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Limit Theorems for Sums of Dependent Random Variables

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Summary. In Lai and Stout [7] the upper half of the law of the iterated logarithm (LIL) is established for sums of strongly dependent stationary Gaussian random variables. Herein, the upper half of the LIL is established for strongly dependent random variables $\{X_i\}$ which are however not necessarily Gaussian. Applications are made to multiplicative random variables and to $\sum f(Z_i)$ where the Z_i are strongly dependent Gaussian. A maximal inequality and a Marcinkiewicz-Zygmund type strong law are established for sums of strongly dependent random variables X_i satisfying a moment condition of the form $E|S_{a,n}|^p \leqq g(n)$, where $S_{a,n} = \sum_{a+1}^{a+n} X_i$, generalizing the work of Serfling [13, 14].

1. Introduction

Let $\{X_n, n \geqq 1\}$ be a stationary sequence of zero-mean random variables with finite variances and let $S_n = \sum_1^n X_i$, $n \geqq 1$. Most of the laws of the iterated logarithm and related strong limit theorems for $\{S_n\}$ in the literature apply only to the case where the sequence $\{X_n\}$ is weakly dependent (cf. [10]) so that $\{S_n\}$ behaves in a very strong sense like Brownian motion. We have recently considered in [7] for the stationary Gaussian case the situation where there is much stronger dependence. One of our results shows that the upper half of the law of the iterated logarithm holds under very weak assumptions on the dependence structure; more specifically, we have proved

Theorem 1. *Let $\{X_n, n \geqq 1\}$ be a zero-mean stationary Gaussian sequence. Let $S_n = \sum_1^n X_i$, $g(n) = ES_n^2$. Suppose that*

$$\liminf_{n \to \infty} g(Kn)/g(n) > 1 \quad \textit{for some integer } K \geqq 2, \tag{1.1}$$

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and

$$\forall \varepsilon>0,\ \exists \rho=\rho(\varepsilon)<1 \quad \text{such that } \limsup_{n\to\infty}\{\max_{\rho n\leqq i\leqq n} g(i)/g(n)\}<1+\varepsilon. \tag{1.2}$$

Then

$$\limsup_{n\to\infty} |S_n|/\{2g(n)\log_2 n\}^{1/2}\leqq 1 \text{ a.s.} \tag{1.3}$$

In the above we have let $\log_2 n$ denote $\log\log n$, and we shall also set $\log_k n=\log(\log_{k-1} n)$. The assumption (1.1) is closely related to the concept of dominated variation introduced by Feller [4] as a one-sided analogue of Karamata's regular variation. Note that if either $g(n)$ is nondecreasing or $\max_{i\leqq n} g(i)\sim g(n)$, then (1.2) holds. The assumptions (1.1) and (1.2) cover a wide spectrum of dependence situations, e.g., the independent case, the case where $\mathrm{Cov}(X_i, X_j)\geqq 0$ for all i, j, and the case where ES_n^2 is regularly varying with exponent $\alpha>0$.

In this paper we shall extend Theorem 1 to the non-Gaussian and also to the nonstationary case. *Thoughout the sequel, $\{X_i\}$ denotes a general sequence of random variables and $S_n=\sum_1^n X_i$; no assumption on stationarity or joint normality is made unless otherwise stated. We shall also let $S_{a,n}$ denote the delayed sum $\sum_{i=a+1}^{a+n} X_i$ and set $S_{a,0}=0$. Further, $g(n)$ does not necessarily satisfy $g(n)=ES_n^2$.* Noting that in Theorem 1 the stationary Gaussian assumption and the assumption that $g(n)=ES_n^2$ imply that $E\exp(tS_{a,n})=\exp(\frac{1}{2}t^2 g(n))$ for all a, n, and t, we shall prove in Sect. 2 the following generalization of Theorem 1.

Theorem 2. *Suppose $g\colon\{1, 2, \ldots\}\to(0,\infty)$ satisfies* (1.1) *and* (1.2). *Let $\{X_n\}$ be a sequence of random variables such that*

$$E\exp(t S_{a,n})\leqq C\exp\{\tfrac{1}{2}t^2(1+\varepsilon_n)\,g(n)\} \quad \text{for all } a\geqq a_0,\ n\geqq 1 \text{ and } |t|\leqq u_n/g^{1/2}(n), \tag{1.4}$$

where C, a_0, ε_n, and u_n are positive constants such that $\varepsilon_n\to 0$ and $u_n\sim(2\log_2 n)^{1/2}$ as $n\to\infty$. Then (1.3) *holds.*

The condition (1.4) in Theorem 2 implies the finiteness of $E\exp(t_n|X_n|)$ for all n, where $\{t_n, n\geqq 1\}$ is a sequence of positive constants. For some of our applications in Sect. 3, however, $E\exp(t|X_n|)=\infty$ for all $t>0$ and $n\geqq 1$. In the classical independent case, infinite moment generating functions can be circumvented by truncation and by considering the moment generating function of the truncated random variables. However, for dependent random variables like those considered in the applications in Sect. 3, the moment generating function of the sum of the truncated random variables is often very difficult to handle. Therefore for dependent random variables it is sometimes more convenient to replace (1.4) by conditions on the moments. Returning to the stationary Gaussian case of Theorem 1, we have $E|S_{a,n}/g^{1/2}(n)|^p=E|N(0,1)|^p$, and therefore

$$\exists B>0 \quad \text{such that } E|S_{a,n}/g^{1/2}(n)|^p\leqq B(pe^{-1})^{p/2} \quad \forall a\geqq 0,\ n\geqq 1 \text{ and } p>0. \tag{1.5}$$

In general, in the absence of the Gaussian assumption, the condition (1.5) implies (1.4), as can be easily shown by using power series expansions for both sides of (1.4). Hence by Theorem 2, the upper half of the law of the iterated logarithm (1.3) also holds under the assumption (1.5). This suggests the following generalization of Theorem 1 without the assumption of finiteness of $E\exp(tX_n)$.

Theorem 3. *Suppose* $g\colon \{1, 2, \ldots\} \to (0, \infty)$ *satisfies* (1.1) *and* (1.2). *Let* $\alpha > 0$, $\beta > 0$, *and let* $\{p(n)\}$ *be a sequence of positive constants such that*

$$p(n) \sim \beta^{-1} \log_2 n. \tag{1.6}$$

Let $\{X_n\}$ *be a sequence of random variables such that*

$$\limsup_{a\to\infty} E|S_{a,n}|^p < \infty \quad \textit{for every} \;\; n \geqq 1 \textit{ and } p > 0, \tag{1.7}$$

and

$$E|S_{a,n}/g^{1/2}(n)|^{p(n)} \leqq B(\alpha p(n))^{\beta p(n)} \quad \textit{for all} \;\; a \geqq a_0 \textit{ and } n \geqq 1, \tag{1.8}$$

where B *and* a_0 *are positive constants. Then*

$$\limsup_{n\to\infty} |S_n|/\{(\alpha e \beta^{-1} \log_2 n)^\beta g^{1/2}(n)\} \leqq 1 \;\; a.s. \tag{1.9}$$

We note that (1.5) implies (1.8) with $\alpha = e^{-1}$ and $\beta = 1/2$, in which case (1.9) reduces to (1.3). In Sect. 3 we shall give some applications of Theorems 2 and 3. Clearly Theorem 2 also includes the upper half of the classical law of the iterated logarithm for the i.i.d. second moment case via the Hartman-Wintner truncation scheme [6]. In [17], Taqqu proved the weak convergence (to certain semi-stable processes) of sums of nonlinear functions of stationary Gaussian random variables that exhibit a long range dependence, and he asked about their almost sure limiting behavior. In Sect. 3 we shall apply Theorem 3 to solve this problem. Theorem 3 also generalizes an earlier result of Gál [5, Théorème 2] and of Philipp [9, Satz 2] who considered the special case $g(n) = n$ and $\beta = 1/2$ and who respectively indicated some interesting applications in this special case to lacunary trigonometric series and to stationary mixing sequences.

The proof of Theorems 2 and 3 in Sect. 2 makes use of a dyadic expansion argument due to Gál [5]. A simple probabilistic exposition of this useful technique will be given in the proof of Lemma 1. We shall also use this kind of dyadic expansion argument in Sect. 4 to obtain a maximal inequality and a Marcinkiewicz-Zygmund type strong law for dependent random variables which satisfy moment restrictions of the form

$$E|S_{a,n}|^p \leqq g(n) \quad \text{for all } \; a = 0, 1, \ldots \text{ and } n = 1, 2, \ldots, \tag{1.10}$$

where $p > 0$ and g satisfies regularity conditions of the type (1.1) and (1.2). These results generalize some of the related results in the literature due to Serfling [13], [14].

Throughout the sequel, we shall use Vinogradov's symbol $\ll$ instead of Landau's 0 notation. We shall sometimes also write $S(a, n)$ instead of $S_{a,n}$.

2. Proof of Theorems 2 and 3

Let $A = \liminf_{n\to\infty} g(Kn)/g(n)$. As shown in Lemma 1(i) and Lemma 2 of [7], the assumptions (1.1) and (1.2) imply that

$$g(n) \to \infty, \tag{2.1}$$

$$\max_{i \leq n} g(i) \ll g(n), \tag{2.2}$$

and that given $0 < \gamma < (\log A)/(\log K)$, there exists N such that

$$g([an])/g(n) > a^{\gamma} \quad \text{for all } a \geqq N \text{ and } n \geqq N. \tag{2.3}$$

As we have remarked in Sect. 1, the condition (1.5) on the moments of $S_{a,n}$ implies the assumption (1.4) on the moment generating function of $S_{a,n}$. It is interesting to note that the moment condition (1.5) provides the following exponential inequality for the tail probabilities of $S_{a,n}$: For all $t > 0$, $a \geqq 0$, and $n \geqq 1$,

$$\begin{aligned} P[|S_{a,n}/g^{1/2}(n)| \geqq t] &\leqq B \inf_{p>0} t^{-p} (p\, e^{-1})^{p/2}, \quad \text{by (1.5)}, \\ &= B \inf_{p>0} \exp\{-p \log t + \tfrac{1}{2} p \log(p\, e^{-1})\} = B \exp(-\tfrac{1}{2} t^2). \end{aligned} \tag{2.4}$$

More generally, if the term $(p\,e^{-1})^{p/2}$ in (1.5) is replaced by $(\alpha\, p)^{\beta p}$, where α, β are positive constants, then (2.4) becomes

$$\begin{aligned} P[|S_{a,n}/g^{1/2}(n)| \geqq t] &\leqq B \inf_{p>0} t^{-p} (\alpha\, p)^{\beta p} \\ &= B \exp\{-(\beta\, t^{1/\beta})/(\alpha\, e)\}. \end{aligned} \tag{2.5}$$

The proof of Theorems 2 and 3 depends on similar exponential bounds for certain large probabilities which are obtained under the weaker assumption (1.4) of Theorem 2 or (1.8) of Theorem 3. Making use of the properties (1.1) and (1.2) (and therefore (2.1)–(2.3) as well) of g, the following lemma relates these exponential bounds to the almost sure asymptotic behavior of S_n.

Lemma 1. *Suppose* g: $\{1, 2, \ldots\} \to (0, \infty)$ *satisfies* (1.1) *and* (1.2). *Let* $\beta > 0$, $\theta > 0$, *and let*

$$t_n = (\theta^{-1} \log_2 n)^{\beta}. \tag{2.6}$$

Let $\{X_n\}$ *be a sequence of random variables satisfying* (1.7) *and the following two conditions:*

(i) *Given* $0 < \varepsilon < 1$, $\exists \varepsilon' > 0$ *and* a_0, n_0 *such that for all* $a \geqq a_0$ *and* $n \geqq n_0$,

$$P[|S_{a,n}/g^{1/2}(n)| \geqq (1+\varepsilon)\, t_n] \leqq \exp\{-(1+\varepsilon')\, \theta\, t_n^{1/\beta}\};$$

(ii) $\exists d > 0$, $B > 1$, *and* a_1, n_1 *such that for all* $a \geqq a_1$ *and* $n \geqq n_1$,

$$P[|S_{a,n}/g^{1/2}(n)| \geqq x\, t_n] \leqq \exp\{-d t_n^{1/\beta} \log x\} \quad \textit{if } x \geqq B.$$

Then

$$\limsup_{n\to\infty} |S_n|/\{t_n\, g^{1/2}(n)\} \leqq 1 \quad a.s. \tag{2.7}$$

Proof. Let $0<\varepsilon<1$. Take $0<\delta<1$ (to be specified later). Let $n_k=[2^{k^\delta}]$ and define

$$A_k=[|S_{n_k}|\geqq(1+\varepsilon)\,t_{n_k}\,g^{1/2}(n_k)],$$

$$B_k=[\max_{n_k\leqq n\leqq n_{k+1}}|S_n-S_{n_k}|\geqq 2\varepsilon\,t_{n_k}\,g^{1/2}(n_k)].$$

By condition (i), $\sum P(A_k)<\infty$ and so $P[A_k \text{ i.o.}]=0$. Moreover, in view of (1.2), we can choose δ sufficiently small such that for all k sufficiently large and $n_k\leqq n\leqq n_{k+1}$, $g(n_k)<(1+\varepsilon)\,g(n)$. Therefore it remains to show that

$$P[B_k \text{ i.o.}]=0. \tag{2.8}$$

To prove (2.8), we use a dyadic expansion argument due to Gál [5]. Take $j=j(k)$ such that $2^{j-1}\leqq n_{k+1}-n_k<2^j$. Therefore

$$j(k)=k\,\delta+O(1), \tag{2.9}$$

and

$$n_k\geqq 2^{j(k)-1}/(2^\delta-1)+O(1), \qquad \text{since } n_{k+1}-n_k=n_k(2^\delta-1)+O(1). \tag{2.10}$$

For $n_k\leqq n<n_{k+1}$, since $n-n_k<2^j$, it then follows that

$$n-n_k=e_{j-1}\,2^{j-1}+e_{j-2}\,2^{j-2}+\cdots+e_0,$$

where $e_{i-1}=0$ or 1 $(i=1,\ldots,j)$, and therefore

$$S_n=S_{n_k}+\sum_{i=1}^{j}S(n_k+m_i\,2^i,\,2^{i-1}\,e_{i-1}), \tag{2.11}$$

where $m_j=0$, $m_i=e_{j-1}\,2^{j-i-1}+\cdots+e_i$ $(i=1,\ldots,j-1)$, so that $0\leqq m_i<2^{j-i}$. For $i=1,\ldots,j(k)$ and $m=0,1,\ldots,$ define

$$B_k(i,m)=[|S(n_k+m\,2^i,\,2^{i-1})|>\varepsilon(j(k)-i+1)^{-2}\,t_{n_k}\,g^{1/2}(n_k)].$$

Since $\sum_{i=1}^{j}(j-i+1)^{-2}<2$, we obtain from (2.11) that

$$\Big(\bigcup_{1\leqq i\leqq j(k)}\ \bigcup_{0\leqq m<2^{j(k)-i}}B_k(i,m)\Big)\supset B_k. \tag{2.12}$$

Take $0<\eta<1$, and let $0<\tilde{\varepsilon}<\varepsilon/2$. By choosing δ sufficiently small, we obtain from (2.3) and (2.10) that $g(n_k)/g(2^{i-1})\geqq\frac{1}{2}\{2^{j(k)-i}/(2^\delta-1)\}^\gamma$ and $\log_2 n_k\geqq\log_2(2^{i-1})$ for $i_0\leqq i\leqq j(k)$. Hence it follows from condition (ii) that for all large k

$$\begin{aligned}
&\sum_{\eta j(k)\leqq i\leqq j(k)}\ \sum_{0\leqq m<2^{j(k)-i}}P(B_k(i,m))\\
&\quad\leqq\sum_{\eta j(k)\leqq i\leqq j(k)}2^{j(k)-i}\exp\{-d\theta^{-1}(\log i)\,[\log\tilde{\varepsilon}-2\log(j(k)-i+1)\\
&\qquad+\tfrac{1}{2}\gamma\log(2^{j(k)-i}/(2^\delta-1))]\}\\
&\quad\leqq j(k)\max_{v\geqq 0}(\exp\{v\log 2-d\theta^{-1}(\log\eta\,j(k))\,[\tfrac{1}{2}\gamma\log(2^v/(2^\delta-1))-2\log(v+1)\\
&\qquad+\log\tilde{\varepsilon}]\})\\
&\quad\ll k^{-2}, \qquad \text{by choosing } \delta \text{ sufficiently small and using (2.9).}
\end{aligned} \tag{2.13}$$

Moreover, choosing $N > i_0$ such that (2.3) holds, we obtain by a modification of (2.13) that for all large k

$$\sum_{N < i \leqq \eta j(k)} \sum_{0 \leqq m < 2^{j(k)-i}} P(B_k(i,m))$$
$$\ll 2^{j(k)} j(k) \exp\{-d\theta^{-1}(\log N)[\tfrac{1}{4}\gamma \log(2^{(1-\eta)j(k)}) - 2\log j(k)]\}$$
$$\ll k^{-2}, \quad \text{by choosing } N \text{ large enough.} \tag{2.14}$$

Finally, by (1.7), $\sup\{E|S_{a,n}|^p : n \leqq N,\ a \geqq a_p\} < \infty$ for every $p > 0$, and therefore choosing p large enough, we have

$$\sum_{i=1}^{N} \sum_{0 \leqq m < 2^{j(k)-i}} P(B_k(i,m)) \ll 2^{j(k)}(j(k))^{2p}(g(n_k))^{-p/2}$$
$$\ll k^{-2}, \quad \text{by (2.3) and (2.9), if } p \text{ is large enough.} \tag{2.15}$$

From (2.12)–(2.15), $\sum_1^\infty P(B_k) < \infty$ and therefore (2.8) holds. □

Proof of Theorem 2. It is easy to show that (1.4) implies that the conditions of Lemma 1 are satisfied with $\theta = \beta = \frac{1}{2}$. In particular, to check that condition (ii) holds, we note that

$$P[|S_{a,n}/g^{1/2}(n)| \geqq x\,t_n] \leqq \{\exp(-u_n x\,t_n)\}\,\{E\exp(u_n S_{a,n}/g^{1/2}(n)) + E\exp(-u_n S_{a,n}/g^{1/2}(n))\}.$$

Hence the desired conclusion follows from Lemma 1. □

Proof of Theorem 3. By the Chebyshev inequality and (1.8),

$$P[|S_{a,n}/g^{1/2}(n)| \geqq x\,t_n] \leqq B\{\alpha\,p(n)/(x\,t_n)^{1/\beta}\}^{\beta p(n)},$$

and it then follows that the assumptions of Lemma 1 are satisfied with $\theta = \beta/(\alpha e)$. □

3. Some Applications of Theorems 2 and 3

Clearly Theorem 2 includes Theorem 1 as a special case. Another special case of Theorem 2 is Corollary 1 below on uniformly bounded multiplicative sequences. Recall that a sequence $\{X_n, n \geqq 1\}$ is said to be multiplicative if

$$E(X_{i_1} \ldots X_{i_n}) = 0 \quad \text{for all } n \geqq 1 \text{ and all } 1 \leqq i_1 < \ldots < i_n. \tag{3.1}$$

Suppose that the sequence $\{X_n\}$ is multiplicative and that $|X_n| \leqq A$ for all n, where A is a positive constant. Then it has been shown in [1] that for every real number t and for every double array $\{b_{mk}: 1 \leqq k \leqq m,\ m \geqq 1\}$ of real constants,

$$E\exp\left(t\sum_{k=1}^{m} b_{mk} X_k\right) \leqq \exp\left\{\tfrac{1}{2}A^2 t^2 \sum_{k=1}^{m} b_{mk}^2\right\}. \tag{3.2}$$

Hence (1.4) holds with $g(n) = A^2 n$, $C = 1$ and $\varepsilon_n = 0$. Therefore the following theorem of Serfling [14] and Takahashi [16] is a special case of Theorem 2.

Corollary 1. *Let $\{X_n, n \geqq 1\}$ be multiplicative with $|X_n| \leqq A$ a.s. for some positive constant A and all $n \geqq 1$. Then*

$$\limsup_{n\to\infty} |S_n|/(2n \log_2 n)^{1/2} \leqq A \text{ a.s.} \tag{3.3}$$

Takahashi [16] has sharpened the result (3.3) under stronger assumptions on the sequence $\{X_n\}$. A multiplicative sequence $\{X_n\}$ is said to be an equinormed strongly multiplicative sequence (ESMS) if $EX_n^2 = 1$ for all $n \geqq 1$ and

$$E\left(\prod_{j=1}^{n} X_{i(j)}^{r(j)}\right) = \prod_{j=1}^{n} EX_{i(j)}^{r(j)} \quad \text{for all } 1 \leqq i(1) < \ldots < i(n) \tag{3.4}$$

and all $r(j)$ such that $r(j)=1$ or $2(j=1, \ldots, n)$ and all $n \geqq 1$.

It is known (see Lemma 2.2 of [15] and its proof there) that if $\{X_n\}$ is a uniformly bounded ESMS then for given $\delta > 0$ there exists $t_\delta > 0$ such that

$$E\exp(tS_{a,n}) \leqq \exp\{\tfrac{1}{2}(1+\delta)t^2 n\} \quad \text{for all } a \geqq 0,\ n \geqq 1,\ |t| \leqq t_\delta. \tag{3.5}$$

Hence the following refinement of (3.3) due to Takahashi [16] is another special case of Theorem 2.

Corollary 2. *Let $\{X_n, n \geqq 1\}$ be a uniformly bounded ESMS. Then*

$$\limsup_{n\to\infty} |S_n|/(2n \log_2 n)^{1/2} \leqq 1 \text{ a.s.} \tag{3.6}$$

By making use of Theorem 3, we can extend (3.3) to the case where the sequence $\{X_n, n \geqq 1\}$ need not be uniformly bounded. The following strengthening of the multiplicative criterion (3.1) in terms of higher order product moments is due to Dharmadhikari and Jogdeo [2]:

$$E\left(\prod_{j=1}^{n} X_{i(j)}^{r(j)}\right) = 0 \quad \text{for all } n \geqq 1 \text{ and all } 1 < i(1) < \ldots < i(n) \tag{3.7}$$

and all positive integers $r(j)$ such that $\min_{1 \leqq j \leqq n} r(j) = 1$.

We shall say that a sequence $\{X_n\}$ is *higher-order multiplicative* if (3.7) holds. If $|X_n| \leqq A$ a.s. for all n, then obviously $E|X_n|^p \leqq A^p$ for all $n \geqq 1$ and $p > 0$. Theorem 3 gives the following extension of (3.3) to the case where $\{X_n\}$ need not be uniformly bounded.

Corollary 3. *Let $\{X_n, n \geqq 1\}$ be a higher-order multiplicative sequence such that there exist $A > 0$, $C > 0$ and $\gamma \geqq 0$ for which*

$$E|X_n|^p \leqq CA^p p^{\gamma p} \quad \textit{for all } n \geqq 1 \textit{ and even integers } p. \tag{3.8}$$

Then letting $\beta = \gamma + \frac{1}{2}$ and $\alpha = (\frac{1}{2}A^2 e)^{1/(1+2\gamma)}$,

$$\limsup_{n\to\infty} |S_n|/\{n^{1/2}(\alpha e \beta^{-1} \log_2 n)^\beta\} \leqq 1 \text{ a.s.} \tag{3.9}$$

Remark. If $\{X_n\}$ is uniformly bounded, then $\gamma = 0$ and therefore $\beta = \frac{1}{2}$ in (3.9), which then implies that $\limsup_{n\to\infty} |S_n|/(n \log_2 n)^{1/2} < \infty$ a.s., in agreement with (3.3).

Proof. Clearly (3.8) implies (1.7). Let $\{p(n)\}$ be a sequence of positive even integers such that $p(n) \sim \beta^{-1} \log_2 n$. We now show that (1.8) holds with $g(n)=n$. The condition (3.7) implies that for $m=1, 2, \ldots$, and $a \geqq 0$, $n \geqq 1$,

$$E|S_{a,n}|^{2m} \leqq \sum_{k=1}^{m} (k^{2m-1}/(k-1)!)\, n^{k-1} \sum_{i=a+1}^{a+n} E|X_i|^{2m} \tag{3.10}$$

(cf. [2], page 1507, line 9). Setting $2m=p(n) \sim \beta^{-1} \log_2 n$ in (3.10) and making use of (3.8), we obtain (1.8) with $g(n)=n$ and B being a sufficiently large positive number. Hence the desired conclusion follows from Theorem 3. □

A random variable X is said to be generalized Gaussian with parameter $\lambda > 0$ if $E\exp(tX) \leqq \exp(\frac{1}{2}\lambda t^2)$ for all real t. Obviously the following result of Stout [15] which was proved by using the maximal inequalities of Serfling is another special case of Theorem 2.

Corollary 4. *Let $S_{a,n}$ be generalized Gaussian with parameter λn for all $a \geqq 0$ and $n \geqq 1$. Then*

$$\limsup |S_n|/(2\lambda n \log_2 n)^{1/2} \leqq 1 \ a.s. \tag{3.11}$$

We now apply Theorem 3 to study a problem of Taqqu [17] concerning the almost sure limiting behavior of $S_n = \sum_1^n f(Z_i)$, where $f(\cdot)$ is an arbitrary Borel function such that for some $p \geqq 2$

$$E|f(Z_1)|^p < \infty \quad \text{and} \quad Ef(Z_1)=0, \tag{3.12}$$

and $\{Z_i, i \geqq 1\}$ is a mean zero, unit variance stationary Gaussian sequence that exhibits a long-range dependence in the following sense:

$$r(k)=EZ_1 Z_{k+1} \sim k^{-d} L(k) \quad \text{as} \quad k \to \infty, \tag{3.13}$$

with $d > 0$ and $L(\cdot)$ being a positive slowly varying function. Let

$$g(n)=ES_n^2. \tag{3.14}$$

Let ν denote the measure on the real line R defined by

$$d\nu(x)=(2\pi)^{-1/2} \exp(-\tfrac{1}{2}x^2)\, dx.$$

Since $Ef(Z_1)=0$ and $Ef^2(Z_1) < \infty$, the function $f(x)$ may be uniquely expanded in $L^2(R, \nu)$ in terms of the Hermite polynomials

$$H_k(x)=(-1)^k e^{x^2/2} \frac{d^k}{dx^k} e^{-x^2/2}, \tag{3.15}$$

so that the series

$$\sum_{k=1}^{\infty} J(k)\, H_k(x)/k!, \qquad \text{where } J(k)=Ef(Z_1)\, H_k(Z_1), \tag{3.16}$$

converges to $f(x)$ in $L_2(R, \nu)$. As defined in [17], the *Hermite rank* of f is the smallest positive integer k such that $J(k) \neq 0$, where $J(k)$ is as defined in (3.16). Taqqu [17] has noted that if f has Hermite rank m and (3.13) holds with $0<d<1/m$, then as $n \to \infty$

$$ES_n^2(=g(n)) \sim \{2J^2(m)/(m!(1-md)(2-md))\}\, n^{2-md} L^m(n). \tag{3.17}$$

For the case $f(x)=x$, $\{S_n\}$ is itself a Gaussian sequence to which Theorem 1 above and the laws of the iterated logarithm in [7] and [17] are applicable. These results can be extended to more general functions f with Hermite rank 1, in view of the following strong reduction lemma due to Taqqu [17].

Lemma 2 ([17], p. 206). *Let m be a positive integer, $0<d<1/m$, and let p be the smallest even integer satisfying $p>2\max\{d^{-1}, (1-md)^{-1}\}$. Let L: $[1, \infty) \to R$ be slowly varying (at ∞). Suppose that $\{Z_i, i \geqq 1\}$ is a stationary Gaussian sequence such that $EZ_1=0$, $EZ_1^2=1$ and* (3.13) *holds. Suppose also that f: $R \to R$ is a Borel function satisfying* (3.12) *and having Hermite rank m. Let $S_n=\sum_1^n f(Z_i)$* and $g(n) = ES_n^2$. *Then with probability* 1

$$\left|\sum_{i=1}^{n} \{f(Z_i)-(J(m)/m!)\, H_m(Z_i)\}\right| \Big/ g^{1/2}(n) \to 0 \quad as \quad n \to \infty, \tag{3.18}$$

where $H_m(x)$ is the m^{th} Hermite polynomial (*see* (3.15)) *and $J(m)$ is as defined in* (3.16).

When the Hermite rank $m=2$, the requirement $d<1/m$ in Lemma 2 turns out to imply that $S_n/g^{1/2}(n)$ has a limiting distribution which is non-Gaussian ([17], Theorem 3). This result was first discovered by Rosenblatt [12] and was used by him to show that the sequence $\{Z_i\}$ fails to be strong mixing if $d<\frac{1}{2}$. With a highly dependent structure and in the absence of a limiting Gaussian distribution, it is not obvious what analogue of the law of the iterated logarithm would $\{S_n\}$ exhibit, and Taqqu has raised this problem in [17]. We now apply Theorem 3 to this problem in the following:

Corollary 5. *With the same notations and assumptions as in Lemma* 1, *there exists a positive constant A such that*

$$\limsup_{n \to \infty} |S_n|/\{n^{2-md} L^m(n)(\log_2 n)^m\}^{1/2} \leqq A \ a.s. \tag{3.19}$$

Proof. Let $\tilde{S}_n=\sum_1^n H_m(Z_i)$. We first note that for every positive even integer p,

$$E\tilde{S}_n^p \leqq (EH_m^p(Z_1)) \left(n \sum_{j=-n}^{n} |r(j)|^m\right)^{p/2}. \tag{3.20}$$

This inequality follows from Corollary 4.2 and Lemma 4.4 of [17] together with the formula in the fifth display on p. 227 of [17]. By (3.13),

$$n \sum_{j=-n}^{n} |r(j)|^m \sim 2n^{2-md} L^m(n)/(1-md) \quad \text{as} \quad n \to \infty. \tag{3.21}$$

Moreover, for even integers p,

$$EH_m^p(Z_1) \sim 2^{1/2}(c_m p)^{mp/2} \quad \text{as} \quad p \to \infty, \tag{3.22}$$

where c_m is a positive constant depending only on m (cf. [17], p. 228). Let $\psi(n) = n^{2-md} L^m(n)$. From (3.20), (3.21), (3.22), and the stationarity of $\{Z_i\}$, we obtain that for all even integers $p \geqq p_0$ (sufficiently large) and all $a \geqq 0$ and $n \geqq 1$,

$$E|\tilde{S}_{a,n}/\psi^{1/2}(n)|^p = E|\tilde{S}_n/\psi^{1/2}(n)|^p \leqq B(\alpha p)^{mp/2}, \tag{3.23}$$

for some positive constants B and α. Clearly the function ψ satisfies (1.2) and (1.3). Hence by Theorem 3 the conclusion (3.19) of Corollary 5 holds for $\tilde{S}_n$. In view of Lemma 2 and (3.17), the desired conclusion therefore also holds for S_n. □

4. A Maximal Inequality and a Marcinkiewicz-Zygmund Type Strong Law

Throughout this section we shall let

$$M_{a,n} = \max_{1 \leqq k \leqq n} |S_{a,n}|, \qquad M_n = M_{0,n}.$$

In [13], Serfling has obtained the following useful maximal inequality.

Theorem 4 ([13], p. 1231). *Let f: $\{1, 2, \ldots\} \to (0, \infty)$ be a nondecreasing function such that*

$$2f(n) \leqq f(2n) \quad \textit{for all} \quad n \geqq 1, \tag{4.1}$$

and

$$f(n)/f(n+1) \to 1 \quad \textit{as} \quad n \to \infty. \tag{4.2}$$

Let $p > 2$. If

$$E|S_{a,n}|^p \leqq f^{p/2}(n) \quad \textit{for all} \quad a \geqq 0 \textit{ and } n \geqq 1, \tag{4.3}$$

then there exists a positive constant C such that

$$EM_{a,n}^p \leqq Cf^{p/2}(n) \quad \textit{for all} \quad a \geqq 0 \textit{ and } n \geqq 1. \tag{4.4}$$

Serfling's proof of the above theorem in [13] is based on induction on n and depends heavily on the assumption that $p > 2$ and the conditions (4.1) and (4.2). By making use of a similar dyadic expansion argument as in the proof of Lemma 1, we can drop the assumption that $p > 2$ and relax (4.1) and (4.2) into more general conditions of the type (1.1) and (1.2). Thus we generalize Serfling's maximal inequality in the following:

Theorem 5. *Let $p > 0$. Suppose g: $\{1, 2, \ldots\} \to (0, \infty)$* satisfies (1.2) and

$$\liminf_{n \to \infty} g(Kn)/g(n) > K \quad \textit{for some integer} \quad K \geqq 2. \tag{4.5}$$

If

$$E|S_{a,n}|^p \leqq g(n) \quad \textit{for all} \ \ a \geqq 0 \ \textit{and} \ n \geqq 1, \tag{4.6}$$

then there exists a positive constant C such that

$$EM_{a,n}^p \leqq C g(n) \quad \textit{for all} \ \ a \geqq 0 \ \textit{and} \ n \geqq 1. \tag{4.7}$$

Remarks. (i) Put $g(n) = f^{p/2}(n)$ in Theorem 4. Then since $p > 2$, (4.1) implies that g satisfies (4.5) with $K = 2$. Moreover, since f is nondecreasing, so is g and therefore (1.2) obviously holds. Hence Theorem 4 is a special case of Theorem 5.

(ii) Recently Longnecker and Serfling [9] have established a result which has some of the flavor of Theorem 5 although it is more distantly related to Theorem 5 than Theorem 4 is. In this result of [8], (4.6) is replaced by the existence of a function $f(\cdot, \cdot)$ such that for some $\gamma > 1$ and $p > 0$,

$$E|S_{a,n}|^p \leqq (f(a+1, a+n))^\gamma \quad \text{for all} \ \ a \geqq 0 \text{ and } n \geqq 1. \tag{4.8}$$

Furthermore, (1.2) and (4.5) are replaced either by

$$f(i,j)/f(1,n) \leqq (j-i+1)/n \quad \text{for all} \ \ 1 \leqq i \leqq j \leqq n, \tag{4.9}$$

or by

$$f(i,j) + f(j+1, n) \leqq f(i,j) \quad \text{for all} \ \ 1 \leqq i \leqq j \leqq n. \tag{4.10}$$

Then the conclusion (4.7) still holds with $g(n)$ replaced by $(f(1,n))^\gamma$. As pointed out in [8], either (4.9) or (4.10) implies the existence of nonnegative constants $u_1, u_2, \ldots$ such that $f(i,j) \leqq \sum_{k=i}^{j} u_k$ for all $1 \leqq i \leqq j$ with equality if $i = 1, j = n$. Therefore clearly the function g in Theorem 5 cannot be reduced to the form of the Longnecker-Serfling theorem. On the other hand, like Theorem 5, the Longnecker-Serfling theorem treats the general case $p > 0$ instead of just $p > 2$ which has been assumed in the earlier papers.

Proof of Theorem 5. We shall only consider EM_n^p, as the same argument works for the more general $EM_{a,n}^p$. We first consider the case $n = 2^h$ $(h = 1, 2, \ldots)$. Letting $n_k = 2^k$ in (2.11) and using an argument as in (2.12), we have for $x > 0$ and $h = 1, 2, \ldots,$

$$P[M_{n_h} \geqq 4x] \leqq P[|X_1| \geqq x] + \sum_{k=1}^{h} \Big\{ P[|S_{n_k}| \geqq x] + \sum_{i=1}^{k} \sum_{m=0}^{2^{k-i}} P[|S(n_k + m2^i, 2^{i-1})| \geqq (k-i+1)^{-2} x] \Big\}. \tag{4.11}$$

Let $A = \liminf_{n \to \infty} g(Kn)/g(n) (> K)$, and choose $1 < \gamma < (\log A)/(\log K)$. Since $E|S_{a,n}|^p \leqq g(n)$, it follows from (1.2) and (2.3) that there exists a positive constant c such that for $h = 1, 2, \ldots,$

$$\sum_{k=1}^{h} \int_0^\infty p x^{p-1} P[|S_{n_k}| \geqq x]\, dx \leqq \sum_{k=1}^{h} g(2^k)$$
$$\leqq c g(2^h) \sum_{k=1}^{h} 2^{-(k-h)\gamma} \leqq c' g(2^h), \tag{4.12}$$

where $c' = c \sum_1^\infty 2^{-j\gamma}$, and

$$\sum_{k=1}^{h} \sum_{i=1}^{k} \sum_{m=0}^{2^{k-i}} \int_0^\infty p x^{p-1} P[|S(n_k + m 2^i, 2^{i-1})| \geqq (k-i+1)^{-2} x]\, dx$$
$$= \sum_{k=1}^{h} \sum_{i=1}^{k} \sum_{m=0}^{2^{k-i}} (k-i+1)^{2p} \int_0^\infty p y^{p-1} P[|S(n_k + m 2^i, 2^{i-1})| \geqq y]\, dy$$
$$\leqq \sum_{k=1}^{h} \sum_{i=1}^{k} 2^{k-i+1} (k-i+1)^{2p} g(2^{i-1})$$
$$\leqq c g(2^h) \sum_{k=1}^{h} \sum_{i=1}^{k} 2^{(k-i+1)-\gamma(h-i+1)} (k-i+1)^{2p}$$
$$\leqq c g(2^h) \sum_{k=1}^{h} 2^{-\delta(h-k+1)} \sum_{i=1}^{k} 2^{(1-\eta)(k-i+1)} (k-i+1)^{2p} \leqq c'' g(2^h), \tag{4.13}$$

where we take $\delta > 0$ and $\eta > 1$ such that $\gamma = \delta + \eta$ and we let

$$c'' = c \left(\sum_1^\infty 2^{-\delta j} \right) \left(\sum_1^\infty 2^{-(\eta-1) j} j^{2p} \right).$$

From (4.11), (4.12), and (4.13), we obtain (4.7) for $a=0$ and $n=2^h$. Clearly the same argument also establishes (4.7) for $a=2h$ and $n=2^h$.

Now let $2^h \leqq n \leqq 2^{h+1}$. We note that

$$EM_n^p \leqq 2^p \{ EM_{2^h}^p + EM_{2^h, 2^h}^p \} \leqq 2^{p+1} C g(2^h) \leqq C_1 g(n), \tag{4.14}$$

where C_1 is a positive constant. The last inequality in (4.14) follows from (2.2). Therefore we have proved (4.7) for $a=0$ and $n=1, 2, \ldots$. □

While the moment inequality in Theorem 5 relates $EM_{a,n}^p$ to the upper bound $g(n)$ of $ES_{a,n}^p$, a straightforward modification of its proof also yields in the following theorem an analogous maximal inequality relating the tail probability of $M_{a,n}$ to that of $S_{a,n}$.

Theorem 6. *Let* $p > 0$. *Suppose* $g: \{1, 2, \ldots\} \to (0, \infty)$ *satisfies* (1.2) *and* (4.5). *If*

$$P[|S_{a,n}| \geqq x] \leqq x^{-p} g(n) \quad \textit{for all } x \geqq 0,\ a \geqq 0 \textit{ and } n \geqq 1, \tag{4.15}$$

then there exists a positive constant C *such that*

$$P[M_{a,n} \geqq x] \leqq C x^{-p} g(n) \quad \textit{for all } x > 0,\ a \geqq 0 \textit{ and } n \geqq 1. \tag{4.16}$$

As a corollary of Theorem 6, we now obtain a Marcinkiewicz-Zygmund type strong law for dependent random variables. First if $X_1, X_2, \ldots$ are i.i.d. and

$0<p<2$, then the classical Marcinkiewicz-Zygmund strong law says that the following two statements are equivalent:

$$n^{-1/p} S_n \to 0 \quad \text{a.s.}; \tag{4.17}$$

$$E|X_1|^p < \infty \qquad \text{and in the case } \; p \geqq 1, \; EX_1 = 0. \tag{4.18}$$

Let $g(n) = 2n\,E|X_1|^p$. Using the Esseen-von Bahr inequality [3] for the case $1<p<2$ and the i.i.d. structure, it is easy to see that (4.18) implies

$$E|S_{a,n}|^p \leqq g(n) \qquad \text{for all } \; a \geqq 0, \; n \geqq 1. \tag{4.19}$$

Moreover, (4.17) can be rewritten as

$$(g(n))^{-1/p} S_n \to 0 \quad \text{a.s.} \tag{4.20}$$

In the general situation without the i.i.d. assumption, it is natural to ask whether for $0<p<2$, (4.19) is strong enough to guarantee (4.20). The answer turns out to be negative. For example, even in the i.i.d. case, if $EX_1=0$ and $EX_1^2<\infty$, then $E|S_n|^p \ll n^{p/2}$ for $p \leqq 2$ and therefore (4.19) also holds with $g(n)=cn^{p/2}$, where c is some sufficiently large positive constant. However, $(cn^{p/2})^{-1/p} S_n = S_n/(c^{1/p} n^{1/2})$ clearly does not converge to 0 a.s. Although (4.19) does not necessarily imply (4.20), the following theorem says that a somewhat weaker assumption than (4.19) implies a slightly weaker conclusion than (4.20).

Theorem 7. *Let* $p>0$. *Suppose that* $g\colon \{1, 2, \ldots\} \to (0, \infty)$ *satisfies* (1.2) *and* (4.5). *If* (4.15) *holds, then for every* $\delta>0$ *and* $k=1, 2, \ldots$,

$$\lim_{n\to\infty} S_n/\{g(n)(\log n)\ldots(\log_k n)^{1+\delta}\}^{1/p} = 0 \quad \textit{a.s.} \tag{4.21}$$

Remarks. (i) Obviously (4.19) implies (4.15) by the Čebyšev inequality.

(ii) For $p=2$, the conclusion (4.21) is not much weaker than the iterated logarithm result (1.3).

(iii) Since $g(n) \gg n^\gamma$ by (2.3), we can replace $\log_h n$ by $\log_h g(n)$ $(h=1, 2, \ldots, k)$ in (4.21) (and also in (1.3) and (1.9)). Hence (4.21) is only slightly weaker than (4.20).

(iv) Under the assumptions of Theorem 4 and assuming further that $f(2n)/f(n)$ is bounded, Serfling [14] has proved that (4.21) holds with $g(n) = f^{p/2}(n)$. $p>2$, $k=2$, and $\delta=1$. In Theorem 7 we are able to drop the assumption that $p>2$ so that the result is closer in spirit to the classical Marcinkiewicz-Zygmund law. Moreover, our conditions on g are considerably weaker, and in particular, we are able to drop the boundedness assumption on $g(2n)/g(n)$ by using the argument in the following proof which is different from that of Serfling [14].

Proof of Theorem 7. Let $b(n)=g(n)(\log n)\ldots(\log_k n)^{1+\delta}$. We shall write $S(n)$ instead of S_n and $M(a, n)$ instead of $M_{a,n}$. Let $\varepsilon>0$. By Theorem 6,

$$P[|S(2^j)| \geqq \varepsilon b^{1/p}(2^j)] \leqq \varepsilon^{-p}(b(2^j))^{-1}\, g(2^j), \tag{4.22}$$

and

$$P[M(2^j, 2^j) \geqq \varepsilon b^{1/p}(2^j)] \leqq C\varepsilon^{-p}(b(2^j))^{-1} g(2^j). \tag{4.23}$$

Therefore by the Borel-Cantelli lemma,

$$\max_{2^j \leqq n \leqq 2^{j+1}} |S_n|/b^{1/p}(2^j) \to 0 \quad \text{a.s.} \tag{4.24}$$

From (2.2) and (4.24), the desired conclusion (4.21) follows. □

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