Z. Wahrscheinlichkeitstheorie verw. Gebiete
51, 27–38 (1980)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

Local Theorems for Euclidean Motions. I

V.M. Maximov

Institute of Chemical Physics, Acad. of Sciences of the USSR,
Vorobyevskoe Chaussee 2б, Moscow 117334, USSR

1. Introduction

Let $\mathbb{G}$ be some topological group, ν a probability measure on $\mathbb{G}$, ν^{*n} the n-fold convolution of ν. The value $\nu^{*n}(\mathscr{D})$ is interesting for some applications where $\mathscr{D}$ is any bounded domain in $\mathbb{G}$. Every estimate of the value $\nu^{*n}(\mathscr{D})$, $n \to \infty$ will be called a local theorem (l.t.) on the group $\mathbb{G}$. Accordingly the estimate of the value $\nu^{*n}(\mathscr{D}_1)/\nu^{*n}(\mathscr{D}_2)$ for any bounded domains $\mathscr{D}_1$ and $\mathscr{D}_2$ is called a relation local theorem (r.l.t.). Local theorems have a short history. They have been proved in the case of additive groups $\mathbb{R}^1$ and $\mathbb{R}^d$ by Shepp [11] and Stone [12] respectively in 1965. Kazdan [5] was the first who began to study r.l.t. for the group of Euclidean plane motions. This result refers to a particular case of a measure ν whose support contains four motions $\mathscr{A}$, $\mathscr{A}^{-1}$, $\mathscr{B}$, $\mathscr{B}^{-1}$. This result was obtained by functional methods and didn't possess a probability character. A r.l.t. has also been proved for local compact unimodal groups by Le Pege [6] and for a wide class of commutative groups by Stone [13].

The first l.t. for the group of Euclidean plane motions was reported in the 3rd Soviet Japanese symposium on probability in 1975 [10]. Independently P. Baldi, Ph. Bougerol, P. Crépel obtained analogous results under wider conditions for Euclidean motions of the d-dimensional space $\mathbb{R}^d$ using the Plancherel formula for the Euclidean group of motions [1, 2].

By $\mathbb{M}(d)$ we denote the group of Euclidean motions $\mathbb{R}^d$.

Some details from [10] were given by the author in [8] and [9], especially with reference to the group $\mathbb{M}(2)$. In particular the concept of "lattice motions" of $\mathbb{M}(2)$ in [9] together with l.t. [1] or with the theorem of the present work give l.t. for any case of measures ν on $\mathbb{M}(2)$ with finite support. It is clear that Kazdan's theorem follows from the general cases of l.t. with a finite support of the measure ν. The purpose of the present work is to obtain the same [1] by the direct probability method. This method uses a variant of Stone's l.t. [12] applied to independent and nonidentically distributed values and it is sensitive to complementary conditions. In particular, the method can give some other variants of l.t. that do not follow from [1]. They will be obtained in the second part of this work.

0044-3719/80/0051/0027/$02.40

2. The Formulation of the Main Result

The group of Euclidean motions $\mathbb{M}(d)$ is the cross-group of the additive group $\mathbb{R}^d$ with the rotations group $\mathbb{SO}(d)$. Every element of this group is uniquely determined by the pair (x, α), where x is the shift and α is the rotation. The product of the motions is defined by $(x, \alpha)(y, \beta)=(x+y\alpha, \alpha\beta)$, where $\alpha\beta$ is the product of the rotations and $y\alpha$ is the image $y\in\mathbb{R}^d$ under $\alpha\in\mathbb{SO}(d)$. Let v be a measure defined on $\mathbb{M}(d)$. Taking its natural desintegration we obtain for every $\alpha\in\mathbb{SO}(d)$ the measure $v(\alpha)$ on $\mathbb{R}^d$ and the measure $\mathbf{v}$ on $\mathbb{SO}(d)$ which is the conditional distribution v. The random elements $\xi_1, \xi_2, \dots, \xi_n, \dots$ in $\mathbb{M}(d)$ correspond to the random independent pairs $(\varkappa_1, \alpha_1), \dots, (\varkappa_n, \alpha_n), \dots$. For the pairs $(\varkappa_k, \alpha_k)$, $k=1, 2, \dots$ the random values $\varkappa_k$ and α_k may be dependent. Then the product $\xi_1 \xi_2 \dots \xi_n$ corresponds to $(\varkappa_1+\varkappa_1 \alpha_1+\dots+\varkappa_n \alpha_1 \dots \alpha_{n-1}, \alpha_1 \dots \alpha_n)$. Consequently, the product $\xi_1 \xi_2 \dots \xi_n$ may be reduced to the study of the joint distribution of $\mathscr{S}_n=\varkappa_1+\varkappa_2 \alpha_1+\dots+\varkappa_n \alpha_1 \dots \alpha_{n-1}\in\mathbb{R}^d$ and $\alpha_1 \alpha_2 \dots \alpha_n\in\mathbb{SO}(d)$. Under general conditions the product $\alpha_1 \alpha_2 \dots \alpha_n$ converges to the uniform distribution on $\mathbb{SO}(d)$ and becomes asymptotically independent with $\mathscr{S}_n$. Consequently the l.t. in $\mathbb{M}(d)$ is equivalent to the estimate of the probability $\mathbf{P}\{\mathscr{S}_n\in\mathscr{D}\}$. Therefore we will estimate this probability. Let ξ be arbitrary $\mathbb{M}(d)$ - valued random variable having the distribution v determined by the pair $(\varkappa, \alpha)$. By $\varkappa(\alpha)$ we denote the conditional variable with fixed α, $\mathbf{v}$ - the distribution α, $\mathscr{F}_v$ - the distribution $\varkappa$, $v(\alpha)$ - the distribution $\varkappa(\alpha)$ and $v'(\alpha)$ - the distribution obtained by symmetrization of $v(\alpha)$.

Theorem (l.t.) *Let $\xi_1, \xi_2, \dots, \xi_n, \dots$ be independent $\mathbb{M}(d)$ - valued random variables with identical distribution v and satisfying the conditions:*

1. *the n-fold convolution v^{*n} converges weakly as $n\to\infty$ to the uniform distribution on $\mathbb{SO}(d)$.*
2. *The distribution $\mathscr{F}_v$ on $\mathbb{R}^d$ has finite second moments.*
3. *There exists some number $n_0>0$ and a set K, $K\subseteq\mathbb{SO}(d)$ of measure $\mathbf{v}(K)>0$ so that for any $\alpha\in K$ the support of distribution $v^{*n_0}(\alpha)$ does not contain any hyperplanes. In what follows for the sake simplicity we shall suppose $n_0=1$.*

Then for any bounded Jordan measurable domain $\mathscr{D}\in\mathbb{R}^d$,

$$\mathbf{P}\{\mathscr{S}_n\in\mathscr{D}\}=c\,\mathrm{mes}(\mathscr{D})\,n^{-d/2}+O(n^{-d/2}) \tag{1}$$

where $\mathrm{mes}(\mathscr{D})$ *is the Lebesgue measure of the domain $\mathscr{D}$, c is a constant which does not depend on $\mathscr{D}$ and will be evaluated at the end of this paper.*

It is easy to see that the formulations of the proposition [1] and our theorem coincide.

The main tool of the direct probability method for studying local properties of the sum $\mathscr{S}_n$ is the reduction to the conditional sums $\mathscr{S}_n(\alpha_1, \dots, \alpha_n)$ of independent but not identically distributed random variables where

$$\mathscr{S}_n(\alpha_1, \dots, \alpha_n)=\varkappa(\alpha_1)+\varkappa(\alpha_1)\,\alpha_1+\dots+\varkappa(\alpha_n)\,\alpha_1 \dots \alpha_{n-1} \tag{2}$$

for fixed collections $(\alpha_1, \alpha_2, \dots, \alpha_n)$.

For fixed $(\alpha_1, \alpha_2, \dots, \alpha_n)$ the conditional variables are independent but not identically distributed. If for the "majority" of these sums the l.t. in $\mathbb{R}^d$ may be applied, i.e. the value $\mathbf{P}\{\mathcal{S}_n(\alpha_1,\dots,\alpha_n)\in\mathcal{D}\}$ may be effectively estimated, then we have by integration

$$\mathbf{P}\{\mathcal{S}_n\in\mathcal{D}\}=\int\mathbf{P}\{\mathcal{S}_n(\alpha_1,\dots,\alpha_n)\in\mathcal{D}\}\,\mathbf{v}(d\alpha_1)\dots\mathbf{v}(d\alpha_n). \tag{3}$$

In spite of the essential difference of the methods that have been applied here and in [1] they use effectively the same condition 3. For example whenever this condition or analogous ones are absent the l.t. has not yet been proved, if $d\geqq 3$ and the measure ν has a finite support without additional conditions.

Finally we remark that by virtue of condition 2 we get the following condition 2′: the variables $z(\alpha)$, $\alpha\in\mathbb{SO}(d)$ have second moments which are absolutely integrable. The expectation $\mathbf{M}\,z(\alpha)=m(\alpha)$ has second moments also. We denote the characteristic function of $z(\alpha)$ by $f_\alpha(t)$.

3. A Special Formulation of the l.t. in $\mathbb{R}^d$

To apply the l.t. to the sum (2) we need the following variant of Stone's l.t.

Lemma 1. *Let $\xi_1(\tau),\xi_2(\tau),\dots,\xi_n(\tau),\dots$ be a sequence of $\mathbb{R}^d$-valued nonidentically distributed variables depending on the parameter τ on a measurable compact $\mathbb{T}$ with probability measure μ. For any $\tau\in\mathbb{T}$ the variables $\xi_k(\tau)$ are independent and possess second moments with respect to the measure μ. By $f_n(t,\tau_n)$ we denote the characteristic function of the sum $\xi_1(\tau_n)+\dots+\xi_n(\tau_n)$. If for all n, $n>0$ there exists a set $\mathcal{K}_n\subseteq\mathbb{T}$, $\mu(\mathcal{K}_n)\to 1$ and we have uniformly for any sequence $\{\tau_n\}$, $\tau_n\in\mathcal{K}_n$:*

1) *the distribution of the variable $\frac{1}{\sqrt{n}}\{\xi_1(\tau_n)+\dots+\xi_n(\tau_n)-m(n,\tau)\}$ converges weakly as $n\to\infty$ to the normal distribution with mean zero and covariance matrix Ω where*

$$m(n,\tau_n)=M\{\xi_1(\tau_n)+\dots+\xi_n(\tau_n)\},$$

2) *the integral $\sqrt{n}\int_b^B|f_n(t,\tau_n)|\,dt$ tends to zero as $n\to\infty$ for any positive numbers b, B,*

3) *the integral $\sqrt{n}\int_{\frac{A}{\sqrt{n}}\leqq\|t\|\leqq b}|f_n(t,\tau_n)|\,dt$ tends to zero as $n\to\infty$, $A\to\infty$, $b\to 0$,*

then for any bounded Jordan measurable domain $\mathcal{D}$ and for all sequences $\{\tau_n\}$, $\tau_n\in\mathcal{K}_n$ we have

$$\mathbf{P}\{\xi_1(\tau_n)+\dots+\xi_n(\tau_n)\in\mathcal{D}\}=(\sqrt{n})^{-d}\int_{\mathcal{D}}\mathbf{N}_\Omega\left(\frac{y}{\sqrt{n}}-\frac{m(n,\tau_n)}{\sqrt{n}}\right)dy$$
$$+o_n(A,b,\tau_n)(\sqrt{n})^{-d}, \tag{4}$$

where $\mathbf{N}_\Omega(x)$ is the density of the normal distribution with mean zero and covariance matrix Ω, and the value $o_n(A,b,\tau_n)$ tends to zero uniformly as $n\to\infty$, $A\to\infty$, $b\to 0$.

The scheme of Stone's proof [12] remains valid when applied to the proof of Lemma 1. Since we intend to apply Lemma 1 to the sum $\mathscr{S}_n(\alpha_1, \ldots, \alpha_n)$ it is necessary to represent the terms of the sum in accordance with Lemma 1. In order to do this we take the compact space $\mathbb{T} = \mathbb{SO}(d)^\infty$ with measure μ, where $\mathbb{SO}(d)^\infty$ is equal to a countable product of the group $\mathbb{SO}(d)$ and the measure μ is the direct product of the measure $\mathbf{v}$ on $\mathbb{SO}(d)$. For any $\tau = (\alpha_1, \alpha_2, \ldots) \in \mathbb{T}$ we construct the sequence of $\mathbb{R}^d$-valued variables

$$\xi_1(\tau) = \varkappa(\alpha_1), \quad \xi_2(\tau) = \varkappa(\alpha_2)\alpha_1, \ldots, \xi_n(\tau) = \varkappa(\alpha_n)\, \alpha_1 \ldots \alpha_{n-1} \tag{5}$$

which are independent and nonidentically distributed.

Then for any $\tau \in \mathbb{T}$ we can define the random variables

$$\mathscr{S}_n(\tau) = \mathscr{S}_n(\alpha_1, \ldots, \alpha_n) = \xi_1(\tau) + \ldots + \xi_n(\tau).$$

We shall apply the equality (4) to the sum $\mathscr{S}_n(\alpha_1, \ldots, \alpha_n)$. For this it is enough to verify that the conditions 1), 2), 3) are valid for $\mathscr{S}_n(\alpha_1, \ldots, \alpha_n)$.

4. The Normal Convergence of the Sequence $\frac{1}{\sqrt{n}}\{\mathscr{S}_n(\tau_n) - \mathbf{M}(\mathscr{S}_n(\tau_n))\}$

Condition 1 will hold for $\mathscr{S}_n(\alpha_1, \ldots, \alpha_n)$ if the distribution of the sequence $\frac{1}{\sqrt{n}}$ $\{\mathscr{S}_n(\tau_n) - \mathbf{M}(\mathscr{S}_n(\tau_n))\}$, $\tau_n \in \mathscr{K}_n$ converges to the normal distribution with density $\mathbf{N}_{\Omega_1}(x)$ where $\mathscr{K}_n$ are some subsets of $\mathbb{SO}(d)^\infty$, $\mu(\mathscr{K}_n) \to 1$. We need the following auxilliary proposition:

Lemma 2. *Let $\alpha_1, \alpha_2, \ldots$ be independent $\mathbb{SO}(d)$-valued variables with the same distribution $\mathbf{v}$. Let $f(\alpha)$ be the linear sum of elements taken from the matrix of nontrivial irreducible representation of $\mathbb{SO}(d)$, $a(\alpha)$ a real function on $\mathbb{SO}(d)$ with $\int |a(\alpha)|\, d\mathbf{v} \leqq \infty$. Then*

$$\frac{1}{n} \sum_1^n a(\alpha_k)\, f(\alpha_1 \ldots \alpha_{k-1}) \tag{6}$$

tends to zero weakly.

Proof. At first we suppose that $a(\alpha)$ has second moments and we shall prove that

$$\frac{1}{n^2} \mathbf{M} \left\{ \sum_1^n a(\alpha_k)\, f(\alpha_1 \ldots \alpha_{k-1}) \right\}^2 \to 0. \tag{7}$$

The sum (7) is equal to

$$\frac{1}{n^2} \sum_1^n \mathbf{M}\, \alpha^2(\alpha_k)\, \mathbf{M}\, f^2(\alpha_1 \ldots \alpha_{k-1})$$
$$+ \frac{1}{n^2} \sum_{1 \leqq k < m \leqq n} \mathbf{M}\, \alpha(\alpha_k)\, a(\alpha_m)\, f(\alpha_1 \ldots \alpha_{k-1})\, f(\alpha_1 \ldots \alpha_{k-1}\, \alpha_k \ldots \alpha_{m-1}).$$

Since the function $f(\alpha)$ is continuous, $|f(\alpha)|<c_1$ on $\mathbb{SO}(d)$. Let us set $\mathbf{M}\,a^2(\alpha)=c_2$, $\mathbf{M}\,|a(\alpha)|=c_3$. Under condition 1) for the measure $\mathbf{v}$ it was proved in [7] that $|\mathbf{M}\,f(\alpha\alpha_1 \ldots \alpha_k)|<c_4\lambda^k$ where c_4, λ are constants not depending on α, and in addition $0<\lambda<1$. Therefore the first sum remains less than $\frac{1}{n}c_2 c_1^2$ and the second one is less than $\frac{1}{n^2}\sum_{1\leqq k<m\leqq n} c_3^2 c_1 c_4 \lambda^{m-k}$. Consequently (7) is true, and $\frac{1}{n}\sum a(\alpha_k) f(\alpha_1 \ldots \alpha_{k-1})$ tends to zero weakly if $\mathbf{M}\,a^2(\alpha)<\infty$. If $\mathbf{M}\,a^2(\alpha)=\infty$ we can write $a(\alpha)=a_N(\alpha)+\varepsilon_N(\alpha)$ where $|a_N(\alpha)|<N$ and $\mathbf{M}\,|\xi_N|<\delta$. Then the sum (6) is equal to

$$\frac{1}{n}\sum_1^n a_N(\alpha_k) f(\alpha_1 \ldots \alpha_{k-1})+\frac{1}{n}\sum_1^n \varepsilon_N(\alpha_k) f(\alpha_1 \ldots \alpha_{k-1}).$$

It is proved above that the first sum tends to zero and the second sum is less than $c_1\frac{1}{n}\sum_1^n|\varepsilon_N(\alpha_k)|$. But $c_1\frac{1}{n}\sum_1^n|\varepsilon_N(\alpha_k)|$ tends to $c_1\mathbf{M}\,|\varepsilon_N|$ which is less than $c_1\delta$. Since δ may be chosen arbitrarily we get the required result.

Lemma 3. *Under the conditions of Lemma 2 the value* $\frac{1}{n}\sum_1^n a(\alpha_k) f^2(\alpha_1 \ldots \alpha_{k-1})$ *converges weakly to* $\mathbf{M}\,a(\alpha)\int f^2(\alpha)\,d\alpha$ *where the integral* $\int f^2(\alpha)\,d\alpha$ *is taken with respect to the invariant measure on* $\mathbb{SO}(d)$.

Proof. The function $f^2(\alpha)$ is an element of the tensor square of the corresponding representation. Therefore

$$f^2(\alpha)=\gamma_0+\sum_1^d \gamma_k f_k(\alpha) \tag{8}$$

where γ_1 is constant, $f_k(\alpha)$ are elements of the some irreducible nontrivial representations $\mathbb{SO}(d)$ and $\int f^2(\alpha)\,d\alpha=\gamma_0$ because $\int f_k(\alpha)\,d\alpha=0$. Consequently applying Lemma 2 to the sum (8) we obtain Lemma 3.

By $(\varkappa_1(\alpha), \ldots, \varkappa_n(\alpha))$ we denote the coordinates of the random vector $\varkappa(\alpha)$ with mean zero, and by $g(w)=\|g_{ij}(w)\|$ we denote the orthogonal matrix of the rotation $w\in\mathbb{SO}(d)$. The map $w\to g(w)$ is continuous and $g(w, w_2)=g(w_1)\,g(w_2)$, $g(w)$ is an irreducible representation of $\mathbb{SO}(d)$. Let $\mathbf{M}(\alpha)$ be the covariance matrix of $\varkappa(\alpha)$ and $\varkappa(\alpha)\,w$ be the image of $\varkappa(\alpha)\in\mathbb{R}^d$ under $w\in\mathbb{SO}(d)$. Then the covariance of $\varkappa(\alpha)\,w$ is equal to $g(w)^{\mathbf{T}}\,M(\alpha)\,g(w)$, where $g(w)^{\mathbf{T}}$ is the transposed matrix. However it follows that the covariance matrix for the random variable $\frac{1}{\sqrt{n}}\{\mathscr{S}_n-\mathbf{M}\mathscr{S}_n(\tau)\}$, where $\tau=(\alpha_1, \alpha_2, \ldots)$ is equal to

$$\frac{1}{n}\sum_{k=1}^n g^{\mathbf{T}}(\alpha_1 \ldots \alpha_{k-1})\,\mathbf{M}(\alpha_k)\,g(\alpha_1 \ldots \alpha_{k-1})=\Omega(n;\,\alpha_1, \ldots, \alpha_n). \tag{9}$$

By virtue of condition 2′ the expectation $\mathbf{M}\{\mathbf{M}(\alpha)\}=\|\sigma_{ij}\|$ exists. Set $\sigma=\frac{1}{d}(\sigma_{11}+\ldots+\sigma_{dd})$. It follows from condition 1, that $\sigma>0$.

Lemma 4. *The value $\Omega(n; \alpha_1, \ldots, \alpha_n)$ converges weakly to $\sigma\mathbb{I}$ where I is the identical matrix.*

Proof. The coordinates of the vector $\varkappa(\alpha)\, w$ are equal to

$$\varkappa_j(\alpha, w) = \varkappa_1(\alpha)\, g_{j1}(w) + \ldots + \varkappa_d(\alpha)\, g_{jd}(w), \qquad j = 1, 2, \ldots, d.$$

Then the (i, j)-th element of the matrix Ω is equal to

$$\frac{1}{n} \sum_1^n \left\{ \sum_{m,r} M \varkappa_m(\alpha_k)\, \varkappa_r(\alpha_k)\, g_{mi}(\alpha_1 \ldots \alpha_{k-1})\, g_{rj}(\alpha_1 \ldots \alpha_{k-1}) \right\}. \tag{10}$$

The functions $g_{ij}(w)$ are the elements of an irreducible non-trivial unitary representation for which the following ortogonal relations are well-known:

$$\int g_{ij}^2(w)\, dw = \frac{1}{d}; \qquad \int g_{mi}(w)\, g_{rj}(w) = 0 \quad \text{if } (m, i) \neq (r, j).$$

The functions $g_{ij}(w)\, g_{rj}(w)$ are the elements of the tensor square representation of $g(w)$. Therefore to every sum of the kind

$$\frac{1}{n} \sum_{k=1}^n M\, \varkappa_m(\alpha_k)\, \varkappa_r(\alpha_k)\, g_{mi}(\alpha_1 \ldots \alpha_{k-1})\, g_{rj}(\alpha_1 \ldots \alpha_{k-1})$$

Lemmas 2 or 3 may be applied. This proves Lemma 4.

Now we will indicate the set $\mathscr{K}_n \subset \mathbb{SO}(d)^\infty$, $\mu(\mathscr{K}_n) \to 1$, for which the distribution of the variable $\frac{1}{\sqrt{n}} \{\mathscr{S}_n(\tau_n) - \mathbf{M}\mathscr{S}_n(\tau_n)\}$, $\tau_n \in \mathscr{K}_n$ converges weakly and uniformly to the normal distribution $\mathbf{N}_{\Omega_1}(x)$, $\Omega_1 = \sigma\mathbb{I}$.

At first we set $\varphi_\lambda(x) = \sup_{w, x_0} \int_{\|x - x_0\| > 2} \|x w\|^2\, dv(\alpha)$ where $\|x\| = \sqrt{x_1^2 + \ldots + x_d^2}$. Evidently the function is measurable and $\varphi_\lambda(\alpha) \leqq \int \|x\|^2\, dv(\alpha)$.

Since the right hand side of the inequality is integrable in $\mathbf{v}$, we have for all $\lambda > 0$, $\int \varphi_\lambda(\alpha)\, d\mathbf{v} = \varphi(\lambda) < c$ where the constant c does not depend on λ. For any α the function $\varphi_\lambda(\alpha)$ decreases to zero as $\lambda \to \infty$. Consequently $\varphi(\lambda) \to 0$ as $\lambda \to \infty$, and there exists a sequence λ_N, $\varphi(\lambda_N) = \varepsilon_N \downarrow 0$. Since the sum $\frac{1}{n} \sum_1^n \varphi_\lambda(\alpha) \to \varphi(\lambda)$ in μ measure, we may define the sets $\mathscr{K}_N$, $\mathscr{K}_N \subset \mathscr{K}_{N+1} \subset \mathbb{SO}(d)^\infty$, $\mu(\mathscr{K}_N) \uparrow 1$ such that for any $(\alpha_1, \alpha_2, \ldots) \in \mathscr{K}_N$ the following inequality holds: $\limsup \frac{1}{n} \sum_{k=1}^n \varphi_{\lambda_n}(\alpha_k) \leqq \varepsilon_{N-1}$. In virtue of the decreasing $\varphi_\lambda(\alpha)$ as $\lambda \to \infty$ for any α and $\tau = (\alpha_1, \alpha_2, \ldots) \in \mathscr{K}_N$ we have $\limsup \frac{1}{n} \sum_{k=1}^n \varphi_{\ln n}(\alpha_k) < \varepsilon_{N+m}$ for any integer N, $m > 0$ or $\limsup \frac{1}{n} \sum_{k=1}^n \varphi_{\ln n}(\alpha_k) = 0$. This implies that

$$\lim \frac{1}{n} \sum_1^n \int_{\|x - m(\alpha_k)\alpha_1 \ldots \alpha_{k-1}\| > \varepsilon\sqrt{n}} \|x\|^2\, dv(\alpha_k)\alpha_1 \ldots \alpha_{k-1} \leqq \limsup \frac{1}{n} \sum_1^n \varphi_{\ln n}(\alpha_k) = 0. \tag{11}$$

The last inequality holds for arbitrary but fixed $\mathcal{K}_N$, any $\varepsilon>0$ and from some positive integer N_ε on.

The right-hand side of the inequality (11) is a necessary and sufficient condition for the convergence in the weak sense of $\frac{1}{\sqrt{n}}\{\mathcal{S}_n(\tau_n)-\mathbf{M}\mathcal{S}_n(\tau_n)\}$ to the normal distribution under $\tau_n\in\mathcal{K}_N$.

On the other hand the covariance matrix from Lemma 4 $\frac{1}{\sqrt{n}}\{\mathcal{S}_n(\tau_n)-\mathbf{M}\mathcal{S}_n(\tau_n)\}$ converges to $\Omega_1=\sigma\mathbb{I}$. Since the normal distribution with zero is determined by the covariance matrix uniquely and $\mu(\mathcal{K}_N)\to 1$ as $N\to\infty$, condition 1) of Lemma 1 for the variables (5) is satisfied.

Remark. It is well-known by [4] that the distribution $\frac{1}{\sqrt{n}}\mathcal{S}_n$ converges weakly to the normal one with density $\mathbf{N}_\Omega(x)$. The matrix Ω was defined in [4], too. On other hand we have

$$\mathcal{S}_n=\frac{1}{\sqrt{n}}\{\mathcal{S}_n(\tau)-\mathbf{M}\mathcal{S}_n(\tau)\}+\frac{1}{\sqrt{n}}\mathbf{M}\mathcal{S}_n(\tau)$$

for any collection $(\alpha_1,\alpha_2,\dots)=\tau$.

It was shown above that the distribution $\frac{1}{\sqrt{n}}\{\mathcal{S}_n(\tau_n)-\mathbf{M}\mathcal{S}_n(\tau_n)\}$, $\tau_n\in\mathcal{K}_n$ converges to the normal one with density $\mathbf{N}_{\Omega_1}(x)$. The distribution of the variable $\frac{1}{\sqrt{n}}\mathbf{M}\mathcal{S}_n(\tau)=\frac{1}{\sqrt{n}}\sum m(\alpha_k)\,\alpha_1\dots\alpha_{k-1}$ converges to the normal distribution with the density $\mathbf{N}_{\Omega_2}(x)$ [4] since by condition 2'. the value of $\int\|m(\alpha)\|^2\,d\mathbf{v}$ is bounded, and the random variables $\alpha_1,\alpha_2,\dots$ are independent with common distribution $\mathbf{v}$, satisfying condition 1. Consequently $\mathbf{N}_\Omega(x)=\mathbf{N}_{\Omega_1+\Omega_2}(x)$. This equality will be used to calculate the constant c in the theorem.

5. Proof of Conditions 2), 3) for the Sum $\mathcal{S}_r(\alpha_1,\dots,\alpha_n)$

To check conditions 2), 3) for the sum $\mathcal{S}_n(\alpha_1,\dots,\alpha_n)$ we need the following proposition.

Lemma 5. *Let $f_\alpha(t)$ be the characteristic function of the variable $z(\alpha)$. Put $w_k=(\alpha_1\dots\alpha_k)^{-1}$, w_0 – the identity of $\mathbb{SO}(d)$. Then for any segment $[b,B]$, $0<b<B$ there exists θ, $0<\theta<1$ such that the inequality*

$$\mathbf{M}\left\{\prod_{k=1}^{n}|f_{\alpha_k}(t\,w_{k-1})|\right\}<\theta^n$$

holds uniformly for any $t\in[b,B]$ as $n\to\infty$.

Proof. For $\alpha\in K$ (see condition 3.) the support of the distribution $v(\alpha)$ does not contain any hyperplane. Thus, if $\alpha\in K$, then the function $|f_\alpha(t)|$ may be equal to $\mathbb{1}$

only for the points laying on some lines t_α passing through the origin $\mathbb{O}$. Let β be a random variable on $\mathbb{SO}(d)$ with distribution $\mathbf{v}$. The convergence of $\mathbf{v}^{*n}$ to the uniform distribution on $\mathbb{SO}(d)$ involves the convergence of $(\mathbf{v}\,w)^{*n}$ to one under fixed $w \in \mathbb{SO}(d)$. Then $\mathbf{M}_\beta |f_\alpha(t\,\beta\,w)| < 1$, since the distribution $v(\alpha)$ must be degenerate in the contrary case. Thus $\mathbf{M}_\beta |f_\alpha(t\,\beta\,w)| = v(t, \alpha, w) < 1$ for $t \neq 0$, $\alpha \in K$, $v(t, \alpha, w)$ is a continuous function and $\sup_w v(t, \alpha, w) = v(t, \alpha) < 1$, $\mathbf{M}_\alpha v(t, \alpha) = v(t) < 1$ for $t \neq 0$, $\alpha \in K$.

Hence we can find a θ_1, $0 < \theta_1 < 1$ such that $v(t) < \theta_1$ for all $t \in [b, B]$. Since $|f_\alpha(t)| \leqq 1$ we have the following inequalities:

$$\mathbf{M}\left\{\prod_{k=1}^{n} |f_{\alpha_k}(t\,w_{k-1})|\right\} \leqq \mathbf{M}\left\{\prod_{k=1}^{[n/2]} |f_{\alpha_{2k}}(t\,w_{2k-1})|\right\} \tag{12}$$

and

$$\begin{aligned}&\mathbf{M}_{\alpha_{2k}}(\mathbf{M}_{2k-1} |f_{\alpha_{2k}}(t\,\alpha_{2k-1}^{-1} \cdot \alpha_{2k-2}^{-1} \dots \alpha_1^{-1})| \\ &\quad = \mathbf{M}_{\alpha_{2k}} v(t, \alpha_{2k}, w_{2k-2}) \leqq \mathbf{M}_{\alpha_{2k}} v(t, \alpha_{2k}) = v(t) < \theta_1.\end{aligned} \tag{13}$$

We remark that the random pairs $(\alpha_{2k}, \alpha_{2k-1})$ and $(\alpha_{2m}, \alpha_{2m-1})$ are independent for any integers k, m, $k \neq m$. Thus it follows from (13) and $\mathbf{M}_\alpha \mathbf{M}_\beta |f_\alpha(t\,\beta\,w| < \theta_1$ that the right hand side of (12) can be estimated recursively, i.e.

$$\mathbf{M}\left\{\prod_{k=1}^{[n/2]} |f_{\alpha_{2k}}(t\,w_{2k-1})|\right\} \leqq \mathbf{M}\left\{\prod_{k=1}^{[n/2]-1} |f_{\alpha_{2k}}(t\,w_{2k-1})|\right\} \theta_1 \leqq \dots \leqq \theta_1^{[n/2]}.$$

This completes the proof.

To verify that condition 2) is fulfilled for the sum $\mathcal{S}_n(\alpha_1, \dots, \alpha_n)$ it suffices to show that

$$\mathbf{M}\{\sqrt{n} \int_{b \leqq \|t\| \leqq B} \prod |f_{\alpha_k}(t\,w_{k-1})|\, dt\} \to 0.$$

But by Lemma 5 this value is less than $\theta^n \sqrt{n}(B-b)$, $0 < \theta < 1$ and consequently it tends to zero. To verify that condition 3) is fulfilled it is enough to show that

$$\sqrt{n}\,\mathbf{M}\{\int_{\frac{A}{n} \leqq \|t\| \leqq b} \prod |f_{\alpha_k}(t\,w_{k-1})|\, dt\}$$

tends to zero as $n, A \to \infty$, $b \to 0$. For this we put $\mathbf{M}|f_\alpha(t)| = u(t)$, $u(t\,w_t) = \max_w u(t\,w)$. We know that the variables α_k, $k = 1, 2, \dots$ are independent and distributed identically, hence we have

$$\mathbf{M} \prod_{k=1}^{n} |f_{\alpha_k}(t\,w_{k-1})| \leqq \left\{\mathbf{M} \prod_{k=1}^{n-1} |f_{\alpha_k}(t\,w_{k-1})|\right\} u(t\,w_t) \leqq u^n(t\,w_t). \tag{14}$$

It is regrettable that the function $u(t)$ cannot be a characteristic function. However, $\{\mathbf{M}|f_\alpha(t)|\}^2 = u(t)^2 \leqq \mathbf{M}|f_\alpha(t)|^2 = \mathbf{U}(t)$ where $\mathbf{U}(t)$ is the characteristic function of the distribution $\int v'(\alpha)\, d\mathbf{v}$. This distribution has finite second moments and by condition 3) does not contain any hyperplanes. Hence it follows that for some neighbourhood of zero we have the estimate

$$\mathbf{U}(t) < 1 - c\|t\|^2.$$

Since $\|t\| = \|t\,w\|$ for any rotation $w \in \mathbb{SO}(d)$, we also have

$$\mathbf{U}(t\,w_t) < 1 - c\|t\|^2. \tag{15}$$

Consequently it follows from (14) and (15) that

$$\sqrt{n}\,\mathbf{M} \int_{\frac{A}{\sqrt{n}} \leq \|t\| \leq b} \prod |f_\alpha(t\,w_{k-1})|\,dt \leq \sqrt{n} \int_{\frac{A}{\sqrt{n}} \leq \|t\| \leq b} \mathbf{U}^{[n/2]}(t)\,dt. \tag{16}$$

According to (15) standard tools can be applied to estimate the integral of the power of the characteristic function in the segment $[A/\sqrt{n}, b]$. In that case we obtain $\sqrt{n}\int \mathbf{U}^{[n/2]}(t)\,dt \to 0$ as $n, A \to \infty$, $b \to 0$.

Consequently, this completes the proof of condition 3) for the sum $\mathscr{S}_n(\alpha_1, \ldots, \alpha_n)$.

6. Essen's Inequality for the Sum $\mathscr{S}_n(\alpha_1, \ldots, \alpha_n)$.

Let $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ be a sequence of random variables which are independent and identically distributed in $\mathbb{R}^d$. In the paper by Esseen [3] the following inequality was obtained for the sum $\mathscr{S}_n = \xi_1 + \ldots + \xi_n$

$$\sup_{x \in \mathbb{R}^d} \mathbf{P}\{\mathscr{S}_n \in \Sigma_r + x\} = c(d, r)\left\{\sup_{u \geq r} u^{-2} \sum_1^n \chi_k(u)\right\}^{-d/2} \tag{17}$$

where Σ_r is the sphere of radius r with the centre at the origin; $c(d, r)$ is a constant depending only on d and r, $\mathbf{P}_k$ is the distribution of ξ_k, $\chi_k(u) = \inf_{\|t\|=1} \int_{\|x\|<u} (t, x)^2\,d\mathbf{P}'_k$ where the distribution $\mathbf{P}'_k$ is obtained by symmetrization of $\mathbf{P}_k$. Let us apply the inequality (17) to the sum

$$\mathscr{S}_n(w; \alpha_1, \ldots, \alpha_n) = x_1(\alpha_1)\,w_1 + \ldots + x_n(\alpha_n)\,w_n$$

where the rotations $w_1, \ldots, w_n$ are arbitrary but fixed. Since for any rotation $w \in \mathbb{SO}(d)$ the equality $(v(\alpha)\,w)' = v'(\alpha)\,w$ holds, we shall have

$$\inf_{\|t\|=1} \int_{\|x\| \leq \lambda_0} (t, x)^2\,d(v(\alpha)\,\tau)' = \inf_{\|t\|=1} \int_{\|x\| \leq \lambda_0} (t\,\tau^{-1}, x)^2\,dv'(\alpha)$$

$$= \inf_{\|t\|=1} \int_{\|x\| \leq \lambda_0} (t, x)^2\,dv'(\alpha). \tag{18}$$

Thus Esseen's inequality for the sum $\mathscr{S}_n(w; \alpha_1, \ldots, \alpha_n)$ does not depend on the arbitrary rotations $w_1, \ldots, w_n \in \mathbb{SO}(d)$. The function $\psi(\alpha, \lambda) = \inf_{\|t\|=1} \int_{\|x\| \leq \lambda} (t, x)^2\,dv'(\alpha)$ is measurable with respect to α and is decreasing with respect to λ for every fixed α. By condition 3. if $\alpha \in K$ then $\lim_{\lambda \to \infty} \inf \psi(\alpha, \lambda) > 0$.

Consequently we can find a subset $K_0 \subset K$ with positive measure $\nu(K_0)=p_0>0$ and some λ_0 such that simultaneously for all $\alpha \in K_0$, $\lambda \geqq 0$ the inequality

$$\inf_{\|t\|=1} \int_{\|x\|<\lambda} (t,x)^2\, d\nu'(\alpha) \geqq \delta_0 > 0 \tag{19}$$

is fulfilled.

Lemma 6. *Let us suppose that among $\alpha_1, \ldots, \alpha_n$ the value $\alpha_k \in K_0$ is taken on $p_0 n/2$ times at least. Then there exists a sphere Σ_r and a constant $c(p_0, \delta, d, r)$ such that*

$$\sup_{x \in \mathbb{R}^d} \mathbf{P}\{\mathscr{S}_n(\alpha_1, \ldots, \alpha_n) \in \Sigma_r + x\} < c(p_0, \delta, d, r)\, n^{-d/2}.$$

In fact this can be obtained by employing (18) and (19) in the inequality (17) for the sum $\mathscr{S}_n(\alpha_1, \ldots, \alpha_n)$.

The next lemma follows from the preceding one.

Lemma 7. *Under the conditions of Lemma 6 for any bounded domain $\mathscr{D}$ in $\mathbb{R}^d$ there exists a constant $c(\mathscr{D}, p_0, \delta, d)$ such that*

$$\sup_{x \in \mathbb{R}^d} \mathbf{P}\{\mathscr{S}_n(\alpha_1, \ldots, \alpha_n) \in \mathscr{D} + x\} \leqq c(\cdot)\, n^{-d/2}. \tag{20}$$

Furthermore the measure of all trajectories $(\alpha_1, \ldots, \alpha_n)$ for which the inequality (20) *may not be true is less than* $\exp\{-c(p_0)\, n\}$ *and the constant $c(p_0)>0$ depends only on p_0.*

Proof. In fact any bounded domain $\mathscr{D}$ of diameter L may be covered by the finite number of spheres of the radius r. This number depends only on L and r. Consequently we can apply Lemma 6. The second part of Lemma 7 follows by elementary combinatorial estimations.

7. The Proof of the Theorem

It follows from above that we can define some sequence of sets $\mathscr{K}_n \subset \mathbb{SO}(d)^\infty$, $\mu(\mathscr{K}_n) \to 1$ as $n \to \infty$ such that the conditions of Lemma 1 are fulfilled for the sum $\mathscr{S}_n(\alpha_1, \ldots, \alpha_n)$ if $\tau_n = (\alpha_1, \ldots, \alpha_n, \ldots) \in \mathscr{K}_n$. Then for every number n we shall represent the compact $\mathbb{SO}(d)^\infty$ as the sum of the sets $\mathscr{K}_n$, $\mathscr{L}_n$ and $\mathscr{E}_n$ where $\mathscr{L}_n$ is a set of τ such that $\tau = (\alpha_1, \ldots, \alpha_n, \ldots) \in \mathbb{SO}(d)^\infty$, so that the value $\alpha_k \in K_0$ is taken on less than $p_0 n/2$ times among the n first components, $\mathscr{E}_n$ is the rest of the elements $\mathbb{SO}(d)$. It is obvious that $\mu(\mathscr{E}_n) \leqq \varepsilon_n$. By virtue of Lemma 7 we have $\mu(\mathscr{L}_n) < \exp(-c(p_0)\, n)$. It is clear from the foregoing definition of the measure μ on $\mathbb{SO}(d)^\infty$ that integral (3) can be written

$$\mathbf{P}\{\mathscr{S}_n \in \mathscr{D}\} = \int \mathbf{P}\{\mathscr{S}_n(\tau) \in \mathscr{D}\}\, d\mu = \int_{\mathscr{K}_n} + \int_{\mathscr{L}_n} + \int_{\mathscr{E}_n}. \tag{21}$$

By Lemma 6 and the evident inequality $0 \leqq \mathbf{P}\{\mathscr{S}_n(\tau) \in \mathscr{D}\} \leqq 1$ we have

$$\int_{\mathscr{K}_n} = (\sqrt{n})^{-d} \int_{\mathscr{K}_n} \Big(\int_{\mathscr{D}} \mathbf{N}_{\Omega_1}(\cdot)\, dy\Big)\, d\mu + \Big(\int \theta_n(\cdot)\, d\mu\Big)\, n^{-d/2}.$$

Since θ_n tends to zero uniformly on $\mathscr{K}_n$ as $n \to \infty$ the second integral is less than $\varepsilon'_n n^{-d/2}$ where $\varepsilon'_n \to 0$.

It is obvious that

$$\int_{\mathscr{K}_n} = n^{-d/2} \mathbf{M}_\tau(\cdot) - n^{-d/2} \int_{\mathscr{L}_n + \mathscr{E}_n}, \quad \text{where} \int_{\mathscr{L}_n + \mathscr{E}n} \mathbf{N}_{\Omega_1}(0) \operatorname{mes} \mathscr{D} \cdot \varepsilon''_n, \varepsilon''_n \to 0.$$

To evaluate $M_\tau(\cdot)$ it suffices to change the integration order and to note that $\operatorname{mes}(\mathscr{D})$ and $\mathbf{N}_{\Omega_1}(x)$ are bounded. Therefore

$$\mathbf{M}_\tau(\cdot) = \operatorname{mes} \mathscr{D} \cdot \left\{ \lim_n \mathbf{M}_\tau \left[N_{\Omega_1} \left(\frac{\mathbf{M} \mathscr{S}_n(\tau)}{\sqrt{n}} \right) \right] \right\} + \varepsilon'''_n, \qquad \varepsilon'''_n \to 0.$$

Thus we obtain

$$\mathbf{P}\{\mathscr{S}_n \in \mathscr{D}\} = \operatorname{mes} \mathscr{D} \{\lim_n \mathbf{M}_\tau [\cdot]\} n^{-d/2} + o_n(1) n^{-d/2}.$$

According to the remark above the distribution of the variable $\mathbf{M} \dfrac{\mathscr{S}_n(\tau)}{\sqrt{n}}$ converges uniformly to the normal distribution with the density $\mathbf{N}_\Omega(x)$. Consequently the constant $\lim_n \mathbf{M}_\tau \left[\mathbf{N}_{\Omega_1} \left(\dfrac{\mathbf{M} \mathscr{S}_n(\tau_n)}{\sqrt{n}} \right) \right] = c$ is equal to $\int_{\mathbb{R}^d} \mathbf{N}_{\Omega_1}(x) \mathbf{N}_{\Omega_2}(x) \, dx = \mathbf{N}_\Omega(0)$, where $\mathbf{N}_\Omega(x)$ is the density of the limit distribution of $\dfrac{1}{\sqrt{n}} \mathscr{S}_n$. Thus the theorem is proved.

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Received August 18, 1977; in revised form June 1, 1979