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A Dissonant Voting Model: Nonergodic Case

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1. Introduction

Let S be a countable set, and let $p(x,y)$ be a stochastic matrix defined on $S \times S$. Think of the elements of S as "voters," each of whom is either in favor of or against a certain issue. Each voter $x \in S$ at random times re-evaluates his position on the issue, according to the following procedure: x chooses a voter $y \in S$ at random according to the probabilities $p(x,\cdot)$, and then takes on the position opposite to that of y. The process, which we will call $\mathscr{V}_-$, continues indefinitely in this manner. This model is similar to the $\mathscr{V}_+$ process considered in Holley and Liggett [1], except that in the re-evaluation procedure in the latter model, x takes on the *same* position as y.

In [5] we found necessary and sufficient conditions for the process described above to be ergodic, that is, to be such that the distribution of the process of time t is guaranteed to converge weakly to a unique invariant measure v, which does not depend on the initial distribution of the process. In this paper we treat the nonergodic case.

In Secr. 2 we first give a brief review of the notation and tools used in [5]. Then, after defining several "collision functions," we state and prove several lemmas concerning these functions. These lemmas are used frequently in Sect. 3 and 4.

Section 3 is concerned with $\mathscr{I}$, the class of invariant measures for the process. The main result characterizes the set $\mathscr{I}_e$ of extreme points of $\mathscr{I}$, in terms of a class of harmonic functions on a certain Markov chain. This characterization may be intuitively described as follows: As noted in [5], if $p(x,y)$ has even period, then there is a one-to-one correspondence between our process and that of Holley and Liggett [1]; thus we will assume throughout that $p(x,y)$ has odd period, as we did in [5]. However, one still may be able to find sections of S on which $p(x,y)$ is "asymptotically" of even period. For example, suppose $S = Z$ and

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$$p(x,y)=\begin{cases}\frac{2}{3}, & \text{if } y=x-2 \text{ and } x<0, \text{ or} \\ & \text{if } y=x+1 \text{ and } x\geqq 0 \\ \frac{1}{3}, & \text{if } y=x+1 \text{ and } x<0, \text{ or} \\ & \text{if } y=x-1 \text{ and } x\geqq 0 \\ 0, & \text{otherwise}\end{cases} \tag{1.1}$$

Then if the Markov chain X_n moves according to $p(x,y)$, X_n will eventually be absorbed into one of the two components $Z^-=\{x<0\}$ and $Z^+=\{x\geqq 0\}$. Even though $p(x,y)$ technically has period one here, the component Z^+ has "asymptotic" period two; thus in this asymptotic sense we may think of Z as consisting of one "odd" component (Z^-) and one "even" component (Z^+). The results in Sect. 3 indicate that on "odd" components, the $\mathscr{V}_-$ process acts very much like it does in the ergodic case, covered in [5]; on "even" components, the $\mathscr{V}_-$ process acts very much like a transformed $\mathscr{V}_+$ process. Thus, near $x=-\infty$, all $v\in\mathscr{I}_e$ look essentially the same; near $x=+\infty$, they look essentially like point masses on $(\ldots,0,1,0,1,\ldots)$ or $(\ldots,1,0,1,0,\ldots)$, corresponding to point masses on $(\ldots,0,0,0,0,\ldots)$ or $(\ldots,1,1,1,1,\ldots)$ for the $\mathscr{V}_+$ process.

Last, in Sect. 4 we find necessary and sufficient conditions on initial distributions μ, such that the resulting distribution μ_t at time t converges as $t\to\infty$ to any given element of $\mathscr{I}_e$.

2. Preliminary Material

Let S be a countable set, and let $p(x,y)$ be the transition matrix for a discrete time Markov chain on S. We will assume throughout that p is irreducible and of odd period, and that $p(x,x)\equiv 0$ (see [5] for the motivation for the latter assumption). We will also assume that p is transient, since the recurrent case is one of those covered in [5].

As before, the state space of the process will be $K=\{0,1\}^S$ (unless otherwise stated the word "process" will refer to $\mathscr{V}_-$). Let $C(K)$ be the usual Banach space of continuous real-valued functions on K, and let $\mathscr{F}$ be the class of functions on K which depend on only finitely many coordinates. The intuitive description of our process in Section 1 motivates the following: For any $f\in\mathscr{F}$, and $\eta\in K$, we define $(\Omega f)(\eta)$ to be

$$\sum_{x\in S}\sum_{\substack{y\in S,\\ \eta(x)=\eta(y)}} p(x,y)[f(\eta_x)-f(\eta)], \quad \text{where } \eta_x(z)=\begin{cases}\eta(z), & z\neq x\\ 1-\eta(x), & z=x.\end{cases}$$

An extension of Ω is then the infinitesimal generator of our process. A typical realization of the process at time t will be denoted by η_t.

The Markov chains $\alpha(t)$ and σ_t below play a key role in our analysis of η_t:

(a) *The chain* $\alpha(t)$. Let $S_1=S\times\{0,1\}$, and define the stochastic matrix Q on $S_1\times S_1$ by

$$Q(\alpha,\beta)=\begin{cases}p(x,y), & \text{if } a+b=1\\ 0, & \text{if not}\end{cases}$$

where $\alpha=(x,a)$ and $\beta=(y,b)$. Intuitively, Q describes the motion of a particle which has a "position" (in S) and a "color" (either 0 or 1). The particle moves on S according to $p(x,y)$, and it changes color each time it moves. Now let $\alpha(t)$ be a continuous time Markov chain on S_1 whose paths are governed by Q and whose holding times at each state each have mean one. Finally, extend $\alpha(t)$ to $\bigcup_{i=1}^{n} S_1^i$ by letting $\boldsymbol{\alpha}(t)=[\alpha_1(t),\ldots,\alpha_i(t)]$ consist of i independent copies of $\alpha(t)$. We will often write $\boldsymbol{\alpha}(t)$ as $[(X_1(t),a_1(t)),\ldots,(X_i(t),a_i(t))]$. Also, we will denote the transition semigroup for $\boldsymbol{\alpha}(t)$ by V_t, so that $V_t f(\boldsymbol{\alpha})=E^{\boldsymbol{\alpha}} f(\boldsymbol{\alpha}(t))$ for any $\boldsymbol{\alpha}\in S_1^i$, and any bounded function f.

(b) *The chain* σ_t. Define L_n to be

$$\{\Delta\}\cup\left[\bigcup_{i=1}^{n} S_1^i\right],$$

where Δ is a "death point" which will be explained below, and let $D_n=\{[x_1,a_1),\ldots,(x_i,a_i)]\colon x_r=x_s \text{ for some } r\neq s \text{ and some } i\leqq n\}$. Now let σ_t be the continuous time Markov chain on $L_n\setminus D_n$ whose infinitesimal parameters reflect the following intuitive description: Let $\sigma_0=\boldsymbol{\alpha}(0)$ be in $S_1^i\setminus D_n$, and let T_{D_n} be the hitting time of D_n for $\boldsymbol{\alpha}(t)$; T_{D_n} is allowed to infinite, and of course it will be infinite if i is equal to one. Now let $\sigma_t=\boldsymbol{\alpha}(t)$ for $0\leqq t<T_{D_n}$. At time T_{D_n}, there is a collision (in S) between two of the particles $\alpha_1(t),\ldots,\alpha_i(t)$. If the two colliding particles are of opposite color when they hit, then all of the particles in σ_t die, and we set $\sigma_t=\Delta\ \forall\, t\geqq T_{D_n}$. If the two colliding particles are of the same color, then the particle of higher index dies, and the other particles continue as before, until the time of the next collision. The process σ_t continues in this manner. (A rigorous construction of σ_t may be found in [5].)

Proposition 2.1 below was proved in [5]. To state this result, let $\mathcal{M}$ be the class of probability measures on K, and for any $\mu\in\mathcal{M}$ define f on L_n by $f([x_1,a_1),\ldots,(x_i,a_i)])\equiv\mu(\eta(x_1)=a_1,\ldots,\eta(x_i)=a_i)$ and $f(\Delta)=0$; we will denote this correspondence by $f\leftrightarrow\mu$. Finally, if $\mu\in\mathcal{M}$ is the initial distribution for the process, then we will denote the resulting distribution at time t by μ_t.

Proposition 2.1. *Fix $\mu\in\mathcal{M}$, and let f and μ_t be as above. Then for all $t\geqq 0$ we have*

$$\mu_t(\eta(x_1)=a_1,\ldots,\eta(x_k)=a_k)=E^{\sigma_0}f(\sigma_t)=U_t f(\sigma_0)$$

for all $\sigma_0=[(x_1,a_1),\ldots,(x_k,a_k)]\in L_n\setminus D_n$, where U_t is the transition semigroup for σ_t.

Let $\mathcal{I}$ denote the class of invariant probability measures μ for the process, that is, those for which $\mu_t\equiv\mu$. Then $\mathcal{I}$ is a compact, convex subset of $C(K)^*$, so that the Krein-Milman theorem implies that $\mathcal{I}$ is the closed convex hull of the set of its extreme points, $\mathcal{I}_e$. In this sense, one may characterize $\mathcal{I}$ through $\mathcal{I}_e$.

In our analysis of η_t in [5], we made use of a "collision function" g, which we may define on L_n as follows: $g(\Delta)=0$, and for $\boldsymbol{\alpha}\in S_1^k$,

$$g(\boldsymbol{\alpha})=P^{\boldsymbol{\alpha}}(X_i(t)=X_j(t) \text{ for some } i\neq j \text{ and some } t>0),$$

where $\boldsymbol{\alpha}(t)=[(X_1(t),a_1(t)),\ldots,(X_k(t),a_k(t))]$. In other words, $g(\boldsymbol{\alpha})$ is the probability that there ever will be a collision (in S) among the k particles in $\boldsymbol{\alpha}(t)$, if $\boldsymbol{\alpha}(0)=\boldsymbol{\alpha}$.

Thus we may informally describe $g(\boldsymbol{\alpha})$ as a measure of the dispersion of the k sites in $\boldsymbol{\alpha}$.

In this paper we will also use several other similar functions:

$$g_S(\boldsymbol{\alpha}) = P^{\boldsymbol{\alpha}}(\alpha_i(t) = \alpha_j(t) \text{ for some } i \neq j, \text{ and some } t > 0),$$

$$g_D(\boldsymbol{\alpha}) = P^{\boldsymbol{\alpha}}(\alpha_i(t) = \alpha_j(t)^{\#} \text{ for some } t > 0), \text{ and}$$

$$g_B(\boldsymbol{\alpha}) = P^{\boldsymbol{\alpha}}(\alpha_{i_1}(t) = \alpha_{j_1}(t) \text{ for some } i_1 \neq j_1 \text{ and some } t > 0, \text{ and } \alpha_{i_2}(u) = \alpha_{j_2}(u)^{\#} \text{ for some } i_2 \neq j_2 \text{ and some } u > 0),$$

where for $\beta = (y, b)$ we define $\beta^{\#}$ to be $(y, 1-b)$.

The letters S, D and B stand for "same," "different," and "both." Thinking of the parity $a_i(t)$ as the "color" of the particle $(X_i(t), a_i(t))$, g tells us the probability that the particles will ever collide, without regard to color, while g_S is the likelihood of the occurrence of a collision in which the two particles will be of the same color. Similarly, g_D tells us the probability of an opposite-color collision, and g_B is the probability that both types of collisions will take place eventually. Note that each of these functions is identically 0 on S_1.

We also define $\bar{g}$, $\bar{g}_S$, $\bar{g}_D$ and $\bar{g}_B$ to be the probabilities that the appropriate type of collision occurs infinitely often, that is, at arbitrarily large times.

Lemma 2.2. (a) *The random variable $g(\alpha_1(t), \ldots, \alpha_n(t))$ converges almost surely to one or zero, depending on whether or not there occur infinitely many collisions (without regard to color).*

(b) $\lim_{t \to \infty} V_t g = \bar{g}$.

(c) *Furthermore, $g(\boldsymbol{\alpha}(t)) - \bar{g}(\boldsymbol{\alpha}(t)) \to 0$ a.s. at $t \to \infty$.*

The same results hold for the other collision functions g_S, g_D and g_B.

Proof. Let Y_t be the indicator function of the event $\{\exists\, s > t\; X_i(s) = X_j(s)$ for some $i \neq j\}$, and let $\mathscr{F}_t = \sigma([\alpha_1(u), \ldots, \alpha_n(u)],\ u \leqq t)$. Then $g(\alpha_1(t), \ldots, \alpha_n(t)) = P^{\boldsymbol{\alpha}(t)}(X_i(u) = X_j(u)$ for some $i \neq j$ and some $u > 0) = P^{\boldsymbol{\alpha}(0)}(X_i(t+u) = X_j(t+u)$ for some $i \neq j$ and some $u > 0 \mid \mathscr{F}_t)$, by the Markov property. The last probability is equal to $E^{\boldsymbol{\alpha}(0)}(Y_t \mid \mathscr{F}_t)$, which converges almost surely to the indicator function 1_A of the event

$$A = \{\exists\, s_n \to \infty \text{ such that } X_i(s_n) = X_j(s_n) \text{ for some } i(n) \neq j(n)\}.$$

Thus $g(\boldsymbol{\alpha}(t)) \to 1_A$ a.s., as claimed. Part (b) is then immediate. Now to show (c), merely note that $\lim[g(\boldsymbol{\alpha}(t)) - \bar{g}(\boldsymbol{\alpha}(t))]$ exists $[\bar{g}(\boldsymbol{\alpha}(t))$ is a martingale], is non-negative, and has mean zero.

The proofs for the other collision functions are completely parallel to the argument for g.

Corollary 2.3. $\lim_{t \to \infty} g(\sigma_t)$ *exists and is equal to zero a.s., where $g(\Delta)$ is defined to be zero.*

Proof. By the strong Markov property we need only consider what happens on the set on which the hitting time of S_1^{n-1} for σ_t is infinite, where $\sigma_0 \in S_1^n$. But the distribution of $\{\sigma_t, t \geqq 0\}$ on this set is the same as that of $\{\boldsymbol{\alpha}(t), t \geqq 0\}$ (where $\boldsymbol{\alpha}(0) = \sigma_0$), so that by Lemma 2.2 we know that $\lim_{t\to\infty} g(\sigma_t) = 0$ a.s.

Lemma 2.4. *Let $\alpha(t)$ move according to V_t. Then for any fixed $\beta \in S_1$ we have $(g - \bar{g})(\alpha(t), \beta) \to 0$ a.s. as $t \to \infty$.*

Proof. The proof is an adaption of the argument in Lemma 2.4 of Liggett [3], applied to the first component of $\alpha(t) = (X(t), a(t))$.

Corollary 2.5. *For any $\alpha, \beta \in S_1$ we have $\lim_{t\to\infty} E^\alpha g(\alpha(t), \beta) = \bar{g}(\alpha, \beta)$.*

Proof. By Lemma 2.2, $V_t \bar{g} = \bar{g}$ on S_1^2, so that Lemma 3.14 of Liggett [2] implies that $E^\alpha \bar{g}(\alpha(t), \beta) = \bar{g}(\alpha, \beta)$. An application of Lemma 2.4 now completes the proof.

Corollary 2.6. *Results analogous to those in Lemma 2.4 and Corollary 2.5 hold for the other collision functions g_S, g_D and g_B.*

Proof. The proof of Lemma 2.4 applies equally well to the chain $\alpha(t)$ itself, so $\lim_{t\to\infty} (g_S - \bar{g}_S)(\alpha(t), \beta) = 0$ a.s. for each $\beta \in S_1$. The result for $g_D - \bar{g}_D$ then follows from the relation $(g_D - \bar{g}_D)(\alpha, \beta) = (g_S - \bar{g}_S)(\alpha, \beta^\#)$. The result for $g_B - \bar{g}_B$ then is immediate from the relation $g = g_S + g_D - g_B$ and its analog for $\bar{g}$.

The proofs of the analogs of Corollary 2.5 are completely parallel to the proof of the latter corollary.

Corollary 2.6 can then be used in an argument parallel to that of Corollary 2.3 to prove

Corollary 2.7. $\lim_{t\to\infty} (g - \bar{g})(\sigma_t, \beta) = 0$ *a.s.*

One of the major reasons for studying the collision functions is that we can use them to compare the semigroups U_t and V_t. For instance, one relationship which we will often use is that

$$|U_t f - V_t f| \leqq g \quad \text{on } L_n \setminus D_n \tag{2.1}$$

for any f: $L_n \setminus D_n \to [0, 1]$. To verify this, let $\boldsymbol{\alpha}(t) = (\alpha_1(t), \ldots, \alpha_n(t))$ start at $\boldsymbol{\alpha} \in L_n \setminus D_n$ and move according to V_t, and let σ_t start $\boldsymbol{\alpha}$ and move according to U_t. The underlying probability space can be constructed in such a way that $\sigma_t = \boldsymbol{\alpha}(t)$ on $\{t < T_{D_n}\}$ where T_{D_n} is the hitting time of D_n for $\boldsymbol{\alpha}(t)$. Then

$$|U_t f(\boldsymbol{\alpha}) - V_t f(\boldsymbol{\alpha})| = |E^{\boldsymbol{\alpha}} f(\sigma_t) - E^{\boldsymbol{\alpha}} f(\boldsymbol{\alpha}(t))| = |E^{\boldsymbol{\alpha}}[f(\sigma_t) - f(\boldsymbol{\alpha}(t)), t \geqq T_{D_n}]|$$
$$\leqq P^{\boldsymbol{\alpha}}(t \geqq T_{D_n}) \leqq g(\boldsymbol{\alpha}).$$

A similar but slightly more delicate argument gives us the following fact: Let w: $L_1 \to [0, 1]$, and extend w to W: $L_2 \to [0, 1]$ by setting $W(\alpha, \beta) \equiv w(\alpha)$, and let $W(\Delta) = 0$. Then

$$|U_t W - V_t W| \leqq g_D \quad \text{on } L_2 \setminus D_2. \tag{2.2}$$

Also, suppose $U_t f = f$ on $S_1^2 \setminus D_2$, $f = w$ on S_1, and f is $[0, 1]$-valued. Suppose also that $V_t W = W$ on S_1^2. Then for $(\delta, \alpha) \in S_1^2 \setminus D_2$ we have

$$
\begin{aligned}
&|f(\delta,\alpha)-W(\delta,\alpha)| \\
&\quad=|U_t f(\delta,\alpha)-V_t W(\delta,\alpha)| \\
&\quad=\lim_{t\to\infty}|U_t f(\delta,\alpha)-V_t W(\delta,\alpha)| \\
&\quad\leqq\lim_{t\to\infty}|U_t f(\delta,\alpha)-U_t W(\delta,\alpha)|+\lim_{t\to\infty}|U_t W(\delta,\alpha)-V_t W(\delta,\alpha)| \\
&\quad\leqq\lim_{t\to\infty}P^{(\delta,\alpha)}(\sigma_t\in S_1^2)+\lim_{t\to\infty}g_D(\delta,\alpha) \\
&\quad=1-g(\delta,\alpha)+g_D(\delta,\alpha).
\end{aligned}
\tag{2.3}
$$

3. Characterization of $\mathscr{I}_e$

Define the following classes of harmonic functions:

$$\mathscr{H}^1=\{f{:}f \text{ is a } [0,1]\text{-valued function on } S, \text{ and } pf=f\},$$
$$\mathscr{H}\;=\{f{:}f \text{ is a } [0,1]\text{-valued function on } S_1, \text{ and } Qf=f\} \text{ and}$$
$$\mathscr{H}_0\;=\{f\in\mathscr{H}{:}f(\alpha)+f(\alpha^{\#})\equiv 1\}.$$

The main result of this section is Theorem 3.3, which puts $\mathscr{I}_e$ in one-to-one correspondence with a subset of $\mathscr{H}_0$. After the proof of this theorem, we interpret the results of the theorem in terms of $\mathscr{H}^1, \mathscr{H}, \mathscr{H}_0$ and the "components" of S mentioned in Section 1. We first present some preliminary results:

Lemma 3.1. (a) *For each $r\in\mathscr{H}_0$, there is a unique invariant measure ν_r with $\nu_r(\eta(x)=a)\equiv r(x,a)$, such that for each*

$$\boldsymbol{\alpha}=[(x_1\,a_1),\ldots,(x_n,a_n)]\in S_1^n/D_n$$

we have

$$|\nu_r(\eta(x_1)=a_1,\ldots,\eta(x_n)=a_n)-\nu_r(\eta(x_1)=a_1)\ldots\nu_r(\eta(x_n)=a_n)|\leqq g(\boldsymbol{\alpha}). \tag{3.1}$$

(b) $\mathscr{I}_e\subset\{\nu_r{:}\ r\in\mathscr{H}_0\}$.

Remark. Since we may think of g as a measure of dispersion among voters, we may interpret (3.1) as saying that voters who are "distant" from each other act approximately independently.

Proof. (a) Take any $r\in\mathscr{H}_0$, and let μ be the product measure such that $\mu(\eta(x)=a)\equiv r(x,a)$. By Remark 3.3 of [5], there exist $\nu\in\mathscr{I}$ and $t_i\to\infty$ so that $\mu_{t_i}\to\nu$. Let $f\leftrightarrow\mu$, and let $\boldsymbol{\alpha}=[\alpha_1,\ldots,\alpha_n]$ be as in the statement of the lemma. Now

$$|U_{t_i}f(\boldsymbol{\alpha})-U_{t_i}f(\alpha_1)\ldots U_{t_i}f(\alpha_n)|=|U_{t_i}f(\boldsymbol{\alpha})-V_{t_i}[f(\alpha_1)\ldots f(\alpha_n)]|\leqq g(\boldsymbol{\alpha}),$$

by (2.1). But then if we let $i\to\infty$ and use the facts that $U_t f(\alpha_j)=U_t r(\alpha_j)=r(\alpha_j)$ and $U_{t_i}f(\boldsymbol{\alpha})\to\nu(\eta(x_1)=a_1,\ldots,\eta(x_n)=a_n)$ (by Proposition 2.1), we see that ν satisfies (3.1). Note also that $\nu(\eta(x)=a)=\lim_i U_{t_i}f(x,a)=r(x,a)$, as claimed.

To show uniqueness, suppose μ_1 and μ_2 are both invariant measures which satisfy (3.1) for some r, and let $f_i\leftrightarrow\mu_i$ $(i=1,2)$. Our proof will use an induction on n to show that $f_1 = f_2$ on all $L_n \backslash D_n$. By assumption, $f_1 = f_2 = r$ on S_1, and we will now also assume that f_1 and f_2 agree on $L_{n-1} \backslash D_{n-1}$ for some $n > 1$. Now for $\gamma \in S_1^n \backslash D_n$ we have

$$\begin{aligned} f_1(\gamma) - f_2(\gamma) &= U_t[f_1(\gamma) - f_2(\gamma)] = E^\gamma[f_1(\sigma_t) - f_2(\sigma_t)] \\ &= E^\gamma[f_1(\sigma_t) - f_2(\sigma_t), \tau < \infty] + E^\gamma[f_1(\sigma_t) - f_2(\sigma_t), \tau = \infty], \end{aligned}$$

where τ is the hitting time of $\{\Delta\} \cup S_1^{n-1}$ (i.e., the time of the first collision) for σ_t. The induction assumption implies that $f_1(\sigma_t) = f_2(\sigma_t)$ on $\{t \geqq \tau\}$, and Corollary 2.3 and Equation (3.1) above imply that $\lim_{t \to \infty} [f_1(\sigma_t) - f_2(\sigma_t)] = 0$ on $\{\tau = \infty\}$. Thus by applying bounded convergence to the above expression for $f_1(\gamma) - f_2(\gamma)$, we see that f_1 and f_2 agree on S_1^n, and the induction is complete.

(b) Fix $(x_0, a_0) \in S_1$. Take any $\mu \in \mathscr{I}_e$, and define $\mu^{(1)}$ and $\mu^{(2)}$ in $\mathscr{M}$ by

$$\mu = \lambda\, \mu^{(1)} + (1 - \lambda)\, \mu^{(2)}, \tag{3.2}$$

where $\lambda = \mu(\eta(x_0) = a_0)$ and

$$\mu^{(1)}(\eta(y_1) = b_1, \ldots, \eta(y_l) = b_l) \equiv \mu(\eta(y_1) = b_1, \ldots, \eta(y_l) = b_l | \eta(x_0) = a_0)$$

($\mu^{(1)}$ and $\mu^{(2)}$ are well defined; see Corollary 4.4 of [4]).

We claim that

$$\lim_{t \to \infty} \mu_t^{(1)} = \lim_{t \to \infty} \mu_t^{(2)} = \mu. \tag{3.3}$$

Again by Remark 3.3 of [5], we may show this by proving that any sequence of times $\{t_i\}$ for which $t_i \to \infty$ has a subsequence on which the indicated limits occur. We may take two successively finer subsequences of $\{t_i\}$ to get a sequence $\{u_j\}$ such that $\mu_{u_j}^{(1)}$ and $\mu_{u_j}^{(2)}$ converge to some $\bar{\mu}^{(1)}$ and $\bar{\mu}^{(2)}$ in $\mathscr{I}$ (Corollary 3.2 of [5]). Then since $\mu_t = \mu$ for all $t \geqq 0$, Eq. (3.2) implies $\mu = \lambda\, \bar{\mu}^{(1)} + (1 - \lambda)\, \bar{\mu}^{(2)}$, so that $\bar{\mu}^{(1)} = \bar{\mu}^{(2)} = \mu$, since $\mu \in \mathscr{I}_e$. This shows (3.3).

Now let $f \leftrightarrow \mu$ and define F by $\frac{1}{\lambda} F \leftrightarrow \mu^{(1)}$. By (3.3) we have

$$\lim_{t \to \infty} U_t \left(\frac{1}{\lambda} F\right) = f \tag{3.4}$$

on each $L_n \backslash D_n$ $(n \geqq 1)$. Also, by Remark 3.3 and Lemma 3.1 of [5], we may find $t_i \to \infty$ such that $\lim_{i \to \infty} V_{t_i} f$ is equal to some h on each S_1^n, with $V_t h = h\, \forall t \geqq 0$.

Fix $n \geqq 1$, and let $\boldsymbol{\alpha} = [\alpha_1, \ldots, \alpha_n]$ and $\boldsymbol{\beta} = [\beta_1, \ldots, \beta_n, \alpha_0]$ be in $S_1^n \backslash D_n$ and S_1^{n+1}, respectively; here $\alpha_0 = (x_0, a_0)$. We will usually not require $\boldsymbol{\beta}$ to be in $S_1^{n+1} \backslash D_{n+1}$, but if so we have

$$|f(\boldsymbol{\beta}) - V_t f(\boldsymbol{\beta})| = |U_t f(\boldsymbol{\beta}) - V_t f(\boldsymbol{\beta})| \leqq g(\boldsymbol{\beta}),$$

by (2.1) and thus

$$|f(\boldsymbol{\beta}) - h(\boldsymbol{\beta})| \leqq g(\boldsymbol{\beta}) \quad \text{on } S_1^{n+1} \backslash D_{n+1}. \tag{3.5}$$

Now let σ_t start at $\gamma \in S_1^n \backslash D_n$ and move according to U_t. Regard (σ_t, α_0) as an element of S_1^{i+1} unless $\sigma_t = \Delta$, in which case set $(\sigma_t, \alpha_0) = \Delta$. Then (3.5) implies that

$$|E^\gamma[f(\sigma_t,\alpha_0)-h(\sigma_t,\alpha_0)]| \leqq E^\gamma |f(\sigma_t,\alpha_0)-h(\sigma_t,\alpha_0)|$$
$$\leqq E^\gamma g(\sigma_t,\alpha_0)+P^\gamma((\sigma_t,\alpha_0)\in D_{n+1}), \tag{3.6}$$

where $h(\Delta)$ is defined to be zero. Now $f(\sigma,\alpha_0)=F(\sigma)$, and we can define G, $\bar{G}$ and H similarly: Let $G(\sigma)=g(\sigma,\alpha_0)$, $\bar{G}(\sigma)=\bar{g}(\sigma,\alpha_0)$, and $H(\sigma)=h(\sigma,\alpha_0)$. With this notation we can rewrite (3.6) as

$$|U_t F(\gamma)-U_t H(\gamma)| \leqq U_t G(\gamma)+P^\gamma((\sigma_t,\alpha_0)\in D_{n+1}). \tag{3.7}$$

Since h is harmonic for V_t, we know from Lemma 3.14 of Liggett [2] that H is harmonic for V_t. Thus

$$|U_t H(\gamma)-H(\gamma)|=|U_t H(\gamma)-V_t H(\gamma)| \leqq g(\gamma) \tag{3.8}$$

by (2.1). By a similar argument

$$|U_t\bar{G}(\gamma)-\bar{G}(\gamma)| \leqq g(\gamma). \tag{3.9}$$

Note that the harmonicity of $\bar{g}$ (and thus the harmonicity of $\bar{G}$) follows from Lemma 2.2. Now (3.7) and (3.8) give us

$$|U_t F(\gamma)-H(\gamma)| \leqq U_t G(\gamma)+P^\gamma((\sigma_t,\alpha_0)\in D_{n+1})+g(\gamma),$$

which by (3.9) implies

$$|U_t F(\gamma)-H(\gamma)| \leqq |U_t G(\gamma)-U_t\bar{G}(\gamma)|+2g(\gamma)+P^\gamma(\sigma_t,\alpha_0)\in D_{n+1})+\bar{G}(\gamma).$$

Now as $t\to\infty$, we know that $|U_t[G(\gamma)-\bar{G}(\gamma)]|\to 0$, by Corollary 2.7, and $P^\alpha((\sigma_t,\alpha_0)\in D_{n+1})$ also goes to 0, by the transience of $p(x,y)$. Thus we have by (3.4) that

$$|\lambda f(\gamma)-h(\gamma,\alpha_0)| \leqq 2g(\gamma)+\bar{G}(\gamma)=2g(\gamma)+\bar{g}(\gamma,\alpha_0) \leqq 3g(\gamma,\alpha_0).$$

Finally, the last inequality and (3.5) imply that if $(\gamma,\alpha_0)\in S_1^{n+1}\backslash D_{n+1}$, then

$$|f(\gamma,\alpha_0)-f(\gamma)f(\alpha_0)|=|f(\gamma,\alpha_0)-\lambda f(\gamma)| \leqq 4g(\gamma,\alpha_0). \tag{3.10}$$

Now suppose that in the last paragraph we have $n\geqq 2$, and write γ as $(\boldsymbol{\sigma},\gamma_n)$. By using (3.10) twice we get

$$\begin{aligned}|f(\boldsymbol{\sigma},\gamma_n,\alpha_0)-f(\boldsymbol{\sigma})f(\gamma_n)f(\alpha_0)| &\leqq |f(\boldsymbol{\sigma},\gamma_n,\alpha_0)-f(\boldsymbol{\sigma},\gamma_n)f(\alpha_0)| \\ &\quad +|f(\boldsymbol{\sigma},\gamma_n)f(\alpha_0)-f(\boldsymbol{\sigma})f(\gamma_n)f(\alpha_0)| \\ &\leqq 4g(\boldsymbol{\sigma},\gamma_n,\alpha_0)+f(\alpha_0)[4g(\boldsymbol{\sigma},\gamma_n)] \\ &\leqq 8g(\boldsymbol{\sigma},\gamma_n,\alpha_0).\end{aligned}$$

Thus an induction on n gives us

$$|f(\alpha_1,\ldots,\alpha_n)-f(\alpha_1)\ldots f(\alpha_n)| \leqq 4(n-1)\,g(\alpha_1,\ldots,\alpha_n)$$
$$\text{for all } (\alpha_1,\ldots,\alpha_n)\in S_1^n\backslash D_n. \tag{3.11}$$

Now to get the sharper bound (3.1), note that we showed in our unicity proof above that ν_r is the only invariant measure ν which agrees with ν_r on S_1 and for which

$$\nu(\eta(x_1)=a_1,\ldots,\eta(x_n)=a_n)-\nu(\eta(x_1)=a_1)\ldots\nu(\eta(x_n)=a_n)\to 0 \qquad (3.12)$$

as $g(\alpha_1,\ldots,\alpha_n)\to 0$. (3.12) holds for μ by (3.11), so that $\mu\in\{\nu_r: r\in H_0\}$ and the proof is complete.

Let $(\alpha(t),\ \beta(t))$ move on S_1^2 according to V_t. Now for any ν_r, $r(\alpha(t))$ is a martingale, since $U_t r=r$ for all $t\geqq 0$. Thus since r is bounded, we know that $r(\alpha(t))$ converges almost surely. We now define $\mathscr{A}$ to be the class of all ν_r such that $\lim_{t\to\infty} r(\alpha(t))=0$ or 1 a.s. on $\{g(\alpha(t),\beta(t))\to 1,\ g_B(\alpha(t),\beta(t))\to 0\}$. Note that on the latter set we have infinitely many collisions in which the colliding particles are of the same color, and only finitely many opposite-color collisions, or vice versa.

Lemma 3.2. $\mathscr{I}_e\subset\mathscr{A}$.

Proof. Take any $\mu\in\mathscr{I}_e$, and let $f\leftrightarrow\mu$. Define W on $S_1^2\cup\{\Delta\}$ by $W(\alpha,\beta)\equiv f(\alpha)$ and $W(\Delta)=0$. Then $V_t W=W$ and $U_t f=f$ on $S_1^2\backslash D_2$ for all $t\geqq 0$.

Let $(\alpha(t),\ \beta(t)$ move on S_1^2 according to V_t, and let β be a fixed element of S_1. From Eq. (3.4) we have $\lim_{t\to\infty} E^\alpha f(\alpha(t),\beta)=f(\alpha)f(\beta)$, and of course

$$\lim_{t\to\infty} E^\alpha W(\alpha(t),\beta)=\lim_{t\to\infty} U_t f(\alpha)=U_t f(\alpha)=f(\alpha) \qquad (3.13)$$

Thus (2.3) and (3.13) imply that

$$\begin{aligned}&|f(\alpha)f(\beta)-f(\alpha)|=\lim_{t\to\infty}|E^\alpha[f(\alpha(t),\beta)-W(\alpha(t),\beta)]|\\&\leqq\overline{\lim_{t\to\infty}}\,E^\alpha\,|f(\alpha(t),\beta)-W(\alpha(t),\beta)|\leqq\lim_{t\to\infty}E^\alpha[1-g(\alpha(t),\beta)+g_D(\alpha(t),\beta)\\&\quad+P((\alpha(t),\beta)\in D_2)]=1-\bar{g}(\alpha,\beta)+\bar{g}_D(\alpha,\beta),\end{aligned} \qquad (3.14)$$

by Corollaries 2.5 and 2.6.

As required by the definition of $\mathscr{A}$, we must show that $\lim_{t\to\infty} f(\alpha(t))=0$ or 1 a.s. on

$$E=\{g(\alpha(t),\beta(t))\to 1,\ g_B(\alpha(t),\beta(t))\to 0\}.$$

Now by Lemma 2.2, E can be written as the disjoint union (a.s.)

$$E_1\cup E_2=[E\cap\{g_S\to 1\}]\cup[E\cap\{g_D\to 1\}].$$

It will suffice to prove our assertion on E_1. To see why this is enough, rewrite E_2 as $E\cap\{g_S(\alpha(t)^{\#},\ \beta(t))\to 1\}$. If we knew that the assertion is true for E_1, we could then conclude that $\lim_{t\to\infty} f(\alpha(t)^{\#})=0$ or 1 and thus $\lim_{t\to\infty} f(\alpha(t))=1$ or 0.

Thus we may concentrate on E_1. On this set $g\to 1$ and thus the same is true for $\bar{g}$, by Lemma 2.2(c). Lemma 2.2 also implies that $g_D\to 0$ and thus $\bar{g}_D\to 0$ (by applying parts (a) and (c) to g_D and $\bar{g}_D$). Thus (3.14) implies that

$$\lim_{t\to\infty} f(\alpha(t))=\lim_{t\to\infty} f(\alpha(t))f(\beta(t))=\lim_{t\to\infty}[f(\alpha(t))]^2,$$

since $\alpha(t)=\beta(t)$ for arbitrarily large t on $\{g_S\to 1\}$. This gives us the desired result, since $\mu=\nu_f$ by Lemma 3.1.

Theorem 3.3. $\mathscr{I}_e=\mathscr{A}$

Proof. Take any $\nu_r\in\mathscr{A}$. By Lemma 3.2, we must show that $\nu_r\in\mathscr{I}_e$. Now Choquet's Theorem implies that there exists a probability measure γ on $\{s: \nu_s\in\mathscr{I}_e\}$ such that $\nu_r=\int \nu_s\,d\gamma(s)$, from which it follows that

$$f_r(\delta)=\int f_s(\delta)\,d\gamma(s) \quad \text{on } S_1, \tag{3.15}$$

and

$$f_r(\delta_1,\delta_2)=\int f_s(\delta_1,\delta_2)\,d\gamma(s) \quad \text{on } S_1^2. \tag{3.16}$$

Now let $(\alpha(t),\beta(t))$ move on S_1^2 according to V_t. By (3.16) we have $f_r(\alpha(t),\beta)=\int f_s(\alpha(t),\beta)\,\gamma(s)$ and thus

$$E^\alpha f_r(\alpha(t),\beta)=\int E^\alpha f_s(\alpha(t),\beta)\,d\gamma(s), \tag{3.17}$$

by applying Tonelli's Theorem to the product measure $\gamma\times P^\alpha$. Since each $\nu_s\in\mathscr{I}_e$, we have from (3.4) that $E^\alpha f_s(\alpha(t),\beta)\to f_s(\alpha)f_s(\beta)$ as $t\to\infty$, and thus

$$m(\alpha,\beta)=\lim_{t\to\infty} E^\alpha f_r(\alpha(t),\beta)$$

exists and is equal to $\int f_s(\alpha)f_s(\beta)\,d\gamma(s)$, by (3.17). Also

$$\begin{aligned} V_t m(\alpha,\beta) &= V_t\int f_s(\alpha)f_s(\beta)\,d\gamma(s)=\int V_t[f_s(\alpha)f_s(\beta)]\,d\gamma(s) \\ &=\int f_s(\alpha)f_s(\beta)\,d\gamma(s)=m(\alpha,\beta), \end{aligned} \tag{3.18}$$

where the third equality is an immediate consequence of Lemma 3.14 of Liggett [2] and the fact that each ν_s is an invariant measure for our process.

Now from Lemma 3.1(a) we have

$$|f_r(\delta_1,\delta_2)-f_r(\delta_1)f_r(\delta_2)|\leqq g(\delta_1,\delta_2) \quad \text{on } S_1^2\setminus D_2,$$

so that

$$\begin{aligned} |E^\alpha[f_r(\alpha(t),\beta)-f_r(\alpha(t))f_r(\beta)]| &\leqq E^\alpha|f_r(\alpha(t),\beta)-f_r(\alpha(t))f_r(\beta)| \\ &\leqq E^\alpha g(\alpha(t),\beta)+P^\alpha((\alpha(t),\beta)\in D_2). \end{aligned}$$

Thus by Corollary 2.5, the invariance of ν_r (Lemma 3.1) and the transience of $p(x,y)$, we have

$$|m(\alpha,\beta)-f_r(\alpha)f_r(\beta)|\leqq \bar{g}(\alpha,\beta)\leqq g(\alpha,\beta). \tag{3.19}$$

By (3.18), $m(\alpha(t),\beta(t))$ is a bounded martingale and thus converges almost surely. The same is true for $f_r(\alpha(t))f_r(\beta(t))$, again by Lemma 3.14 of Liggett [2]. We will show that

$$\lim_{t\to\infty} m(\alpha(t),\beta(t))=\lim_{t\to\infty} f_r(\alpha(t))f_r(\beta(t)) \quad \text{a.s.} \tag{3.20}$$

and thus $m(\alpha,\beta)=f_r(\alpha)f_r(\beta)$, by applying $E^{(\alpha,\beta)}$ to both sides of (3.20).

By (3.19) we see that (3.20) certainly holds on $\{g(\alpha(t),\beta(t))\to 0\}$, so we need to find out what happens on the set $\{g\to 1\}$. Now if μ is any invariant measure for our process and $f\leftrightarrow\mu$, then

$$\lim_{t\to\infty} f(\alpha(t))=\lim_{t\to\infty} f(\beta(t))=\lim_{t\to\infty} f(\beta(t)^{\#})=\tfrac{1}{2} \text{ a.s.}$$

on $\{g_B\to 1\}$, since on this set we have $\alpha(s)=\beta(s)$ and $\alpha(t)=\beta(t)^{\#}$ for arbitrarily large s and t, and since $f(\beta(t))+f(\beta(t)^{\#})\equiv 1$. Thus the integral representation of m just before (3.18) implies that (3.20) holds a.s. on $\{g_B\to 1\}$.

It remains to consider the set $E=\{g\to 1,\ g_B\to 0\}$. Since $\nu_r\in\mathscr{A}$ and since each ν_s under consideration is also in $\mathscr{A}$ (Lemma 3.2), we know that $f_r(\alpha(t))$, $f_r(\beta(t))$, $f_s(\alpha(t))$, and $f_s(\beta(t))$ each converge almost surely to 0 or 1 on E. However, we must be careful, since the negligible sets may depend on s. Let

$$A(s)=\{\omega\colon f_s(\alpha(t)(\omega)) \text{ and } f_s(\beta(t)(\omega)) \text{ converge as } t\to\infty\}.$$

Then since the indicated quantities do converge almost surely $(P^{\alpha,\beta})$ for each fixed s, we have $\int E^{\alpha,\beta}\, 1_{A(s)}\, d\gamma(s)=\int 1\, d\gamma(s)=1$, so $1=E^{\alpha,\beta}\int 1_{A(s)}\, d\gamma(s)$. Thus $\int 1_{A(s)}\, d\gamma(s)=1$ a.s.; that is, for any fixed ω outside of a set of $P^{\alpha,\beta}$-measure zero, we have $f_s(\alpha(t)(\omega))$ and $f_s(\beta(t)(\omega))$ converging as $t\to\infty$ for almost all s $[\gamma]$.

Now pick any ω in E which is outside the above-mentioned null set, and for which $\lim_{t\to\infty} f_r(\alpha(t)(\omega))$ and $\lim_{t\to\infty} f_r(\beta(t)(\omega))$ are equal to zero or one. Then from the above comments, Eq. (3.15), and the fact that $0\leqq f_s\leqq 1$, we see that

$$\lim_{t\to\infty} f_s(\alpha(t)(\omega))=\lim_{t\to\infty} f_r(\alpha(t)(\omega)) \quad \text{and}$$

$$\lim_{t\to\infty} f_s(\beta(t)(\omega))=\lim_{t\to\infty} f_r(\beta(t)(\omega))$$

for almost all s. Thus since $m(\alpha,\beta)=\int f_s(\alpha) f_s(\beta)\, d\gamma(s)$, we see that (3.20) holds for the above ω, and hence almost surely on E.

By remark immediately following (3.20), we now have

$$\int f_s(\alpha) f_s(\beta)\, d\gamma(s)\equiv f_r(\alpha) f_r(\beta)=[\int f_s(\alpha)\, d\gamma(s)][\int f_s(\beta)\, d\gamma(s)].$$

Thus $\int [f_s(\alpha)]^2\, d\gamma(s)-[\int f_s(\alpha)\, d\gamma(s)]^2=0$. Since a random variable with zero variance must be constant, $f_s(\alpha)$ must be constant in s a.e. $[\gamma]$, and of course this constant must be $f_r(\alpha)$. This is true for all $\alpha\in S_1$, so the unicity section of Lemma 3.1 implies $\nu_r=\nu_s$ a.e., that is, $\nu_r\in\mathscr{I}_e$.

Below we give a brief informal interpretation of Theorem 3.3 and its relation to the results of Holley and Liggett [1] (details may be found in [4]). The interpretation relies on the fact that the classes of harmonic functions $\mathscr{H}'$, $\mathscr{H}$ and $\mathscr{H}_0$ are closely related to the "components" of S mentioned in Sect. 1.

For the sake of concreteness, we will mainly limit our discussion to the transition function $p(x,y)$ in (1.1). Recall that in that example, S can be thought of as having two components, $Z^-=\{x<0\}$ and $Z^+=\{x\geqq 0\}$. Correspondingly, it can be proved that $\mathscr{H}'$ is generated by the functions h_{Z^-} and h_{Z^+} (together with 1 and 0), where $h_C(x)$ is defined to be $P^x(\exists N\ni X_n\in C\ \forall n\geqq N)$.

We noted in Section 1 that the components Z^- and Z^+ may be considered "odd" and "even", respectively, according to their asymptotic periodicities.

Intuitively the odd component Z^- should correspond to a single component $\Gamma_1 = \{(x,a): x<0\}$ of S_1, while the even component Z^+ should correspond to two components of S_1: $\Gamma_2 = \{(x,a): x \geqq 0 \text{ and } a = \frac{1}{2}[1+(-1)^x]\}$ and $\Gamma_3 = \Gamma_2^{\#}$. We first consider Z^- and Z^+ separately, in order to compare Theorem 3.3 with the results of [1] (in the latter, $\mathscr{H}'$ is assumed to consist only of constants, so that S has only one component).

To discuss the Z^- component, temporarily redefine $p(x,y)$ at $x=-1$ so that p is now a transition function on the state space Z^-, and redefine Q accordingly. Consider the $\mathscr{V}_-$ process for Z^-. Since the Markov chain $\alpha_n = (X_n, a_n)$ corresponding to Q has only one component, $\mathscr{H}$ will consist only of constants and thus $\mathscr{H}_0 = \{\frac{1}{2}\}$. The latter condition is precisely the one given for ergodicity in [5], so the $\mathscr{V}_-$ process for Z^- has only one invariant measure $v_{\frac{1}{2}}$. Note that this result is consistent with Theorem 3.3: On an odd component, $g_B(\alpha(t), \beta(t))$ can not approach zero unless $g(\alpha(t), \beta(t))$ does so. Thus for any odd component, $\mathscr{A}$ is equal to all of $\{v_r: r \in \mathscr{H}_0\}$, so that Theorem 3.3 does imply that $\mathscr{I}_e = \{v_{\frac{1}{2}}\}$.

Now consider the Z^+ component. Again temporarily redefine p and Q (keeping the technical odd periodicity of p). For this example, the set $\{g \to 1, g_B \to 0\}$ has probability one; in fact, we have $\bar{g}(\alpha,\beta) \equiv 1$ and $\bar{g}_B(\alpha,\beta) \equiv 0$. This corresponds to Case I in Sect. 5 of [1]. Now since Z^+ has two components (Γ_2 and Γ_3) with respect to Q, $\mathscr{H}$ is generated by H_{Γ_2} and H_{Γ_3} (plus 1 and 0), where H_Γ is the analog of h_C for Q. From this we see that $\mathscr{A} = \{v_{r_2}, v_{r_3}\}$, where $r_i = H_{\Gamma_i}$ ($i=2,3$). Thus by Theorem 3.3, $\mathscr{I}_e$ consists of just two elements. This is consistent with the corresponding result for the $\mathscr{V}_+$ process (Theorem 5.6(a) of [1]). Furtherfore, inspection of r_2 and r_3 shows that v_{r_2} and v_{r_3} have the forms mentioned in Section 1, again consistent with those for the $\mathscr{V}_+$ process.

Now consider the original version of $p(x,y)$ given in (1.1). If $\boldsymbol{\alpha}_n \to -\infty$ and $\boldsymbol{\beta}_n \to +\infty$, then $g(\boldsymbol{\alpha}_n, \boldsymbol{\beta}_n) \to 0$. Thus (3.1) implies that for any $v \in \mathscr{I}_e$, voters in Z^- act asymptotically independently of those in Z^+. Thus $\mathscr{I}_e$ consists of measures which are essentially "product measures", whose factors are those mentioned above when Z^- and Z^+ were discussed separately.

We remark that there is also a third case, not discussed above. This is the case in which S consists only of one even component, for which $\bar{g}(\alpha,\beta) \equiv 0$. In this situation, $\mathscr{A}$ is equal to all of $\{v_h = h \in H_0\}$, again corresponding to [1] (Corollary 5.15).

Thus we may summarize as follows: Take any $v \in \mathscr{I}_e$ for the $\mathscr{V}_-$ process.

(a) On odd components of S, v is asymptotically unique, corresponding to the ergodic case discussed in [5].

(b) On even components of S, v looks asymptotically like a corresponding member of $\mathscr{I}_e$ for the $\mathscr{V}_+$ process.

(c) v looks asymptotically like a product measure, with one factor for each component of S.

4. Asymptotic Behavior of the Process

Let μ denote an arbitrary initial distribution for the process, and let $f \leftrightarrow \mu$; we will find necessary and sufficient conditions under which μ_t will converge to a given member of $\mathscr{I}_e$.

The first result is motivated by the fact that any extremal invariant measure must satisfy Lemma 3.1(a).

Theorem 4.1. *Let* $\nu \in \mathscr{I}_e$, *and* $h \leftrightarrow \nu$. *Then if* μ *satisfies* (3.1), *and if* $\lim_{t\to\infty} V_t f(\alpha) = h(\alpha)$ *on* S_1, *then* $\mu_t \to \nu$ *as* $t \to \infty$.

Proof. As before, it will suffice to show that if $\lim_i \mu_{t_i} = \bar{\mu}$ exists for some $t_i \to \infty$, then $\bar{\mu} = \nu$. Then $\bar{\mu}(\eta(x)=a) = \lim_i U_{t_i} f(x,a) = \lim_i V_{t_i} f(x,a) = \nu(\eta(x)=a)$, so that $\bar{\mu}(\eta(x)=a) \equiv \nu(\eta(x)=a)$, and we also know that $\bar{\mu}$ is invariant. Thus the last paragraph in the proof of Lemma 3.1 implies that we just need to show that $\bar{\mu}$ satisfies (3.1).

Let $\varphi(\alpha_1, \ldots, \alpha_1) = f(\alpha_1) \ldots f(\alpha_i)$ $(i \leqq n)$, and extend φ to L_n by setting $\varphi(\Delta) = 0$. Then by assumption we have $|f - \varphi| \leqq g$ on $L_n \setminus D_n$, and thus on $S_1^n \setminus D_n$ we have $|U_{t_i} f - U_{t_i} \varphi| \leqq U_{t_i} g$. Thus if we let $\bar{f} \leftrightarrow \bar{\mu}$ and let $i \to \infty$, we see that $\lim_{i\to\infty} U_{t_i} \varphi$ exists and is equal to $\bar{f}$, by Corollary 2.3. On the other hand, $|U_{t_i} \varphi - V_{t_i} \varphi| \leqq g$ on $L_n \setminus D_n$ and

$$\lim_{t\to\infty} V_{t_i} \varphi(\alpha_1, \ldots, \alpha_n) = \lim_{i\to\infty} V_{t_i} f(\alpha_i) \ldots V_{t_i} f(\alpha_n)$$
$$= h(\alpha_1) \ldots h(\alpha_n) = \bar{f}(\alpha_1) \ldots \bar{f}(\alpha_n).$$

Hence

$$|\bar{f}(\alpha_1, \ldots, \alpha_n) - \bar{f}(\alpha_1) \ldots \bar{f}(\alpha_n)| \leqq g(\alpha_1, \ldots, \alpha_n), \tag{4.1}$$

as desired.

Corollary 4.2. *The conclusion of Theorem* 4.1 *is still true if we replace the hypothesis that* μ *satisfies* (3.1) *by the assumption that* $\lim_{t\to\infty} V_t f(\alpha, \alpha) = [h(\alpha)]^2$ *for all* $\alpha \in S_1$.

Proof. Let $Q(t, \delta_1, \delta_2)$ be the transition function corresponding to V_t on S_1. Then

$$\int \Big[\sum_{\beta=(y,b)} Q(t,\alpha,\beta)\, 1_{\{b\}}(\eta(y)) - h(\alpha) \Big]^2 d\mu(\eta)$$
$$= \sum_{\beta=(y,b)} Q(t,\alpha,\beta) \sum_{\gamma=(z,c)} Q(t,\alpha,\gamma) \int 1_{\{(b,c)\}}[(\eta(y), \eta(z))]\, d\mu(\eta)$$
$$- 2h(\alpha) \sum_{\beta=(y,b)} Q(t,\alpha,\beta) \int 1_{\{b\}}(\eta(y))\, d\mu(\eta) + [h(\alpha)]^2. \tag{4.2}$$

Now

$$\int 1_{\{b\}}(\eta(y))\, d\mu(\eta) = f(\beta), \quad \text{and} \quad \int 1_{\{(b,c)\}}[(\eta(y), (\eta(z))]\, d\mu(\eta) = f(\beta, \gamma),$$

so by our hypothesis and (4.2) we have

$$\lim_{t\to\infty} \int [W_t(\alpha)(\eta) - h(\alpha)]^2\, d\mu(\eta) = [h(\alpha)]^2 - 2[h(\alpha)]^2 + [h(\alpha)]^2 = 0,$$

where

$$W_t(\alpha)(\eta) = \sum_{\beta=(y,b)} Q(t,\alpha,\beta)\, 1_{\{b\}}(\eta(y)).$$

Thus by Čebyšev's Inequality we see that $W_t(\alpha)\to h(\alpha)$ in probability (with respect to μ) as $t\to\infty$. Hence

$$W_t(\alpha_1)\dots W_t(\alpha_n)\xrightarrow{pr} h(\alpha_1)\dots h(\alpha_n),$$

which implies

$$\int W_t(\alpha_1)\dots W_t(\alpha_n)\,d\mu(\eta)\to h(\alpha_1)\dots h(\alpha_n) \tag{4.3}$$

by bounded convergence. Note that (4.3) may be written as

$$\lim_{t\to\infty} V_t f(\alpha_1,\dots,\alpha_n)=h(\alpha_1)\dots h(\alpha_n). \tag{4.4}$$

Thus since $|U_{t_i}f-V_{t_i}f|\leqq g$ (using the notation in the proof of Theorem 4.1), and since $\bar{f}=h$ on S_1, (4.4) implies that (4.1) holds, which completes the proof.

Theorem 4.3. *Suppose* $\lim_{t\to\infty}\mu_t=\nu\in\mathscr{I}_e$, *and let* $h\leftrightarrow\nu$. *Then* $\lim_{t\to\infty}V_t f(\alpha)=h(\alpha)$ *and* $\lim_{t\to\infty}V_t f(\alpha,\beta)=h(\alpha)\,h(\beta)$ *for all* $(\alpha,\beta)\in S_1^2$.

Proof. Since $U_t f=V_t f$ on S_1, it is immediate from $\mu_t\to\nu$ that $V_t f\to h$ on S_1. Thus it will suffice to show that $V_{t_i}f\to\bar{f}$ on S_1^2 for some $t_i\to\infty$, then $\bar{f}=h$.

Now by Lemma 3.1 of [5], $\bar{f}(\alpha,\beta)$ is harmonic on S_1^2 for V_t, and since $V_t h=h$ on S_1, we also have $V_t[h(\alpha)\,h(\beta)]=h(\alpha)\,h(\beta)$ on S_1^2, from Lemma 3.14 of Liggett [2]. Thus if we consider the particles $\alpha(t)$ and $\beta(t)$ in the definition of $\mathscr{A}$ then $\bar{f}(\alpha(t),\ \beta(t))$ and $h(\alpha(t))\,h(\beta(t))$ are bounded martingales and hence have limits almost surely. Moreover, the martingale property implies that these limits have expected values $\bar{f}(\alpha,\beta)$ and $h(\alpha)\,h(\beta)$, respectively. Thus if we show that

$$\lim_{t\to\infty}\bar{f}(\alpha(t),\beta(t))=\lim_{t\to\infty} h(\alpha(t))\,h(\beta(t)), \tag{4.5}$$

then we will have $\bar{f}=h$ on S_1^2 and the proof will be complete.

Recall from Lemma 3.1 that on $S_1^2\backslash D_2$ we have $|h(\alpha,\beta)-h(\alpha)\,h(\beta)|\leqq g(\alpha,\beta)$. Thus since we also have $|U_t f(\alpha,\beta)-V_t f(\alpha,\beta)|\leqq g(\alpha,\beta)$ and since $U_t f\to h$ as $t\to\infty$, we know that

$$|\bar{f}(\alpha,\beta)-h(\alpha)\,h(\beta)|\leqq 2g(\alpha,\beta)\qquad\text{on } S_1^2\backslash D_2. \tag{4.6}$$

$$f(\alpha,\beta)=f(x,a),(y,b))=\mu(\eta(x)=a,\eta(y)=b)\geqq\mu(\eta(x)=a)$$
$$+\mu(\eta(x)=b)-1=f(\alpha)+f(\beta)-1$$

we get

$$1\geqq\lim_t\bar{f}(\alpha(t),\beta(t))=\lim_t\lim_i V_{t_i}f(\alpha(t),\beta(t))$$
$$\geqq\lim_t\lim_i[V_{t_i}f(\alpha(t))+V_{t_i}f(\beta(t))-1]=\lim_t[h(\alpha(t)+h(\beta(t))-1]$$
$$=1\ \text{ a.s.},$$

so that (4.5) holds almost surely on A_1. Similarly, on A_2 we have $\lim_t h(\alpha(t))\,h(\beta(t))=0$ a.s., since $\alpha(t)^{\#}=\beta(t)$ for arbitrarily large t and $h(\alpha(t)^{\#})=1-h(\alpha(t))$. But then (4.6) holds here too, since

$$0\leqq \bar{f}(\alpha(t),\beta(t))\leqq \min\{\bar{f}(\alpha(t)),\bar{f}(\beta(t))\}=\min\{h(\alpha(t)),h(\beta(t))\}\to 0 \quad \text{as } t\to\infty.$$

Now on A_3, the right-hand side of (4.5) is equal to 1/4 a.s., by the remarks in the proof of Theorem 3.3, so we must show the same is true for $\lim_t \bar{f}(\alpha(t),\beta(t))$. Fix $\omega\in A_3$ outside an appropriate set of probability zero which will be clear from the context. By appealing once more to Lemma 2.2(a), we can find a sequence of times $\{u_n\}$ such that $\alpha(u_n)=\alpha_n$ and $\beta(u_n)=\alpha_n^{\#}$, and since $g_S\geqq g_B\to 1$, we choose the u_n in such a way that $g_S(\alpha_n,\alpha_n^{\#})\geqq 1-1/n$.

Now let $[\delta_1(t),\ \delta_2(t),\delta_3(t)]$ move on S_1^3 according to V_t, starting at $[\alpha_n,\alpha_n^{\#},\alpha_n^{\#}]$. Then

$$\begin{aligned}\bar{f}(\alpha_n,\alpha_n^{\#})-\bar{f}(\alpha_n^{\#},\alpha_n^{\#})&=\lim_i[V_{t_i}f(\alpha_n,\alpha_n^{\#})-V_{t_i}f(\alpha_n^{\#},\alpha_n^{\#})]\\&=\lim_i[Ef(\delta_1(t_i),\delta_3(t_i))-Ef(\delta_2(t_i),\delta_3(t_i))].\end{aligned}$$

Now let τ be the hitting time of the diagonal of S_1^2 for $[\delta_1(t),\ \delta_2(t)]$. Then the strong Markov property implies that

$$E[f(\delta_1(t_i),\delta_3(t_i))-f(\delta_2(t_i),\delta_3(t_i)),\tau\leqq t_i]=0$$

so that from above we see that

$$|\bar{f}(\alpha_n,\alpha_n^{\#})-\bar{f}(\alpha_n^{\#},\alpha_n^{\#})|\leqq \lim_i P(\tau>t_i)=P(\tau=\infty)=1-g_S(\alpha_n,\alpha_n^{\#})\leqq\frac{1}{n}.$$

A similar argument gives us $|\bar{f}(\alpha_n,\alpha_n^{\#})-\bar{f}(\alpha_n,\alpha_n)|\leqq 1/n$, and of course $\bar{f}(\alpha_n,\alpha_n^{\#})=\bar{f}(\alpha_n^{\#},\alpha_n)$, since f is symmetric on S_1^2. Thus

$$\begin{aligned}4\bar{f}(\alpha_n,\alpha_n^{\#})-\frac{2}{n}&\leqq 2\bar{f}(\alpha_n,\alpha_n^{\#})+\bar{f}(\alpha_n^{\#},\alpha_n^{\#})+\bar{f}(\alpha_n,\alpha_n)\\&\leqq 4\bar{f}(\alpha_n,\alpha_n^{\#})+\frac{2}{n}.\end{aligned}\tag{4.8}$$

The middle term in this series is equal to one, since

$$\sum_{(a,b)\in\{0,1\}} f[(x,a),(y,b)]=1 \qquad \text{for any } (x,y)\in S_1^2.$$

Letting $n\to\infty$ in (4.8), we find that $\lim_{n\to\infty}\bar{f}(\alpha(u_n),\beta(u_n))=\frac{1}{4}$, which yields (4.5).

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