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Uniform Variation Results for Brownian Motion

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1. Introduction

Let $X(t,\omega)=X(t)$ denote the standard Brownian motion process and consider the uniform modulus of continuity $S(h)$ say, where

$$S(h)=\sup_{0\leq t\leq 1}\left[\sup_{t\leq u<v<t+h}|X(v)-X(u)|\right].$$

Sirao's result [8] concerning the large values of $S(h)$ sharpened Lévy's earlier result [6]. The Chung-Erdös-Sirao results [1] however settled completely the problem of large values of $S(h)$ in R^1:

Theorem 1.1 (Chung-Erdös-Sirao). *With probability one*

(i) $S(h)>h^{\frac{1}{2}}\left(2\log\frac{1}{h}+5\log_2\frac{1}{h}+2\log_3\frac{1}{h}+\ldots+2\log_{n-1}\frac{1}{h}+2\log_n\frac{1}{h}\right)^{\frac{1}{2}}$

for some arbitrarily small h.

(ii) *for each $\varepsilon>0$ there is a $\delta>0$ such that*

$$S(h)\leq h^{\frac{1}{2}}\left(2\log\frac{1}{h}+5\log_2\frac{1}{h}+2\log_3\frac{1}{h}+\ldots+2\log_{n-1}\frac{1}{h}+(2+\varepsilon)\log_n\frac{1}{h}\right)^{\frac{1}{2}}$$

for $0<h<\delta$.

Theorem 1.1 above (Theorem 2 in [1]) includes the earlier results of Sirao and Lévy.

In Sect. 3 of this paper we investigate the small values of $S(h)$ as $h\to 0$ for Brownian motion in R^k, $k\geq 1$. We obtain some terms of the asymptotic expansion for the small values of $S(h)$, complete solution being hindered by serious independence difficulties. We then procced in Sect. 4 to examine instead of $S(h)$, the variable

$$\lambda(h)=\inf_{0\leq t\leq 1}\left[\sup_{0\leq s\leq h}|X(t+s)-X(t)|\right].$$

We obtain some interesting results from which we deduce the asymptotic behaviour of $\lambda(h)$ as $h\to 0$. We conclude each of Sects. 3 and 4 with a conjecture.

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2. Preliminaries

Throughout this paper c, C_0, c', $C_1, \ldots$ will denote a finite positive constant whose value is not important and not necessarily the same at different occurrences unless otherwise stated; ω_s^+ will denote the shifted Brownian path such that $\omega_s^+: t \to X(t+s, \omega)$. Other notations are standard.

The next two definitions will recur frequently in this paper.

Definition 2.1. We shall say that a non-negative, continuous and monotone non-decreasing function $\psi(t)$ belongs to the lower class (or $\psi(t) \in L$) if with probability one there exists $h_0(\omega) > 0$ such that

$$S(h) > \phi(h) \quad \text{for all } h \leqq h_0(\omega), \quad \text{where } \phi(t) = t^{\frac{1}{2}} \psi(1/t).$$

Otherwise $\psi(t)$ belongs to the upper class (or $\psi(t) \in U$).

Definition 2.2. Define

$S_k(x, \alpha)$ as the sphere in R^k with centre x, radius α;
$T_k(\alpha, \omega)$ as the total time spent by the path ω in $S_k(o, \alpha)$ for $k \geqq 3$;
$P_k(\alpha, \omega)$ as the first passage time out of $S_k(o, \alpha)$ for $k \geqq 1$;
C_k as the unit cube in R^k.

Next we state some useful results.

Lemma 2.1. *Let U and V be standard Gaussian random variables in R^k, $k \geqq 1$ and suppose that $E(U_i V_j) = \rho \delta_{ij}$. Then there is a positive constant c' such that*

$$P(|U| > \alpha, |V| > \alpha) \leqq c' \exp\{-(1-\rho^2)\alpha^2/8\}. \quad P(|U| > \alpha).$$

See Lemma 1.6 in [7].

Lemma 2.2 (Kochen-Stone [5]). *Let $\{E_n\}$ be a sequence of events. Then*

(i) $\Sigma_n P(E_n) < \infty \Rightarrow P\{E_n \text{ occur i.o.}\} = 0$.

(ii) $\Sigma_n P(E_n) = +\infty$ *and* $\liminf_n \dfrac{\sum_{j=1}^{n} \sum_{k=1}^{n} P(E_j \cap E_k)}{\sum_{j=1}^{n} \sum_{k=1}^{n} P(E_j) P(E_k)} \leqq C_1$

$$\Rightarrow P\{E_n \text{ occur i.o.}\} \geqq C_1^{-1}.$$

All subsequent results in this section are proved in [2].

Lemma 2.3. *For almost all $\omega \in \Omega$, $k \geqq 3$*

$$\limsup_{\alpha \to 0^+} \frac{T_k(\alpha, \omega)}{\alpha^2 \log\log \alpha^{-1}} = \frac{2}{p_k^2},$$

where p_k is the first positive root of the Bessel function $J_\mu(z)$ with $\mu = \frac{k}{2} - 2$.

Lemma 2.4. *For almost all* $\omega\in\Omega$, $k\geqq 1$

$$\limsup_{\alpha\to 0+}\frac{P_k(\alpha,\omega)}{\alpha^2\log\log\alpha^{-1}}=\frac{2}{q_k^2},$$

where q_k *is the first positive zero of the Bessel function* $J_\nu(z)$ *with* $\nu=\frac{k}{2}-1$.

Lemma 2.5. $P\{T_k(\alpha,\omega)>y\}=d_k\exp\left(-\frac{p_k^2}{2\alpha^2}y\right)[1+0(\exp(-\mu_k y))]$ *as* $y\to\infty$ *for suitable positive constants* d_k, μ_k *where* p_k *is as in Lemma* 2.3.

Lemma 2.6. $T_{k+2}(\alpha,\omega)$ *and* $P_k(\alpha,\omega)$ *have precisely the same distribution for* $k=1,2,\ldots$.

Note that Lemma 2.6 is not true for regions more general than the sphere and then only if the Brownian path starts at the centre of the sphere.

3. The Small Values of $S(h)$

Theorem 3.1. *If* $\psi(t)=(2\log t+C\log_2 t)^{\frac{1}{2}}$ *then* $\psi(t)\in L$ *for* $C<1$, *and* $\psi(t)\in U$ *for* $C>1$.

Proof. Let $R(X;t,t+h)=\sup_{t\leqq u<v<t+h}|X(v)-X(u)|$ so that

$$S(h)=\sup_{0\leqq t\leqq 1}R(X;t,t+h).$$

Subdivide the unit interval (0, 1) into subintervals each of length $1/\lambda^2$. Then

$$P\{S(1)>\lambda\}\leqq P\left\{R\left(X;i/\lambda^2,1+\frac{i+1}{\lambda^2}\right)>\lambda\quad\text{for some } 0\leqq i\leqq[\lambda^2]\right\}.$$

Note that the distribution of $R(X;t,t+h)$ is the same as that of $R(h)$ where $R(h)=\sup_{0\leqq u<v<h}|X(v)-X(u)|$ and by the scaling property, that

$$P\{R(h)>\lambda h^{\frac{1}{2}}\}=P\{R(1)>\lambda\}.$$

The tail of the distribution of $R(1)$ was obtained by Feller [4] for Brownian motion in R^1 and leads immediately to

$$P\left\{R\left(X;i/\lambda^2,1+\frac{i+1}{\lambda^2}\right)>\lambda\right\}<\frac{c'}{\lambda}e^{-\lambda^2/2}\quad\text{for large }\lambda.$$

Since there are $[\lambda^2]+1$ such subintervals, we have proved the upper bound half of

Lemma 3.1. *There exist positive constants* c_1, c_2 *such that*

$$c_1\lambda e^{-\lambda^2/2}\leqq P\{S(1)>\lambda\}\leqq c_2\lambda e^{-\lambda^2/2}\quad\text{as }\lambda\to\infty.$$

For the lower bound, define the event $E_i=\{\omega: |X(1+i/\lambda^2)-X(i/\lambda^2)|>\lambda\}$. Then the event $\{S(1)>\lambda\}\supseteq\bigcup_{i=1}^{[\lambda^2]} E_i$ where $P(E_i)=P\{|X(1)|>\lambda\}$ does not depend on i. Moreover,

$$\operatorname{cov}(E_i, E_j)=\rho_{ij}=1-\frac{|i-j|}{\lambda^2}$$

and

$$1-\rho_{ij}^2=(1-\rho_{ij})(1+\rho_{ij})\geqq\frac{|i-j|}{\lambda^2}$$

so that Lemma 2.1 gives for any i, j,

$$P(E_i\cap E_j)\leqq c\exp(-|i-j|/8). \quad P(E_i).$$

Therefore for each positive integer n,

$$P\bigcup_{i=1}^{[\lambda^2]} E_i\geqq\sum_{i=1}^{[\lambda^2/n]} P(E_{in})-c\sum_{i=1}^{[\lambda^2/n]} P(E_{in})\sum_{\substack{j=1\\ j\neq i}}^{[\lambda^2/n]}\exp\{-|i-j|\,n/8\}$$

and on choosing n sufficiently large we have $\sum_{\substack{j=1\\ j\neq i}}^{\infty}\exp\{-|i-j|\,n/8\}<\frac{1}{2c}$.

For such values of n we have

$$P\bigcup_{i=1}^{[\lambda^2]} E_i\geqq\tfrac{1}{2}\sum_{i=1}^{[\lambda^2/n]} P(E_{in})=\tfrac{1}{2}[\lambda^{2/}n]\,P(E_i).$$

$P(E_i)$ can be estimated from the tail of the standard normal distribution to complete the proof of Lemma 3.1.

We now proceed to prove Theorem 3.1 in two parts.

Lower Class

Define the event $F_n=\{\omega: S(h_n)<h_{n-1}^{\frac{1}{2}}\psi(1/h_{n-1})\}$, where $h_n=e^{-n^\rho}$, and $0<\rho<\frac{1}{2}$. Then the events $\{S_i(h_n)<h_{n-1}^{\frac{1}{2}}\psi(1/h_{n-1})\}$, for the independent and identically distributed random variables

$$S_i(h_n)=\sup_{2ih_n\leqq t<(2i+1)h_n} R(X; t, t+h),\ i=0, 1, 2, \ldots, \left[\frac{1}{2h_n}\right];$$

are independent and have equal probabilities. Moreover since

$$F_n\subseteq\bigcap_{i=0}^{[1/2h_n]}\left\{S_i(h_n)<h_{n-1}^{\frac{1}{2}}\psi\left(\frac{1}{h_{n-1}}\right)\right\},$$

$$P(F_n)\leqq(P\{S_0(h_n)<h_{n-1}^{\frac{1}{2}}\psi(1/h_{n-1})\})^{[1/2h_n]}.$$

By Lemma 3.1 and the scaling property,

$$\begin{aligned}&P\{S_0(h_n)<h_{n-1}^{\frac{1}{2}}\psi(1/h_{n-1})\}\\ &\quad=P\{S(1)<h_n^{-\frac{1}{2}}h_{n-1}^{\frac{1}{2}}\psi(1/h_{n-1})\}\leqq 1-C_0\lambda_n e^{-\lambda_n^2/2},\end{aligned}$$

where $\lambda_n = \left(\frac{h_{n-1}}{h_n}\right)^{\frac{1}{2}} \psi(1/h_{n-1})$. Hence

$$P(F_n) \leqq (1 - C_0 \lambda_n e^{-\lambda_n^2/2})^{[1/2h_n]} = (1-u)^N$$

say, $= e^{N \log(1-u)} < e^{-Nu}$ because $\log(1-u) < -u$.

Now $N = \left[\frac{1}{2h_n}\right] \sim \frac{1}{2} e^{n^\rho}$ and $u = C_0 \lambda_n e^{-\lambda_n^2/2}$. But

$$\begin{aligned}\lambda_n^2 &= \left(\frac{h_{n-1}}{h_n}\right)\left(2\log\frac{1}{h_{n-1}} + C\log_2\frac{1}{h_{n-1}}\right)\\ &= \{1+O(n^{\rho-1})\}\{2n^\rho(1+O(n^{-1}) + C\rho(\log n + O(n^{-1}))\}\\ &= 2n^\rho + C\rho\log n + O(n^{2\rho-1}) = 2n^\rho + C\rho\log n + o(1) \qquad \text{since } \rho < \tfrac{1}{2}.\end{aligned}$$

Therefore $\lambda_n \sim c_1 n^{\frac{1}{2}\rho}$ and hence

$$Nu \sim \tfrac{1}{2} e^{n^\rho} \cdot C_0 c_1 n^{\frac{1}{2}\rho} \exp\{-n^\rho - \tfrac{1}{2} C\rho\log n + o(1)\} = c_2 n^\delta,$$

where $\delta = \frac{1}{2}(1-C)\rho > 0$ if $C < 1$.

Therefore $P\{F_n\} < e^{-c_3 n^\delta}$ for large n, so that $\sum_{n=1}^{\infty} P(F_n) < \infty$, and, by Lemma 2.2, F_n happens only finitely often. In other words, for almost all ω, there exists an integer $n^*(\omega)$ such that F_n does not happen for $n \geqq n^*(\omega)$. Notice that $h_n \leqq h \leqq h_{n-1}$ implies $S(h) \geqq S(h_n)$ and $h_{n-1}^{\frac{1}{2}} \psi(1/h_{n-1}) \geqq h^{\frac{1}{2}} \psi(1/h)$. Therefore for almost all ω, there exists $h_{n^*(\omega)} > 0$ such that $S(h) \geqq h^{\frac{1}{2}} \psi(1/h)$ for all $h \leqq h_{n^*(\omega)}$. This means that $\psi(t) \in L$ for $C < 1$.

Upper Class

Consider two sets of alternate intervals and let

$$E_1(h) = \bigcap_{i=0}^{\left[\frac{1}{2h}\right]} \{\sup_{2ih \leqq t < (2i+1)h} R(X; t, t+h) \leqq h^{\frac{1}{2}} \psi(1/h)\},$$

$$E_2(h) = \bigcap_{i=0}^{\left[\frac{1}{2h}\right]-1} \{\sup_{(2i+1)h \leqq t < 2(i+1)h} R(X; t, t+h) \leqq h^{\frac{1}{2}} \psi(1/h)\},$$

and

$$F(h) = \{S(h) \leqq h^{\frac{1}{2}} \psi(1/h)\} = \{\sup_{0 \leqq t \leqq 1} R(X; t, t+h) \leqq h^{\frac{1}{2}} \psi(1/h)\}.$$

Then $P\{E_1(h)\} = (P\{S(1) < \psi(1/h)\})^{\left[\frac{1}{2h}\right]+1}$ by the scaling property so that Lemma 3.1 gives $P\{E_1(h)\} \geqq (1 - c_2 \lambda e^{-\lambda^2/2})^{\left[\frac{1}{2h}\right]+1}$, where $\lambda = \psi(1/h)$. Hence $P\{E_1(h)\} \geqq e^{\frac{1}{2h}\log(1-u)}$ where $\log(1-u) \sim -u$ so that $P\{E_1(h)\} \geqq e^{-\tau_h}$ where

$$\tau_h \sim \frac{\left(c_3 \log\frac{1}{h} + \frac{1}{2} C \log_2 \frac{1}{h}\right)^{\frac{1}{2}}}{\left(\log\frac{1}{h}\right)^{C/2}} \to 0 \quad \text{as } h \downarrow 0 \quad \text{for } C > 1.$$

Therefore for $C>1$, $P\{E_1(h)\}\to 1$ as $h\to 0$ and similarly, $P\{E_2(h)\}\to 1$ as $h\to 0$. Moreover since $F(h)\supseteq E_1(h)\cap E_2(h)$, $P\{F(h)\}\to 1$ as $h\to 0$, for $C>1$. We have thus proved that $P\{(F(h))^c\}\to 0$ as $h\downarrow 0$ so that we can find a sequence $h_n\downarrow 0$ such that $P\{(F(h_n))^c\}<\frac{1}{n^3}$. For such a sequence, $\Sigma_n P\{(F(h_n))^c\}<\infty$ so that $P\{(F(h_n))^c$ occur for infinitely-many $n\}=0$ by Lemma 2.2. In other words there exists $n_0(\omega)$ such that $F(h_n)$ occurs for all $n\geqq n_0$, that is $\psi(t)\in U$ for $C>1$.

Remarks 1. Although all the arguments given above are for Brownian motion in R^1, the corresponding results for the small values of $S(h)$ are also true in R^k, $k\geqq 2$. For instance, applying (2.6) and then (2.3) of [9] we obtain

$$P\left\{R\left(X;\, i/\lambda^2,\, 1+\frac{i+1}{\lambda^2}\right)>\lambda\right\}<d_k'\,\lambda^{k-2}\,e^{-\lambda^2/2} \qquad \text{for } \lambda \text{ large,}$$

where X is Brownian motion in R^k, $k\geqq 1$ and d_k' is a positive constant which depends on k. This together with the tail of the standard multivariate normal distribution extends Lemma 3.1 to the general form:

Lemma 3.1.* *For Brownian motion in R^k, $k\geqq 1$ there exist positive constants d_k, d_k' depending on k such that*

$$d_k'\,\lambda^k e^{-\lambda^2/2}\leqq P\{S(1)>\lambda\}\leqq d_k\,\lambda^k e^{-\lambda^2/2} \qquad \text{as } \lambda\to\infty.$$

Using Lemma 3.1* we obtain the general form of Theorem 3.1 as

Theorem 3.1.* *If $\psi(t)=(2\log t+C_k\log_2 t)^{\frac{1}{2}}$ then $\psi(t)\in L$ for $C_k<k$, and $\psi(t)\in U$ for $C_k>k$ for Brownian motion in R^k, $k\geqq 1$.*

2. The independence difficulties are such that it seems hopeless to obtain a complete characterisation of the growth rate for the small values of $S(h)$. I state the

Conjecture. *If $\psi(t)=(2\log t+\log_2 t+C\log_3 t)^{\frac{1}{2}}$ then $\psi(t)\in L$ for $C<-2$, and $\psi(t)\in U$ for $C>-2$.*

Sharpening the arguments used in the proof of Theorem 3.1 yields $\psi(t)\in L$ for $C<-2$ and $\psi(t)\in U$ for $C>0$; I could only obtain a heuristic argument to show that $\psi(t)\in U$ for $-2<C\leqq 0$.

4. The Asymptotic Behaviour of $\lambda(h)$

Instead of looking at $S(h)=\sup_{0\leqq t\leqq 1}[\sup_{0\leqq s\leqq h}|X(t+s)-X(t)|]$ let us now consider $\lambda(h)=\inf_{0\leqq t\leqq 1}[\sup_{0\leqq s\leqq h}|X(t+s)-X(t)|]$.

Define $P_k(X(t),\alpha,\omega_t^+)$ as the first passage time out of $S_k(x,\alpha)$, where $x=X(t)$, for Brownian motion in R^k, $k\geqq 1$; and $T_k(X(t),\alpha,\omega_t^+)$ as the total time spent in $S_k(x,\alpha)$, where $x=X(t)$, by Brownian motion in R^k, $k\geqq 3$. Since

$$\sup_{0\leqq s\leqq h}|X(t+s)-X(t)|<\alpha \text{ IFF } P_k(X(t),\alpha,\omega_t^+)>h, \tag{4.1}$$

it follows that $\inf_{0 \leqq t \leqq 1} [\sup_{0 \leqq s \leqq h} |X(t+s)-X(t)|] = \alpha$ IFF

$$\sup_{\substack{x = X(t) \text{ on path} \\ 0 \leqq t \leqq 1}} P_k(x, \alpha, \omega_t^+) = h. \tag{4.2}$$

We first obtain a result from which we can deduce the asymptotic behaviour of $\lambda(h)$. Define

$$M_k(\alpha) = \sup \{P_k(x, \alpha, \omega_t^+) : x = X(t) \in C_k\},$$
$$U_k(\alpha) = \sup \{T_k(x, \alpha, \omega_t^+) : x = X(t) \in C_k\}.$$

Theorem 4.1. *With probability one,* $\lim_{\alpha \to 0^+} \frac{M_k(\alpha)}{\alpha^2 \log \alpha^{-1}} = \frac{2}{p_{k+2}^2}$ *for* $k \geqq 1$, *where* p_{k+2} *is the first positive root of the Bessel function* $J_\mu(z)$ *with* $\mu = \frac{k}{2} - 1$.

The same method of proof gives another interesting result:

Theorem 4.2. *With probability one,* $\limsup_{\alpha \to 0^+} \frac{U_k(\alpha)}{\alpha^2 \log \alpha^{-1}} \leqq \frac{2}{p_k^2}$ *for* $k \geqq 3$, *where* p_k *is the first positive root of the Bessel function* $J_\mu(z)$ *with* $\mu = \frac{k}{2} - 2$.

Proof of Theorem 4.2. Subdivide the unit cube C_k into smaller cubes each of side length $\frac{1}{n}$. Each of the n^k such cubes in this mesh has an inscribed sphere of diameter $\frac{1}{n}$ and a concentric circumscribing sphere of diameter $\frac{k^{\frac{1}{2}}}{n}$. If $N_n(\omega)$ is the total number of cubes of this mesh hit by Brownian motion in R^k, $k \geqq 3$ then

$$N_n(\omega) < c' n^2 \quad \text{for } n \geqq K^* = K^*(\omega) \tag{4.3}$$

as proved on p. 360 of [3].

For each cube of the mesh hit by Brownian motion, let τ_i be hitting times such that $x_i = X(\tau_i)$ and consider the sphere $S_k\left(x_i, \alpha + \frac{k^{\frac{1}{2}}}{n}\right)$. Then

$$U_k(\alpha) \leqq \sup_{x_i} \left\{ T_k\left(x_i, \alpha + \frac{k^{\frac{1}{2}}}{n}, \omega_{\tau_i}^+\right) : x_i = X(\tau_i) \right\} \tag{4.4}$$

where $\sup_{x_i}$ means the maximum over the finite set of hitting points of cubes of the mesh.

Consider the sequence $\alpha_j = e^{-j/\log j}$, $j \geqq 2$. Clearly $(\alpha_j/\alpha_{j+1}) - 1 \sim (\log j)^{-1}$ as $j \to \infty$. Define

$$F_j = \left\{ \omega : U_k(\alpha) > \frac{2}{p_k^2} (1+\varepsilon) \alpha_{j+1}^2 \log(1/\alpha_{j+1}) \right\},$$
$$G_j = \left\{ \omega : U_k(\alpha) > \frac{2+\varepsilon}{p_k^2} \alpha_j^2 \log(1/\alpha_j) \right\}.$$

For each $\varepsilon>0$, there then exists j_1 such that

$$F_j \subset G_j \quad \text{for } j \geqq j_1. \tag{4.5}$$

Moreover

$$P(G_j) \leqq P\left\{\sup_{x_i} T_k\left(x_i, \alpha_j+\frac{k^{\frac{1}{2}}}{n}, \omega_{\tau_i}^+\right) > \frac{(2+\varepsilon)}{p_k^2}\, \alpha_j^2 \log(1/\alpha_j)\right\}$$

by (4.4). Choose n large, equal to $(Q/\alpha_j)^{\frac{1}{2}}$ where Q is large enough to satisfy $\frac{k^{\frac{1}{2}}}{Q}<\varepsilon$. Since there exists $j_2(\omega)$ such that $(Q/\alpha_j)^{\frac{1}{2}}>K^*$ for $j \geqq j_2(\omega)$, where K^* is as in (4.3), we may apply (4.3) and Lemma 2.5 together with the fact that $n=(Q/\alpha_j)^{\frac{1}{2}}$ to obtain $P(G_j) \leqq c(Q/\alpha_j)\, \alpha_j^{1+\delta}$; $\delta>0$. Hence $\sum_{j=1}^{\infty} P(G_j) \leqq \sum_{j=1}^{\infty} c_1\, Q(\exp\{-j/\log j\})^\delta$ converges for $\delta>0$. This implies by 4.5 that $\sum_{j=1}^{\infty} P(F_j)$ is convergent and Lemma 2.2 further implies that for almost all ω, there exists an integer $N(\omega)$ such that F_j does not occur for $j \geqq N(\omega)$. In other words, there exists α_{j_0} with $j_0>N(\omega)$ such that with probability one,

$$0<\alpha<\alpha_{j_0} \Rightarrow U_k(\alpha)<\frac{2}{p_k^2}\,(1+\varepsilon)\, \alpha^2 \log \alpha^{-1}.$$

Since ε is arbitrary, we have proved Theorem 4.2.

If we subdivide the unit cube C_k in the same manner as before, then

$$M_k(\alpha) \leqq \sup_{x_i} P_k\left(x_i, \alpha+\frac{k^{\frac{1}{2}}}{n}, \omega_{\tau_i}^+\right) \qquad \text{for } k \geqq 1,$$

where $\sup_{x_i}$ means the maximum over the finite set of hitting points x_i of the cubes of the mesh. Considering again the sequence $\alpha_j=e^{-j/\log j} j \geqq 2$, using arguments similar to those used in the last theorem, and taking note of Lemma 2.6 gives

Lemma 4.1. *With probability one,* $\limsup_{\alpha \to 0+} \frac{M_k(\alpha)}{\alpha^2 \log \alpha^{-1}} \leqq \frac{2}{p_{k+2}^2}$ *for* $k \geqq 1$, *where* p_{k+2} *is as in Theorem* 4.1.

Let us now consider, instead of smaller cubes, small spheres each of radius ε_0/n centred on the points $(i_1/n, i_2/n, \ldots)$ of the lattice of side $\frac{1}{n}$. If $N_n^{(s)}(\omega)$ is the total number of small spheres hit by Brownian motion in R^k, $k \geqq 1$, then

$$N_n^{(s)}(\omega)>c_1 n \qquad \text{for } n \geqq K_*=K_*(\omega) \tag{4.6}$$

again by the result on p. 360 of [3] and the fact that (4.6) holds trivially for $k=1,2$. Define $b_j=\frac{1-2\varepsilon_0}{2j}$. Then $b_j/b_{j+1}=1+\frac{1}{j}$. Let x_r, x_s be hitting points of two different small spheres. Then $S_k(x_r, b_j)$ and $S_k(x_s, b_j)$ are disjoint if $n \leqq j$. Choose $n=j$ and let $j_*=K_*+1$. Then for all $j \geqq j_*$, $S_k(x_r, b_j)$ and $S_k(x_s, b_j)$ are disjoint for x_r, x_s hitting points of two distinct spheres. Therefore by the strong

Markov property,

$$A_{i_j}=\left\{\omega: P_k(x_i, b_j, \omega_{\tau_i}^{+})>\frac{2}{p_{k+2}^2}(1-\varepsilon)\, b_j^2 \log(1/b_j)\right\}, \qquad j\geqq j_* \tag{4.7}$$

are independent, where τ_i are hitting times such that $X(\tau_i)=x_i$ are hitting points of different small spheres. $P(A_{i_j})\sim c\, b_j^{(1-\varepsilon)}$ as $j\to\infty$, by Lemma 2.5. Define

$$E_j=\left\{\omega: M_k(b_j)<\frac{2(1-2\varepsilon)}{p_{k+2}^2}\, b_{j-1}^2 \log(1/b_{j-1})\right\},$$

$$D_j=\left\{\omega: M_k(b_j)<\frac{2(1-\varepsilon)}{p_{k+2}^2}\, b_j^2 \log(1/b_j)\right\}.$$

Then for each $\varepsilon>0$, there exists j_1 such that $E_j\subset D_j$ for $j\geqq j_1\geqq j_*$. Since $P(D_j)=P\{\bigcap_i A_{i_j}^c\}$ it follows from (4.6) and the fact that $n=j$ that

$$P(D_j)\leqq\left(1-c\left(\frac{1-2\varepsilon_0}{2j}\right)^{1-\varepsilon}\right)^{c_1 j}.$$

Clearly $P(D_j)\leqq c_1 \exp(-c_0 j^{\varepsilon})$ for $j\to\infty$. Hence $\sum_{j=1}^{\infty} P(D_j)\leqq \sum_{j=1}^{\infty} c_2 \exp(-c_0 j^{\varepsilon})$ which converges for $\varepsilon>0$. Therefore $\sum_{j=1}^{\infty} P(E_j)$ is convergent. We conclude, by Lemma 2.2, that there exists b_{j_0} with $j_0>N(\omega)$ such that

$$0<b<b_{j_0} \Rightarrow M_k(b)>\frac{2(1-2\varepsilon)}{p_{k+2}^2}\, b^2 \log(1/b).$$

Since ε is arbitrary we have proved that with probability one,

$$\liminf_{b\to 0+}\frac{M_k(b)}{b^2 \log b^{-1}}\geqq\frac{2}{p_{k+2}^2} \qquad \text{for } k\geqq 1,$$

where p_{k+2} is as in Theorem 4.1. This result with Lemma 4.1 completes the proof of Theorem 4.1.

We conclude from (4.1) and (4.2) that

$$\lim_{h\downarrow 0}\frac{\inf\limits_{0\leqq t\leqq 1}\ [\sup\limits_{0\leqq s\leqq h} |X(t+s)-X(t)|]}{(h/\log h^{-1})^{\frac{1}{2}}}=c \text{ IFF } \lim_{\alpha\downarrow 0}\frac{\sup\limits_{\substack{x=X(t)\\ 0\leqq t\leqq 1}} P_k(x,\alpha,\omega_t^{+})}{\alpha^2 \log \alpha^{-1}}=c'$$

where c, c' are positive constants with $c^2=1/c'$.

Since a.s. the second limit exists and equals $\frac{2}{p_{k+2}^2}$ we have proved

Theorem 4.3. $\lim_{h\downarrow 0}\frac{\lambda(h)}{(h/\log h^{-1})^{\frac{1}{2}}}=\frac{p_{k+2}}{2^{\frac{1}{2}}}$, a.s., *where* p_{k+2} *is the first positive root of the Bessel function* $J_\mu(z)$ *with* $\mu=\frac{k}{2}-1$. Finally we state a

Conjecture. *With probability one,* $\lim_{\alpha\to 0+} \frac{U_k(\alpha)}{\alpha^2 \log \alpha^{-1}} = \frac{2}{p_k^2}$ *for* $k \geqq 3$, *where* p_k *is the first positive root of the Bessel function* $J_\mu(z)$ *with* $\mu = \frac{k}{2} - 2$.

This conjecture cannot be proved by using the connection between P_{k-2} and T_k. Note however that, with probability one,

$$\frac{2}{p_{k+2}^2} \leqq \liminf_{\alpha\to 0+} \frac{U_k(\alpha)}{\alpha^2 \log \alpha^{-1}} \leqq \limsup_{\alpha\to 0+} \frac{U_k(\alpha)}{\alpha^2 \log \alpha^{-1}} \leqq \frac{2}{p_k^2}$$

for $k \geqq 3$ by Theorem 4.2 and the obvious inequality $T_k(\alpha, \omega) \geqq P_k(\alpha, \omega)$.

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