P(A)>0$ and $A\in\mathscr{B}_{n_0,i_0}$ for some (n_0, i_0), cf. [15] and [2], p. 139. However, $\max_{1\leqq i\leqq i_0} \left|\sum_{j=1}^{i} X_{n,i}\right| \leqq i_0\cdot M_n \xrightarrow{P(\cdot|A)} 0$, and since the sigma-algebras are nested the array $\{X'_{n,i}\}$ defined for $n\geqq n_0$ by $X'_{n,i} = X_{n,i} I(i>i_0)$ is a m.d.a. with respect to $P(\cdot\,|A)$. Writing $S'_n(k) = \sum_{i=1}^{k} X'_{n,i}$, it follows easily from Theorem 2 that $S'\circ\tau_n \xrightarrow{d} B$ in $D(0,T)$ under $P(\cdot\,|A)$, for all $T>0$, which is equivalent to $S'\circ\tau_n \xrightarrow{d} B$ in $D(0,\infty)$ under $P(\cdot\,|A)$, and then (32) follows.

From (28) and (30) we have that $\tau_n^{-1}\circ\eta_n \xrightarrow{P} \eta$ in $D(0,1)$ and since $S\circ\tau_n$ is Rényi-mixing, Theorem 4.5 of [2] gives that

$$(S\circ\tau_n,\ \tau_n^{-1}\circ\eta_n) \xrightarrow{d} (B, \eta) \tag{33}$$

in $D(0,\infty)\times D(0,1)$, with B and η independent. Since B is continuous a.s. (31), and thus (29), now follows from Theorem 3.1 of [19]. Further, above the condition that the sigma-algebras are nested was only used to get joint convergence in (33). If η is non-random this is automatic, and the condition can be deleted.

Finally, Corollary 8 follows by introducing the time-scale

$$\eta_n(t)=\inf\left\{k;\ \sum_{i=1}^{k} X_{n,i}^2 \geqq t \sum_{i=1}^{\eta_n} X_{n,i}^2\right\}$$

and then applying the theorem. □

The conditions of the theorem above can of course easily be replaced by various others, e.g. conditions analogous to those of Theorems 3 and 5, or by somewhat weaker versions of (28) which still permit deducing joint convergence in (33), e.g. in the manner of [9]. The convergence in probability in (28) is as noted only used to go from marginal to joint convergence in (33) and is not necessary, as is easily seen by example. Clearly (33) is sufficient for the result of the theorem, and it may well be necessary too, see [7].

By truncating and recentering it is possible to use Theorem 6 to prove central limit theorems also for arrays which are close to being m.d.a.'s. The following corollary, which improves on a result of [13], contains one set of conditions for this.

Corollary 9. *The conclusions of Theorem 6 and Corollary 8 hold also if the requirements that $\{X_{n,i}\}$ is a m.d.a. and that $E \max_{1\leqq i\leqq \zeta_n} |X_{n,i}|\to 0$ are replaced by the following condition; that there exist constants d, d_n with $d_n \geqq d>0$ for $n\geqq 1$ such that*

$$d_n \log d_n P(\max_{1\leqq i\leqq \zeta_n} |X_{n,i}|>d)\to 0 \qquad \text{as } n\to\infty \tag{34}$$

and such that

$$\sum_{i=1}^{\zeta_n} |E_{i-1}(X_{n,i} I(|X_{n,i}|\leqq d_n))| \xrightarrow{P} 0 \qquad \text{as } n\to\infty. \tag{35}$$

Proof. Since the modified version of Corollary 8 follows from the modified version of Theorem 6 in the same way as above, we will only prove the latter. Put $X'_{n,i}=X_{n,i} I(|X_{n,i}|\leqq d_n)-E_{i-1}(X_{n,i} I(|X_{n,i}|\leqq d_n))$, so that $\{X'_{n,i}\}$ is a m.d.a. and let as usual $S'_n(k)=\sum_{i=1}^{k} X'_{n,i}$. Since $P(\zeta_n\geqq \eta_n(1))\to 1$ we have by (34) and (35) that $\sup_{0\leqq t\leqq 1} |S\circ \eta_n(t)-S'\circ \eta_n(t)| \xrightarrow{P} 0$. Further, by arguments similar to the proof of (4) of Lemma 1 above, $\sup_{0\leqq t\leqq 1}\left|\sum_{i=1}^{\eta_n(t)} X_{n,i}^2-\sum_{i=1}^{\eta_n(t)} X_{n,i}'^2\right| \xrightarrow{P} 0$, and hence the corollary follows from Theorem 6 if we show that

$$E \max_{1\leqq i\leqq \zeta_n} |X'_{n,i}|\to 0 \qquad \text{as } n\to\infty. \tag{36}$$

For brevity put $I_{n,i}=I(d<|X_{n,i}|\leqq d_n)$ and $\tilde{M}_n=\max_{1\leqq i\leqq \zeta_n} |X_{n,i}| I_{n,i}$ and note that $\{E_{i-1}(\tilde{M}_n)\}_{i=1}^{\infty}$ is a martingale. Further let $c=e/(e-1)$ and let $K>0$ be arbitrary. Then inequality VII: (3.7) of [4], with x_j replaced by $KE_{i-1}(\tilde{M}_n)$ gives that

$$E \max_{1 \leqq i \leqq \zeta_n} E_{i-1}(|X_{n,i}| I_{n,i}) \leqq E \max_{1 \leqq i \leqq \zeta_n} E_{i-1}(\tilde{M}_n)$$
$$\leqq c/K + c E\tilde{M}_n \log^+ K\tilde{M}_n$$
$$\leqq c/K + c d_n \log^+ (d_n K) E \max_{1 \leqq i \leqq \zeta_n} I_{n,i},$$

where $\log^+ x = 0 \vee \log x$. Clearly $E\tilde{M}_n \leqq d_n E \max_{1 \leqq i \leqq \zeta_n} I_{n,i}$ and $E \max_{1 \leqq i \leqq \zeta_n} I_{n,i} \leqq P(\max_{1 \leqq i \leqq \zeta_n} |X_{n,i}| > d)$, so writing $X''_{n,i} = X_{n,i} I(|X_{n,i}| \leqq d) - E_{i-1}(X_{n,i} I(|X_{n,i}| \leqq d))$ we have

$$E \max_{1 \leqq i \leqq \zeta_n} |X'_{n,i} - X''_{n,i}|$$
$$\leqq c/K + d_n(1 + c \log^+ d_n K) P(\max_{1 \leqq i \leqq \zeta_n} |X_{n,i}| > d) \to c/K, \quad \text{as } n \to \infty,$$

by (34). Since K is arbitrary this shows that $E \max_{1 \leqq i \leqq \zeta_n} |X'_{n,i} - X''_{n,i}| \to 0$.

Further, from (34) and (35) follows that $\max_{1 \leqq i \leqq \zeta_n} |X'_{n,i}| \xrightarrow{P} 0$, and hence also $\max_{1 \leqq i \leqq \zeta_n} |X''_{n,i}| \xrightarrow{P} 0$. Since $|X''_{n,i}| \leqq 2d$ this implies that $E \max_{1 \leqq i \leqq \zeta_n} |X''_{n,i}| \to 0$ and hence (36) holds. □

The conditions for convergence in [13] do not include the existence of the bounding stopping times $\{\zeta_n\}$. Under the conditions of Theorem 1 of [13] it is easy to find such stopping times, one can e.g. take $\zeta_n = \inf\left\{k;\ \sum_{i=1}^{k} X_{n,i}^2 > 1 + \delta\right\}$. However, Theorem 2 of that paper is incorrectly stated, as can be seen from the following example, where we use the notation of [13]. Let $X_{n,i}$; $i = 1, 2, \ldots$ be independent and let $X_{n,i} = 1/n$ with probability $n/(n+1)$ and -1 otherwise and let $k_n = \inf\{i;\ X_{n,i+1} = -1\} \wedge n$. Then it is easy to see that the conditions of the theorem are satisfied, with $\sigma^2 = 0$ (take e.g. $d_n = d = 2$), but $\sum_{i=1}^{k_n} X_{n,i}$ does not converge to zero as it should by the theorem.

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