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Strong Invariance Principles for Mixing Random Fields

István Berkes[1] and Gregory J. Morrow[2]

[1] Mathematical Institute of the Hungarian Academy of Sciences,
1053 Budapest, Reáltanoda u. 13–15, Hungary
[2] Department of Mathematics, Washington University, Box 1146, St. Louis, Mo. 63130, USA

Summary. Given a random field $\{\xi_\nu, \nu\in Z^q_+\}$ indexed by q-tuples of positive integers and satisfying a strong mixing condition we study the approximation of the partial sum field $\{S_n, n\in Z^q_+\}$ by Brownian sheet. Setting

$$G_\alpha=\{(n_1, \dots, n_q)\in Z^q_+ : n_k \geqq (\prod_{1\leqq i\leqq q,\, i\neq k} n_i)^\alpha,\ k=1, \dots, q\}$$

for $0<\alpha<1$ we show that in the domain G_α the approximation $S_n-W(n)=O([n]^{1/2-\lambda})$ a.s. is possible where $\lambda>0$. We also construct an example showing that in a somewhat larger, similar type domain the above approximation is generally impossible, even with $\lambda=0$.

1. Introduction

Let Z^q (resp. Z^q_+) denote the set of all q-dimensional vectors with integral (resp. positive integral) coordinates. Let $\{\xi_\nu, \nu\in Z^q_+\}$ be a random field with values in R^N, that is a collection of N-dimensional random vectors indexed by Z^q_+. We assume that

$$E\xi_\nu=0, \qquad E|\xi_\nu|^{2+\delta}\leqq C \qquad \nu\in Z^q_+ \tag{1.1}$$

for some $\delta>0$, $C>0$ and that $\{\xi_\nu, \nu\in Z^q_+\}$ satisfies the following mixing condition:

$$\begin{aligned}\rho(S_1, S_2) &\overset{\text{def}}{=} \sup_{\substack{A\in\sigma\{\xi_\nu:\ \nu\in S_1\}\\ B\in\sigma\{\xi_\nu:\ \nu\in S_2\}}} |P(AB)-P(A)\,P(B)|\\ &\leqq C_1\big(\inf_{\substack{\mu\in S_1\\ \nu\in S_2}}|\mu-\nu|\big)^{-q(1+\varepsilon)(1+2/\delta)}\end{aligned} \tag{1.2}$$

for some $0<\varepsilon<1/2$, constant C_1 and any disjoint non-empty sets S_1, $S_2\subset Z^q_+$. (Here $\sigma\{\cdot\}$ denote the σ-field generated by the random variables in the brackets.) Set

$$\xi_\nu=(\xi_\nu^{(1)}, \dots, \xi_\nu^{(N)}) \qquad \nu\in Z^q_+.$$

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We call the random field $\{\xi_\nu, \nu \in Z_+^q\}$ weakly stationary if $E\xi_\mu^{(i)}\xi_\nu^{(j)}$ ($\mu, \nu \in Z_+^q$, $1 \leqq i, j \leqq N$) depends only on $\nu - \mu$ i.e. if there exist real valued functions $r_{i,j}$, $1 \leqq i, j \leqq N$, each having domain Z^q, such that

$$r_{i,j}(\nu-\mu) = E\xi_\mu^{(i)}\xi_\nu^{(j)}, \qquad \mu, \nu \in Z_+^q,\ 1 \leqq i, j \leqq N. \tag{1.3}$$

For any $\mu, \nu \in Z^q$ the relations $\mu \leqq \nu$ (resp. $\mu < \nu$) are defined to mean $\mu_i \leqq \nu_i$ (resp. $\mu_i < \nu_i$) for $1 \leqq i \leqq q$ where $\mu = (\mu_1, \ldots, \mu_q)$, $\nu = (\nu_1, \ldots, \nu_q)$. Set $e = (1, 1, \ldots, 1_q)$, $0 = (0, 0, \ldots, 0_q)$ and define for $\nu \in Z_+^q$

$$[\nu] = \prod_{1 \leqq i \leqq q} \nu_i$$

where $\nu = (\nu_1, \ldots, \nu_q)$. Put finally, for any $d \in (0, 1)$

$$G_d = \bigcap_{k=1}^{q} \{\nu \in Z_+^q : \nu_k \geqq \prod_{1 \leqq l \leqq q,\, l \neq k} \nu_l^d\}. \tag{1.4}$$

Our main goal is to prove the following theorem:

Theorem 1. *Let $q \geqq 2$, $d \in (0, 1)$ and let $\{\xi_\nu, \nu \in Z_+^q\}$ be a random field with values in R^N for some $N \geqq 1$. Suppose that* (1.1), (1.2) *hold and that the field is weakly stationary. These hypotheses imply that the series*

$$\gamma_{i,j} = \sum_{\nu \in Z^q} r_{i,j}(\nu) \tag{1.5}$$

are absolutely convergent for $1 \leqq i, j \leqq N$ where the covariance function $r_{i,j}$ is determined by (1.3). *Moreover, $\Gamma = (\gamma_{i,j})^{N \times N}$ is nonnegative definite. Then, without changing its distribution the random field $\{\xi_\nu, \nu \in Z_+^q\}$ can be redefined on a new probability space together with a q-parameter Wiener process $\{W(\tau), \tau \in [0, \infty)^q\}$ in R^N with covariance matrix Γ such that*

$$\sup_{n \in G_d} [n]^{\lambda - 1/2} \sup_{e \leqq \nu \leqq n} \Big|\sum_{\mu \leqq \nu} \xi_\mu - W(\nu)\Big| < \infty \quad \text{a.s.} \tag{1.6}$$

Here G_d is defined by (1.4) *and λ is a positive constant depending on the field $\{\xi_\nu, \nu \in Z_+^q\}$.*

A q-parameter Wiener process $\{W(\tau), \tau \in [0, \infty)^q\}$ in R^N with covariance matrix Γ means a (Gaussian) process with values in R^N, with independent increments such that $W(\tau) = 0$ if any of the coordinates of τ vanishes and the increment of W over a rectangle R has normal distribution with mean 0 and covariance matrix $|R|\Gamma$ where $|R|$ is the volume of R.

It is not hard to find an explicit value for λ, for instance, $\lambda = \varepsilon^{6q}\delta^5 d^4/8^{q+9}$ will do.

Theorem 1 says that under the given conditions there exists a q-parameter Wiener process W with convariance matrix Γ such that

$$\sup_{e \leqq \nu \leqq n} \Big|\sum_{\mu \leqq \nu} \xi_\mu - W(\nu)\Big| = O([n]^{1/2 - \lambda}) \quad \text{a.s.} \tag{1.7}$$

holds for $n \in G_d$. The stipulation "$n \in G_d$" is essential here: in general there exists no Wiener process W such that (1.7) holds for all $n \in Z_+^q$ (cf. Theorem 3 below). However, $n \in G_d$ can be somewhat relaxed at the cost of having a weaker error term in (1.7). Put, for instance,

$$G_\beta^* = \bigcap_{k=1}^{q} \{v \in Z_+^q : v_k \geqq \log^\beta (\prod_{1 \leqq l \leqq q,\, l \neq k} v_l)\} \tag{1.8}$$

for $q \geqq 2$ and $\beta > 0$. Then we have

Theorem 2. *Theorem* 1 *remains valid if relation* (1.6) *is changed to*

$$\sup_{n \in G_\beta^*} [n]^{-1/2} (\log\log [n])^\lambda \sup_{e \leqq v \leqq n} |\sum_{\mu \leqq v} \xi_\mu - W(v)| < \infty \quad \text{a.s.} \tag{1.9}$$

for suitable positive constants β and λ.

The proof of Theorem 2 yields (1.9) with a large β, if actually (1.9) is valid for every $\beta > 0$ remains open.

On the basis of Theorems 1 and 2 one might perhaps conjecture that under the conditions of Theorem 1 there exists also a Wiener process W such that

$$\sup_{n \in G_\beta^{**}} \varphi([n])^{-1} \sup_{e \leqq v \leqq n} |\sum_{\mu \leqq v} \xi_\mu - W(v)| < \infty \quad \text{a.s.}$$

holds for some function $\varphi(t) = o(t^{1/2})$ $(t \to \infty)$ where

$$G_\beta^{**} = \bigcap_{k=1}^{q} \{v \in Z_+^q : v_k \geqq (\log\log)^\beta (\prod_{1 \leqq l \leqq q,\, l \neq k} v_l)\}$$

(here $(\log\log)^\beta x = (\log\log x)^\beta$). However, as Theorem 3 below shows, this is not the case if $\beta < 1/2$. For simplicity, we consider the case $q = 2$. Set, for any function $0 \leqq f(t) \leqq t$

$$G_f = \{v \in Z_+^2 : v_1 \geqq f(v_2),\, v_2 \geqq f(v_1)\}.$$

Then we have

Theorem 3. *There exists a stationary 2-dependent Gaussian random field $\{\xi_v, v \in Z^2\}$ such that $E\xi_v = 0$, $\gamma_0 = \sum_{v \in Z^2} E\xi_0 \xi_v = 1$ and, for any standard Wiener process $\{W(t), t \in [0, \infty)^2\}$ and any positive nondecreasing function $f(t)$, $t \geqq 0$ satisfying the conditions*

$$f(t) \leqq c_1 (\log\log t)^{1/2} (t \geqq t_0) \quad \textit{for a sufficiently small } c_1 > 0 \tag{1.10}$$

$$\sup_{t \geqq 1} |f(2t) - f(t)| < \infty \tag{1.11}$$

the approximation

$$\lim_{\substack{[n] \to \infty \\ n \in G_f}} ([n] \log\log [n])^{-1/2} f([n]) |\sum_{e \leqq v \leqq n} \xi_v - W(n)| = 0 \quad \text{a.s.} \tag{1.12}$$

cannot hold[1].

[1] Here (1.12) is to be understood sequentially i.e. in the sense that the given limit relation holds with probability one along every sequence $n_k \in Z_+^q$ satisfying $[n_k] \to \infty$, $n_k \in G_f$, the exceptional zero set depending on $\{n_k\}$.

Choosing $f(t)=(\log\log t)^\beta$, $\beta<1/2$ we get our previous remark. Choosing $f(t)=1$ we get that for the field $\{\xi_v, v\in Z^2\}$ there exists no Wiener process W such that

$$\sum_{e\leqq v\leqq n} \xi_v - W(n) = o([n] \log\log [n])^{1/2} \quad \text{a.s. as } [n]\to\infty. \tag{1.13}$$

It should be remarked that Theorem 3 does not imply the impossibility of the approximation

$$\sum_{e\leqq v\leqq n} \xi_v - W(n) = o([n] \log\log [n])^{1/2} \quad \text{a.s. as } n_1 \wedge \ldots \wedge n_q \to\infty \tag{1.14}$$

where $n=(n_1, \ldots, n_q)$; whether (1.14) is possible or not remains open. However, Theorem 3 implies that, if $\varphi(t)$ is any function satisfying $\varphi(t)=o(t\log\log t)^{1/2}$, $t\to\infty$ then

$$\sum_{e\leqq v\leqq n} \xi_v - W(n) = o(\varphi([n])) \quad \text{a.s. as } n_1 \wedge \ldots \wedge n_q \to\infty$$

is in general impossible.

The above results show that though the partial sum process $\{S_n, n\in Z^q_+\}$ of a mixing random field $\{\xi_v, v\in Z^q_+\}$ can be well approximated by Wiener process "far away" from the coordinate planes $p_k=\{n_k=0\}$ $(k=1, \ldots, q)$, the degree of approximation gets, in general, worse as we approach the planes p_k. The reason of this phenomenon is, as we shall see, the irregular behaviour of ES_n^2 close to the coordinate planes: while ES_n^2 is approximately proportional to $EW(n)^2$ away from the planes p_k, this proportionality breaks down gradually as we get closer to these planes. There are, of course, special classes of mixing random fields (e.g. orthogonal fields) for which ES_n^2 is proportional to $EW(n)^2$ everywhere in Z^q_+; it is natural to ask if for such fields a good approximation of the type (1.7) can be obtained for all $n\in Z^q_+$. The answer is in the affirmative; the proof is, however, more involved than that of Theorem 1 and will not be given here. In the present paper we will show only (see the end of Sect. 7) that the approximation (1.13) holds in the simple case when ξ_v are independent. (As a matter of fact, for independent ξ_v, $v\in Z^q_+$ with finite $(2+\delta)$th moments the remainder term $o([n]\log\log[n])^{1/2}$ is far from the best possible one but we do not investigate this question here.) For further related results we mention the forthcoming paper Morrow (1980) where a.s. invariance principles of the form (1.13) are proved for i.i.d.r.v.'s ξ_v, $v\in Z^q_+$, taking values from a separable Banach space B and satisfying the moment condition

$$\xi_v \in L^2 \log^{q-1} L, \qquad E\xi_v = 0. \tag{1.15}$$

As is shown in the just mentioned paper, under these conditions (1.13) holds for some B-valued q-parameter Wiener process W iff ξ_e is pregaussian and

$$\lim_{[n]\to\infty} ([n] \log\log[n])^{-1/2} \sum_{v\leqq n} \xi_v = 0$$

in probability. In the Hilbert space case the first condition of (1.15) can be weakened to $\xi_v\in L^2 \log^{q-1} L/\log\log L$ for $q\geqq 2$ which is then also necessary.

Our next theorem states the analogue of Theorems 1 and 2 for Gaussian fields.

Theorem 4. *Let* $q \geqq 2$ *and let* $\{\xi_\nu, \nu \in Z_+^q\}$ *be a (not necessarily stationary) Gaussian random field satisfying the conditions*

$$E\xi_\nu = 0, \qquad E\xi_\nu^2 \leqq C \qquad \nu \in Z_+^q. \tag{1.16}$$

$$|E\xi_\mu \xi_\nu| \leqq \text{const} \cdot |\mu - \nu|^{-q(1+\varepsilon)} \qquad \mu, \nu \in Z_+^q, \ \mu \neq \nu \tag{1.17}$$

and

$$|E(\sum_{m+e \leqq \nu \leqq m+n} \xi_\nu)^2 - [n]| \leqq \text{const} \cdot [n]^{1-\delta}, \qquad m \geqq 0, \ n \in G_d \tag{1.18}$$

for some $C>0$, $\varepsilon>0$, $\delta>0$, $0<d<1$. *Then the conclusion of Theorem* 1 *holds with* Γ *replaced by* 1. *If we replace condition* (1.18) *by*

$$|E(\sum_{m+e \leqq \nu \leqq m+n} \xi_\nu)^2 - [n]| \leqq \text{const} \cdot [n] \log^{-\gamma}[n], \qquad m \geqq 0, \ n \in G_\beta^*$$

for some $\gamma>0$ *then the conclusion of Theorem* 2 *will hold with* $\Gamma=1$.

Notice that the hypotheses of Theorem 4 are satisfied for any stationary Gaussian field with mean zero whose convariances $r(\nu)$ satisfy

$$|r(\nu)| \leqq \text{const} \cdot |\nu|^{-q(1+\varepsilon)} \qquad \nu \in Z^q, \ \nu \neq 0$$

and

$$\sum_{\nu \in Z^q} r(\nu) = 1$$

for some $\varepsilon>0$. (See the proof of Lemmas 3, 3(*).) Notice also that the Gaussian field $\{\xi_\nu, \nu \in Z^2\}$ of Theorem 3 satisfies these latter conditions and thus the counter-examples mentioned in connection with Theorems 1 and 2 are in force also in connection with Theorem 4.

Theorems 1 and 3 will be proved in Sects. 2–6 and in Sect. 7, respectively. The proofs of Theorems 2 and 4 are very similar to that of Theorem 1 and will only be sketched in Sect. 8.

2. Preliminary Lemmas

We shall eventually require a central limit theorem with remainder for certain normalized rectangular sums of $\{\xi_\nu, \nu \in Z_+^q\}$. The lemmas of the present section will be instrumental in obtaining such a result.

For the purposes of this section we define the symbol $[\nu]$ more generally than in the Introduction: for any $\nu \in Z^q$, $\nu \neq 0$ set

$$[\nu] = \prod_{i:\, \nu_i \neq 0} |\nu_i|.$$

For $\nu \in Z_+^q$ this clearly coincides with our earlier definition. Except (2.1) and the proof of Lemma 3, $[\nu]$ will only be used for $\nu \in Z_+^q$.

Lemma 1 (Dvoretzky 1970). *Let ξ be a (possibly complex-valued) random variable with $|\xi|\leqq 1$ and let $\mathscr{F}$ be the σ-field generated by ξ. Then for any σ-field $\mathscr{G}$*

$$E|E(\xi|\mathscr{G})-E\xi|\leqq 2\pi\cdot \sup_{A\in\mathscr{F},\, B\in\mathscr{G}} |P(AB)-P(A)P(B)|.$$

Lemma 2 (Davidov 1970). *Let ξ and η be (possibly complex-valued) random variables measurable $\mathscr{F}$ and $\mathscr{G}$, respectively. Let $p_1, p_2, p_3\geqq 1$ with $p_1^{-1}+p_2^{-1}+p_3^{-1}=1$. If $\|\xi\|_{p_1}<\infty$ and $\|\eta\|_{p_2}<\infty$ then*

$$|E\xi\eta-E\xi\cdot E\eta|\leqq 10(\sup_{\substack{A\in\mathscr{F}\\ B\in\mathscr{G}}}|P(AB)-P(A)P(B)|)^{1/p_3}\|\xi\|_{p_1}\|\eta\|_{p_2}.$$

A consequence of Lemma 2 is found by setting $p_1=p_2=2+\delta$ and $p_3=(2+\delta)/\delta$ and taking (1.1), (1.2) into account. One obtains

$$|E\xi_\mu^{(i)}\xi_\nu^{(j)}|\leqq \text{const}\cdot|\nu-\mu|^{-q(1+\varepsilon)}\leqq \text{const}\cdot[\nu-\mu]^{-(1+\varepsilon)} \tag{2.1}$$

for any $\mu, \nu\in Z_+^q$, $\mu\neq\nu$ and $1\leqq i, j\leqq N$.

Lemma 3. *Assume the hypotheses of Theorem 1. Let $n=(n_1, \ldots, n_q)\in G_d$ where G_d is defined in* (1.4). *Then,*

$$E(\sum_{e\leqq\mu\leqq n}\sum_{e\leqq\nu\leqq n}\xi_\mu^{(i)}\xi_\nu^{(j)})=[n](\gamma_{i,j}+\text{const}\cdot\theta\cdot[n]^{-\varepsilon d/2}) \tag{2.2}$$

where $\gamma_{i,j}$ is defined in Theorem 1. *Moreover, $\Gamma=(\gamma_{i,j})^{N\times N}$ is non-negative definite. (Here, and in the sequel, θ denotes various numbers satisfying $|\theta|\leqq 1$ and all the constants will depend on N, q and the field $\{\xi_\nu\}$.)*

Proof. By weak stationarity,

$$\begin{aligned} E(\sum_{e\leqq\mu\leqq n}\sum_{e\leqq\nu\leqq n}\xi_\mu^{(i)}\xi_\nu^{(j)}) &= \sum_{-n\leqq\nu\leqq n} r_{i,j}(\nu)\prod_{1\leqq k\leqq q}(n_k-|\nu_k|)\\ &=[n]\Bigg(\sum_{-n\leqq\nu\leqq n} r_{i,j}(\nu)\\ &\quad+\text{const}\cdot\theta\cdot\sum_{k=1}^{q}\sum_{-n\leqq\nu\leqq n,\,\nu\neq 0}|r_{i,j}(\nu)|\sum\frac{|\nu_{l_1}\ldots\nu_{l_k}|}{n_{l_1}\ldots n_{l_k}}\Bigg) \end{aligned} \tag{2.3}$$

where the innermost sum is extended over all $1\leqq l_1<\ldots<l_k\leqq q$. Clearly, the just mentioned sum is

$$\leqq(\sum_{1\leqq i\leqq q}|\nu_i|/n_i)^k\leqq \text{const}\cdot\sum_{1\leqq i\leqq q}|\nu_i|/n_i,$$

since $|\nu_i|\leqq n_i$ for $1\leqq i\leqq q$. Also, by (2.1),

$$|r_{i,j}(\nu)|\leqq \text{const}\cdot[\nu]^{-(1+\varepsilon)} \quad (\nu\neq 0)$$

and an easy calculation shows

$$\sum_{\substack{\nu\in Z^q\\ \nu\neq 0}}[\nu]^{-(1+\varepsilon)}<\infty, \qquad \sum_{\substack{\nu\in Z^q\\ |\nu_k|\geqq L}}[\nu]^{-(1+\varepsilon)}\leqq \text{const}\cdot L^{-\varepsilon}$$

for $L>0$, $1\leqq k\leqq q$. Therefore

$$\sum_{k=1}^{q}\sum_{-n\leqq v\leqq n,\, v\neq 0}[v]^{-(1+\varepsilon)}|v_k|/n_k$$
$$\leqq \sum_{k=1}^{q}\Big(\sum_{\substack{v\in Z^q,\, v\neq 0\\ |v_k|\leqq\sqrt{n_k}}}[v]^{-(1+\varepsilon)}n_k^{-1/2}+\sum_{\substack{v\in Z^q\\ |v_k|>\sqrt{n_k}}}[v]^{-(1+\varepsilon)}\Big)$$
$$\leqq \text{const}\cdot\sum_{k=1}^{q}(n_k^{-1/2}+n_k^{-\varepsilon/2}).$$

Also, observe that

$$n\in G_d \quad \text{implies} \quad n_k\geqq[n]^{d/2} \qquad (1\leqq k\leqq q). \tag{2.4}$$

These considerations show that the last expression in (2.3) is $[n](\gamma_{i,j}+\text{const}\cdot\theta\cdot[n]^{-\varepsilon d/2})$ i.e. (2.2) holds. Apply now the just proved statement of Lemma 3 to the random field $\eta_v=|u|^{-1}\langle u,\xi_v\rangle$ $(u\in R^N)$ to get

$$E\Big(\sum_{e\leqq v\leqq n}\langle u,\xi_v\rangle\Big)^2=[n](\langle u,\Gamma u\rangle+\text{const}\cdot\theta\cdot|u|^2[n]^{-\varepsilon d/2}) \qquad n\in G_d$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product. Thus,

$$\lim_{[n]\to\infty,\, n\in G_d}[n]^{-1}E\Big(\sum_{e\leqq v\leqq n}\langle u,\xi_v\rangle\Big)^2=\langle u,\Gamma u\rangle\geqq 0$$

i.e. Γ is nonnegative definite.

Lemma 4. *Assume that the hypotheses of Theorem 1 hold. Then there exist two constants $0<\alpha\leqq\delta$ and $B>0$ such that*

$$E\Big|\sum_{\mu+e\leqq v\leqq\mu+n}\xi_v\Big|^{2+\alpha}\leqq B[n]^{1+\alpha/2} \tag{2.5}$$

uniformly for $\mu=(\mu_1,\ldots,\mu_q)\geqq 0$ and $n=(n_1,\ldots,n_q)\in Z_+^q$.

Proof. We can assume $N=1$ since (2.5) can be reduced to showing the analogous inequality for the coordinate fields $\{\xi_v^{(i)}, v\in Z_+^q\}$, $1\leqq i\leqq q$. We shall also assume $q=2$ since the proof in this case reveals that the lemma is proved in general by induction on q. Define

$$S_a(y)=n_1^{-1/2}\sum_{a+1\leqq v_1\leqq a+n_1,\, v_2=y}\xi_v$$

for each $y=1,2,\ldots,a\geqq 0$. According to Lemma (2.5) of Kuelbs and Philipp (1979) we have uniformly in a and y

$$E|S_a(y)|^{2+\alpha_1}\leqq B_1$$

for some positive constants $\alpha_1\leqq\delta$ and B_1. Then, because the random variables $\{S_a(y), y=1,2,\ldots\}$ are mixing in y and have zero means and uniformly bounded $(2+\alpha_1)$-th moments, we conclude by the same reasoning that

$$T_{a,b}=n_2^{-1/2}\sum_{b+1\leqq y\leqq b+n_2}S_a(y)$$

satisfies

$$E|T_{a,b}|^{2+\alpha} \leqq B$$

uniformly in a, b where α and B are positive constants.

3. The Characteristic Function

For each $\tau=(\tau_1, \ldots, \tau_q) \geqq 0$, $n=(n_1, \ldots, n_q) \in Z_+^q$ and $u \in R^N$ define

$$f_{n,\tau}(u)=E \exp(i\langle u, [n]^{-1/2} \sum_{\tau+e \leqq v \leqq \tau+n} \xi_v\rangle).$$

The statement and proof of the following lemma encompass the subject matter of this section.

Lemma 5. *Under the hypotheses of Theorem* 1 *there exists a constant* $t \in (0,1)$ *such that*

$$\sup_{|u| \leqq [n]^t} |f_{n,\tau}(u)-\exp(-\tfrac{1}{2}\langle u, \Gamma u\rangle)| \leqq \text{const} \cdot [n]^{-t} \tag{3.1}$$

uniformly for $\tau \geqq 0$ *and* $n \in G_d$ *where* G_d *is defined in* (1.4). *If the random vectors* ξ_v *are independent, one obtains* (3.1) *with the stipulation* $n \in G_d$ *removed.*

Proof. The uniformity in τ is a consequence of the assumptions of zero means and weak stationarity and Lemma 4. This will be evident from our demonstration that (3.1) holds when $\tau=0$. To begin, define:

$$L_v=\{\mu \in Z_+^q : (v_k-1)\, n_k^{1-w} < \mu_k \leqq v_k \cdot n_k^{1-w}; k=1,2,\ldots,q\}$$

for each $v \in Z_+^q$ where w is a number satisfying $0<w<\alpha d/16\delta$ and α is the constant appearing in Lemma 4. Set

$$L=\bigcup_{1 \leqq v_k \leqq n_k^w,\, k=1,2,\ldots,q} L_v$$

(where the index set means $\{v: 1 \leqq v_k \leqq n_k^w$ for $1 \leqq k \leqq q\}$). Next, put

$$K_v=\bigcup_{k=1}^{q} \{\mu \in L_v: v_k\, n_k^{1-w}-\mu_k \leqq n_k^{1/2}\}$$

and set

$$a_v=\sum_{\mu \in L_v \setminus K_v} \xi_\mu, \qquad r_v=\sum_{\mu \in K_v} \xi_\mu. \tag{3.2}$$

From our choice of w (note that $w \leqq d/16$) and the assumption that $n \in G_d$ one finds that

$$\begin{aligned}\sum_{1 \leqq v_k \leqq n_k^w,\, k=1,2,\ldots,q} (\text{card}\, K_v)^{1/2} &\leqq [n]^w \left(\sum_{k=1}^{q} n_k^{1/2} \prod_{l \neq k} n_l^{1-w}\right)^{1/2} \\ &\leqq [n]^w \sum_{k=1}^{q} n_k^{1/4} \prod_{l \neq k} n_l^{(1-w)/2} \\ &\leqq \text{const} \cdot [n]^{1/2-w}. \end{aligned} \tag{3.3}$$

Furthermore, from (1.2), (2.4) and $q \geqq 2$ there results

$$\rho(L_\mu \setminus K_\mu, L_\nu \setminus K_\nu) \leqq \text{const} \cdot (\min_{1 \leqq k \leqq q} n_k^{1/2})^{-q(1+\varepsilon)(1+2/\delta)} \leqq \text{const}\, [n]^{-d(1+\varepsilon)(1+2/\delta)/2} \tag{3.4}$$

if $\mu \neq \nu$ and $n \in G_d$.

Let us now express

$$f_{n,0}(u) - \exp(-\langle u, \Gamma u\rangle/2)$$

as the sum $\varepsilon_1 + \varepsilon_2 + \varepsilon_3$ where

$$\varepsilon_1 = f_{n,0}(u) - E \exp(i\langle u, [n]^{-1/2} \sum_{1 \leqq \nu_k \leqq n_k^w,\, k=1,\dots,q} a_\nu\rangle)$$

$$\varepsilon_2 = E \exp(i\langle u, [n]^{-1/2} \sum_{1 \leqq \nu_k \leqq n_k^w,\, k=1,\dots,q} a_\nu\rangle) - \prod_{1 \leqq \nu_k \leqq n_k^w,\, k=1,\dots,q} E \exp(i\langle u, [n]^{-1/2} a_\nu\rangle)$$

$$\varepsilon_3 = \prod_{1 \leqq \nu_k \leqq n_k^w,\, k=1,\dots,q} E \exp(i\langle u, [n]^{-1/2} a_\nu\rangle) - \exp(-\langle u, \Gamma u\rangle/2)$$

and a_ν is defined in (3.2). From familiar facts about the expectation operator and the inequality

$$|\exp(ia) - \exp(ib)| \leqq |a-b|$$

valid for real numbers a and b, one readily observes that

$$|\varepsilon_1| \leqq |u|\,[n]^{-1/2} E\Big|\sum_{1 \leqq \nu_k \leqq n_k^w,\, k=1,\dots,q} r_\nu + \sum_{e \leqq \nu \leqq n,\, \nu \notin L} \xi_\nu\Big|.$$

The Hölder and Minkowski inequalities thus yield

$$|\varepsilon_1| \leqq \text{const}\,|u|\,[n]^{-1/2}\{E^{1/(2+\alpha)}\Big|\sum_{1 \leqq \nu_k \leqq n_k^w,\, k=1,\dots,q} r_\nu\Big|^{2+\alpha} + E^{1/(2+\alpha)}\Big|\sum_{e \leqq \nu \leqq n,\, \nu \notin L} \xi_\nu\Big|^{2+\alpha}\}.$$

The set K_ν can evidently be written as a union of q disjoint rectangles and thus r_ν in (3.2) can be represented as $\sum_{1 \leqq l \leqq q} r_\nu(l)$ where the $r_\nu(l)$'s are rectangular sums of the random variables ξ_μ. Then, by Minkowski's inequality, Lemma 4 and (3.3) we have

$$E^{1/(2+\alpha)}\Big|\sum_{1 \leqq \nu_k \leqq n_k^w,\, k=1,\dots,q} r_\nu\Big|^{2+\alpha} \leqq \sum_{l=1}^{q} \sum_{1 \leqq \nu_k \leqq n_k^w,\, k=1,\dots,q} E^{1/(2+\alpha)} |r_\nu(l)|^{2+\alpha} \leqq \text{const}\, [n]^{1/2-w}.$$

Similarly, decomposing $\sum_{e \leqq \nu \leqq n,\, \nu \notin L} \xi_\nu$ into a sum of q rectangular sums, we have, using $n \in G_d$,

$$E^{1/(2+\alpha)}\Big|\sum_{e \leqq \nu \leqq n,\, \nu \notin L} \xi_\nu\Big|^{2+\alpha} \leqq \text{const} \sum_{k=1}^{q} (n_k^{1-w} \prod_{l \neq k} n_l)^{1/2} \leqq \text{const}\, [n]^{1/2-w_1}$$

for a suitable small $0 < w_1 < w$. Thus, $|\varepsilon_1| \leqq \text{const} \cdot |u|\,[n]^{-w_1}$.

Next, by repeated applications of Lemma 2 with $p_1 = p_2 = 2+\alpha$ and $p_3 = 1 + 2/\alpha$ one has from (3.4) and our choice of w that

$$|\varepsilon_2| \leqq \text{const} \cdot [n]^{w - d(1+\varepsilon)(1+2/\delta)(1+2/\alpha)^{-1}/2} \leqq \text{const} \cdot [n]^{w - \alpha d/2\delta} \leqq \text{const} \cdot [n]^{-w}.$$

Finally, we apply Lemma 3 to the random field $\eta_v = |u|^{-1} \langle u, \xi_v \rangle$ which clearly satisfies the conditions of Theorem 1. Observing that the volume of L_v / K_v is

$$[n]^{1-w}\left(1 + \theta \sum_{k=1}^{q} n_k^{-(1/2-w)}\right) = [n]^{1-w}(1 + \text{const} \cdot \theta [n]^{-d/8})$$

(see (2.4)) we get

$$\begin{aligned} E\langle u, a_v\rangle^2 &= [n]^{1-w}(1 + \text{const} \cdot \theta [n]^{-d/8})(\langle u, \Gamma u\rangle + c\theta |u|^2 [n]^{-\varepsilon d/2}) \\ &= [n]^{1-w}(\langle u, \Gamma u\rangle + c\theta [n]^{-w'}) \end{aligned} \tag{3.5}$$

for $|u| \leqq [n]^{w'}$ with some $0 < w' < w$ and a constant c'. Therefore, by a well known expansion of the characteristic function of a random variable with a $(2+\alpha)$-th moment (Loève (1977) p. 212)

$$E \exp(i\langle u, [n]^{-1/2} a_v\rangle) = 1 - \tfrac{1}{2}[n]^{-1} E\langle u, a_v\rangle^2 + c_\alpha \theta [n]^{-1-\alpha/2} E|\langle u, a_v\rangle|^{2+\alpha}.$$

But by the Cauchy-Schwarz inequality and Lemma 4

$$E|\langle u, a_v\rangle|^{2+\alpha} \leqq |u|^{2+\alpha} E|a_v|^{2+\alpha} \leqq \text{const} \cdot |u|^{2+\alpha} [n]^{(1-w)(1+\alpha/2)}.$$

Hence, a routine calculation yields, by way of (3.5), that

$$\begin{aligned} |\varepsilon_3| &\leqq \exp(-\tfrac{1}{2}\langle u, \Gamma u\rangle) \cdot |\exp(-\tfrac{1}{2}[n]^{-1} \sum_{1 \leqq v_k \leqq n_k^w,\, k=1,\dots,q} E\langle u, a_v\rangle^2 \\ &\quad + \tfrac{1}{2}\langle u, \Gamma u\rangle + c'' \theta [n]^{-w_2} |u|^{4+2\alpha}) - 1| \\ &\leqq \text{const}\, [n]^{-t} \end{aligned}$$

for some constant $0 < t < 1$ provided that $|u| \leqq [n]^t$ (we use here that Γ is nonnegative definite). Thus, upon recalling the upper bounds computed for $|\varepsilon_1|$ and $|\varepsilon_2|$ we evidently have

$$\sup_{|u| \leqq [n]^t} (|\varepsilon_1| + |\varepsilon_2| + |\varepsilon_3|) \leqq \text{const}\, [n]^{-t}$$

for some $t > 0$ and $n \in G_d$. This proves Lemma 5 in the mixing case. Furthermore, when analyzing the characteristic function in the independent case, we have

$$\begin{aligned} &E \exp(i\langle u, [n]^{-1/2} \sum_{e \leqq v \leqq n} \xi_v\rangle) - \exp(-\langle u, \Gamma u\rangle/2) \\ &= \prod_{e \leqq v \leqq n} (1 - \tfrac{1}{2}[n]^{-1} E\langle u, \xi_v\rangle^2 + c_\delta \theta |u|^{2+\delta} E|\xi_v|^{2+\delta} [n]^{-1-\delta/2}) \\ &= \text{const} \cdot \theta \cdot \exp(-\langle u, \Gamma u\rangle/2) |u|^{4+2\delta} [n]^{-\delta/2} \end{aligned}$$

for $|u|\leqq[n]^{\delta/12}$ by the reasoning used above to establish a bound for $|\varepsilon_3|$. (In this case, of course, $\gamma_{i,j}=E\xi_e^{(i)}\xi_e^{(j)}$ since all the terms of the sum (1.5) vanish, except that belonging to $v=0$.) This completes the proof of Lemma 5.

4. Construction of Blocks

Let $d\in(0,1)$ and put $\rho=d/8$. Define, for any $\mu=(\mu_1,\ldots,\mu_q)\geqq 0$

$$t_\mu(k)=\sum_{l=1}^{\mu_k} l^\beta \qquad k=1,2,\ldots,q$$

for

$$\beta=[240\,N/\rho t\gamma] \tag{4.1}$$

where γ is the constant appearing in Lemma 7 below and t appears in Lemma 5. Put

$$t_\mu=(t_\mu(1),\ldots,t_\mu(q))$$

and

$$H_\mu=\{v\in Z_+^q: t_\mu+e\leqq v\leqq t_{\mu+e}\}$$

for $\mu\geqq 0$. Let

$$L=\{\mu\in Z_+^q: H_\mu\subset G_\rho\}$$

and write

$$H=\bigcup_{\mu\in L} H_\mu.$$

Also, for each $\mu\in L$, put

$$\Delta_\mu=\bigcup_{k=1}^{q}\{v\in H_\mu: t_{\mu+e}(k)-\mu_k^{9N/\rho}<v_k\leqq t_{\mu+e}(k)\} \tag{4.2}$$

and

$$x_\mu=\sum_{v\in H_\mu\setminus\Delta_\mu}\xi_v,\qquad z_\mu=\sum_{v\in\Delta_\mu}\xi_v. \tag{4.3}$$

Define further, for each $n\in H$ and $p=1,\ldots,q$

$$n^{(p)}=(n_1^{(p)},\ldots,n_q^{(p)})$$

by

$$n_k^{(p)}=\delta_{k,p}\min_{v\in H,\ v_l=n_l\ \mathrm{for}\ l\neq k} v_k+(1-\delta_{k,p})\,n_k \qquad k=1,2,\ldots,q.$$

As usual, $\delta_{\cdot,\cdot}$ stands for the Kronecker δ-function. Now let $\{W(\tau),\tau\in[0,\infty)^q\}$ be a q-parameter Wiener process in R^N with covariance matrix Γ and set

$$D_p(n)=\max_{v\leqq n^{(p)}}\Big|\sum_{\mu\leqq v}\xi_\mu\Big|,\qquad \hat{D}_p(n)=\max_{v\leqq n^{(p)}}|W(v)| \tag{4.4}$$

for each $p=1,\ldots,q$ and $n\in H$.

We prove in this section an almost sure bound for each of the above maximum terms.

Lemma 6. *Under the hypotheses of Theorem* 1 *we have*

$$\sup_{n \in G_d} [n]^{(\lambda_0 - 1)/2} \max_{p=1,\dots,q} \max(D_p(n), \hat{D}_p(n)) < \infty \quad \text{a.s.} \tag{4.5}$$

for any $\lambda_0 \in (0, d/16)$.

In the proof of Lemma 6 and also in those of Lemmas 8, 9 below we shall need the maximal inequalities given by the following lemma:

Lemma 7. *Suppose that the conditions of Theorem* 1 *hold and put*

$$S(m,n) = \sum_{m+e \leqq v \leqq m+n} \xi_v$$

$$M(m,n) = \max_{e \leqq v \leqq n} |S(m,v)|$$

for $m, n \geqq 0$, $m, n \in Z^q$. *Then we have*

$$P\{M(m,n) \geqq x[n]^{1/2}\} \leqq \text{const} \cdot x^{-(2+\alpha)} \qquad (x>0)$$

uniformly in m, n *where* α *is the constant appearing in Lemma* 4. *Further, for any* $\lambda \in (0,1)$ *we have*

$$P\{M(m,n) \geqq [n]^{1/2} (\log[n])^{q+1}\} \leqq \text{const} \cdot [n]^{-\gamma} \quad \textit{for } n \in G_\lambda$$

uniformly in m *where* γ *is a positive constant depending on* λ *and the field* $\{\xi_v, v \in Z_+^q\}$.

Proof. As in Lemma 4, it suffices to consider the case when ξ_v are real valued. The first inequality of Lemma 7 is a consequence of Lemma 4 and Theorem 7 of Móricz (1977). To prove the second inequality we use the standard bisection technique. We can assume, without loss of generality, that $[n] \geqq 4^q$. Define, for any $v = (v_1, \dots, v_q) \in Z^q$, $\mu = (\mu_1, \dots, \mu_q) \in Z^q$

$$v \cdot 2^\mu = (v_1 \cdot 2^{\mu_1}, \dots, v_q \cdot 2^{\mu_q}), \qquad 2^\mu = (2^{\mu_1}, \dots, 2^{\mu_q}).$$

Let us choose $N = (N_1, \dots, N_q) \in Z^q$ such that

$$2^{N-e} \leqq n < 2^N \tag{4.6}$$

and put, for μ, $k \in Z^q$, μ, $k \geqq 0$

$$E(v,k) = \{\omega \colon |S(v \cdot 2^k, 2^k)| \geqq [n]^{1/2} \log[n]\}$$

$$E = \bigcup_{0 \leqq k \leqq N} \bigcup_{0 \leqq v < 2^{N-k}} E(v,k).$$

Then we have, setting $\bar{N} = N_1 + \dots + N_q$,

$$P(E(v,k)) \leqq \text{const} \cdot [2^{N-k}]^{-(1+\alpha/2)} \tag{4.7}$$

$$\begin{aligned} P(E(v,k)) &\leqq \text{const} \cdot \{\exp(-\gamma_1 \log^2[n]) + [2^k]^{-\gamma}\} \\ &\leqq \text{const} \cdot \{\exp(-\gamma_2 \bar{N}^2) + [2^k]^{-\gamma}\} \quad \text{if } 2^k \in G_{\lambda/8} \end{aligned} \tag{4.8}$$

for some positive constants γ, γ_1, γ_2. (Clearly, (4.7) is a direct consequence of Lemma 4 while (4.8) follows from Lemma 5 and Esseen's lemma.) Set, for any integer $l \geqq 0$

$$P(l) = \sum_{\{v, k \in Z^q:\, 0 \leqq k \leqq N,\, k_1 + \ldots + k_q = l,\, 0 \leqq v < 2^{N-k}\}} P(E(v,k)).$$

By (4.7) we have for $0 \leqq l \leqq \bar{N}$

$$P(l) \leqq \text{const} \cdot 2^{-(\bar{N}-l)\alpha/2} \bar{N}^q \tag{4.9}$$

since the number of those $k \in Z^q$, $k \geqq 0$ such that $k_1 + \ldots + k_q = l$ is at most $\bar{N}^q$ for $0 \leqq l \leqq \bar{N}$. We further claim that for $(1-\lambda/8)\bar{N} \leqq l \leqq \bar{N}$ we have

$$P(l) \leqq \text{const} \cdot \{\exp(-\gamma_2 \bar{N}^2) + 2^{-\gamma l}\} 2^{(\bar{N}-l)} \bar{N}^q. \tag{4.10}$$

To this end we observe that $n \in G_\lambda$, (2.4), $[n] \geqq 4^q$ and the choice of N imply $N_i \geqq \lambda \bar{N}/4$ for $1 \leqq i \leqq q$. Hence if $k = (k_1, \ldots, k_q)$ is a q-vector such that $0 \leqq k \leqq N$ and $k_i < N_i/2$ for at least one $1 \leqq i \leqq q$ then

$$k_1 + \ldots + k_q < \bar{N} - N_i/2 \leqq (1-\lambda/8)\bar{N}.$$

Thus for $(1-\lambda/8)\bar{N} \leqq l \leqq \bar{N}$ the sum defining $P(l)$ contains only such terms $P(E(v,k))$ for which $k_i \geqq N_i/2 \geqq \lambda \bar{N}/8 \geqq \lambda(k_1 + \ldots + k_q)/8$ for $1 \leqq i \leqq q$ i.e. $2^k \in G_{\lambda/8}$. Consequently, (4.10) follows from (4.8). Let now $0 < c < 1$ be a number sufficiently close to 1. Clearly

$$P(E) \leqq \sum_{l=1}^{\bar{N}} P(l)$$

and applying (4.9) for $0 \leqq l \leqq c\bar{N}$, (4.10) for $c\bar{N} < l \leqq \bar{N}$ we get

$$\begin{aligned} P(E) &\leqq \text{const} \cdot \bar{N}^q \{2^{-(1-c)\bar{N}\alpha/2} + \exp(-\gamma_3 \bar{N}^2) + 2^{-((1+\gamma)c-1)\bar{N}}\} \\ &\leqq \text{const} \cdot \exp(-\gamma_4 \bar{N}) \leqq \text{const} \cdot [n]^{-\gamma_4} \end{aligned}$$

for some $\gamma_3 > 0$, $\gamma_4 > 0$. Now, for every $e \leqq \mu \leqq n$, $S(0, \mu)$ can be written as the sum of at most $N_1 N_2 \ldots N_q$ of the $S(v \cdot 2^k, 2^k)$'s above (use the dyadic expansion of each of the coordinates of n) and hence for $\omega \notin E$

$$M(0,n) \leqq [n]^{1/2} \log[n] N_1 N_2 \ldots N_q \leqq [n]^{1/2} (\log[n])^{q+1}.$$

This proves the second inequality of Lemma 7 for $m=0$; for general m the proof is the same.

Proof of Lemma 6. It suffices to estimate $D_p(n)$, $\hat{D}_p(n)$ for one value of p, say $p=1$. We show the argument for $D_1(n)$; for $\hat{D}_1(n)$ we can proceed similarly. By the first inequality of Lemma 7 we have for $n \in G_d$

$$\begin{aligned} P\{D_1(n) \geqq [n]^{(1-\lambda_0)/2}\} &\leqq \text{const}([n^{(1)}]/[n]^{(1-\lambda_0)})^{1+\alpha/2} \\ &\leqq \text{const} \cdot (n_1^{-(1-\lambda_0)} (n_2 \ldots n_q)^{\rho+\lambda_0})^{1+\alpha/2} \\ &\leqq \text{const} \cdot (n_1^{-(1-\lambda_0)+(\rho+\lambda_0)/d})^{1+\alpha/2} \\ &\leqq \text{const} \cdot n_1^{-3/4} \leqq \text{const}[n]^{-3d/8} \end{aligned} \tag{4.11}$$

where we used the fact $n_1^{(1)} \leqq \text{const} \cdot (n_2 \ldots n_q)^\rho$, (2.4) and the choice of λ_0. Now if $t_{\mu-e} \leqq n < t_\mu$ then clearly

$$D_1(n)/[n]^{(1-\lambda_0)/2} \leqq \text{const} \cdot D_1(t_\mu)/[t_\mu]^{(1-\lambda_0)/2}$$

and for $n = t_\mu$ the estimate (4.11) becomes

$$\begin{aligned} P\{D_1(t_\mu) \geqq [t_\mu]^{(1-\lambda_0)/2}\} &\leqq \text{const} \cdot [t_\mu]^{-3d/8} \\ &\leqq \text{const} \cdot [\mu]^{-3d(\beta+1)/8} \leqq \text{const} \cdot [\mu]^{-2} \end{aligned}$$

by the choice of the parameter β in (4.1). Hence Lemma 6 follows from the Borel-Cantelli lemma.

5. Deviation Estimates for the Partial Sum and Wiener Processes

Let z_μ be as defined in (4.3) and take $W(\Delta_\mu)$ to be the corresponding increment of a q-parameter Wiener process in R^N with covariance matrix Γ. The following lemma establishes that these increments will not be significant in so far attaining the error term in (1.7) concerns us.

Lemma 8.

$$\sup_{\mu \in L} [\mu]^{1-\beta/2} (|z_\mu| + |W(\Delta_\mu)|) < \infty \quad \text{a.s.} \tag{5.1}$$

Proof. We shall treat the term $|z_\mu|$ separately in (5.1); the proof for $|W(\Delta_\mu)|$ is the same since the inequalities we are going to use below are valid for the Wiener process as well. The set Δ_μ in (4.2) can evidently be written as $\bigcup_{k=1}^{q} \Delta_\mu(k)$ where the $\Delta_\mu(k)$'s are disjoint rectangles and the volume of $\Delta_\mu(k)$ is at most

$$\mu_k^{9N/\rho} \prod_{l \neq k} (t_{\mu+e}(l) - t_\mu(l)) = \mu_k^{9N/\rho} \prod_{l \neq k} \mu_l^\beta.$$

Setting

$$z_\mu(k) = \sum_{v \in \Delta_\mu(k)} \xi_v$$

and noting that $\mu \in L$ implies $\mu_k \geqq \text{const} \prod_{l \neq k} \mu_l^\rho$ and consequently $\mu_k \geqq \text{const} \cdot [\mu]^{\rho/2}$ we get by the Markov inequality and Lemma 4

$$\begin{aligned} P\{|z_\mu(k)| \geqq [\mu]^{\beta/2-1}\} &\leqq \text{const} \cdot ([\mu]^{\beta-2}/(\mu_k^{9N/\rho} \prod_{l \neq k} \mu_l^\beta))^{-(1+\alpha/2)} \\ &\leqq \text{const} \cdot (\mu_k^{\beta-2-11N/\rho})^{-(1+\alpha/2)} \\ &\leqq \text{const} \cdot [\mu]^{-(\beta-2-11N/\rho)\rho/2} \\ &\leqq \text{const} \cdot [\mu]^{-2} \qquad \text{for } \mu \in L \end{aligned}$$

by our choice of β in (4.1). Since the sum of the last probabilities is finite, we are done.

Define now, for any nonempty subset J of $\{1,2,\ldots,q\}$ and any $\mu\in Z_+^q$

$$M_\mu^{(J)}=\max_{\substack{t_\mu(k)+1\leqq\omega_k\leqq t_{\mu+e}(k)\\ k\in J}}\Big|\sum_{\substack{1\leqq\nu_k\leqq t_\mu(k),\,k\notin J\\ t_\mu(k)+1\leqq\nu_k\leqq\omega_k,\,k\in J}}\xi_\nu\Big|. \tag{5.2}$$

Define by $\hat{M}_\mu^{(J)}$ the analogous quantity for the Wiener process i.e. the quantity we get if we replace the sum in (5.2) by the increment of W over the given rectangle. We then prove

Lemma 9. *With probability one,*

$$\sup_{t_\mu\in G_\rho}[t_\mu]^{(\lambda-1)/2}\max_J\max(M_\mu^{(J)},\hat{M}_\mu^{(J)})<\infty$$

for any $\lambda\in(0,\rho/4\beta)$.

Proof. Choose the index $1\leqq l\leqq q$ such that $l\in J$. Setting $n=(n_1,\ldots,n_q)$ where $n_k=t_{\mu+e}(k)-t_\mu(k)$ for $k\in J$ and $n_k=t_\mu(k)$ for $k\notin J$ we clearly have

$$[n]\leqq\text{const}\cdot\prod_{k\in J}\mu_k^\beta\prod_{k\notin J}\mu_k^{\beta+1}\leqq\text{const}\cdot[\mu]^{\beta+1}/\mu_l$$
$$[n]\geqq[\mu]^\beta. \tag{5.3}$$

We observe further that $t_\mu\in G_\rho$ implies $n\in G_{\rho/4}$. Applying the second inequality of Lemma 7 for the quantity $M_\mu^{(J)}$ we obtain

$$P\{M_\mu^{(J)}\geqq\text{const}\cdot([\mu]^{\beta+1}/\mu_l)^{1/2}(\log[\mu])^{q+1}\}\leqq P\{M_\mu^{(J)}\geqq[n]^{1/2}(\log[n])^{q+1}\}$$
$$\leqq\text{const}\cdot[\mu]^{-\beta\gamma}\leqq\text{const}\cdot[\mu]^{-2}$$

by (5.3) and the choice of β in (4.1). Hence by the Borel-Cantelli lemma we get

$$M_\mu^{(J)}\leqq\text{const}\cdot([\mu]^{\beta+1}/\mu_l)^{1/2}(\log[\mu])^{q+1}$$
$$\leqq\text{const}\cdot[\mu]^{(\beta+1)(1-\lambda)/2}\leqq\text{const}\cdot[t_\mu]^{(1-\lambda)/2}$$

almost surely for $t_\mu\in G_\rho$. (In the second step we used $0<\lambda<\rho/4\beta$ and the observation that $t_\mu\in G_\rho$ implies $\mu_l\geqq\text{const}\cdot\prod_{k\neq l}\mu_k^\rho$ and consequently $\mu_l\geqq\text{const}\cdot[\mu]^{\rho/2}$.) Repeating the same argument for $\hat{M}_\mu^{(J)}$ we get the statement of the lemma.

6. Proof of Theorem 1

Let $n\in G_d$. Here G_d is defined in (1.4) and $d\in(0,1)$. Define $\mu=\mu_n$ by $t_\mu(k)+1\leqq n_k\leqq t_{\mu+e}(k)$, $k=1,2,\ldots,q$. Put

$$h_\mu=\text{card}(H_\mu\setminus\Delta_\mu)$$

and let

$$\varphi:\{1,2,\ldots\}\to L$$

be a one-to-one mapping of the set of positive integers onto L where L is defined in the beginning of Sect. 4. Let $\varphi(k)=(\varphi_1(k),\ldots,\varphi_q(k))$ and $\rho=d/8$ as before. Consider

$$\lambda_k(u)=E|E\{\exp(i\langle u, x_{\varphi(k)}/h_{\varphi(k)}^{1/2}\rangle)|x_{\varphi(1)},\ldots,x_{\varphi(k-1)}\}-\exp(-\langle u,\Gamma u\rangle/2)|$$

for each $u\in R^N$ and $k=1,2,\ldots$ where $\Gamma=(\gamma_{i,j})^{N\times N}$ is determined by (1.5) and x_μ is defined in (4.3) for $\mu\in L$.

Lemma 10. *One has*

$$\lambda_k(u)\leqq \text{const}\cdot[\varphi(k)]^{-9N}$$

for $|u|\leqq[\varphi(k)]^{9N}$.

Proof. By the triangle inequality we have

$$\begin{aligned}\lambda_k(u)\leqq E|E\{\exp(i\langle u, x_{\varphi(k)}/h_{\varphi(k)}^{1/2}\rangle)|x_{\varphi(1)},\ldots,x_{\varphi(k-1)}\}-E\exp(i\langle u, x_{\varphi(k)}/h_{\varphi(k)}^{1/2}\rangle)|\\ +E|E\exp(i\langle u, x_{\varphi(k)}/h_{\varphi(k)}^{1/2}\rangle)-\exp(-\langle u,\Gamma u\rangle/2)|=\mathrm{I}+\mathrm{II},\quad \text{say.}\end{aligned}$$

Using Lemma 1 and the definition of $\Delta_{\varphi(k)}$ in (4.2), together with the mixing condition (1.2), we obtain

$$\begin{aligned}\mathrm{I}&\leqq 2\pi\rho(H_{\varphi(k)}\setminus\Delta_{\varphi(k)},\bigcup_{l\neq k}(H_{\varphi(l)}\setminus\Delta_{\varphi(l)}))\\ &\leqq \text{const}\cdot(\min_{1\leqq j\leqq q}\varphi_j(k))^{-(9N/\rho)q(1+\varepsilon)(1+2/\delta)}\\ &\leqq \text{const}\cdot[\varphi(k)]^{-9N}\end{aligned}$$

the last inequality following because $\varphi(k)\in L$ and, as we already observed in the proof of Lemma 8, $\mu\in L$ implies $\mu_j\geqq\text{const}\cdot[\mu]^{\rho/2}$ for $1\leqq j\leqq q$.

To estimate II we employ Lemma 5 (with d replaced by ρ) which yields

$$\mathrm{II}\leqq\text{const}\cdot[\varphi(k)]^{-\beta t}$$

for $|u|\leqq[\varphi(k)]^{\beta t}$. Here $\beta t\geqq 9N$ by the choice of β in (4.1) and this completes the proof of Lemma 10.

We now define the quantity

$$T_k=10^8 N[\varphi(k)]^{3/2} \tag{6.1}$$

and apply Theorem 1 of Berkes and Philipp (1979). Said theorem guarantees the existence of a probability space supporting independent $N(0,\Gamma)$ random vectors $\{y_k, k\geqq 1\}$ and a random field $\{\xi_\nu^*, \nu\in Z_+^q\}$ having the same distribution as the field given in Theorem 1 such that

$$P\{|\sum_{\nu\in H_{\varphi(k)}\setminus\Delta_{\varphi(k)}}\xi_\nu^*/h_{\varphi(k)}^{1/2}-y_k|\geqq\beta_k\}\leqq\beta_k$$

where by Lemma 10, (6.1) and the aforementioned theorem of Berkes and Philipp,

$$\beta_k\leqq\text{const}(T_k^{-1}\log T_k+[\varphi(k)]^{-9N/2}T_k^N+P\{|y_k|\geqq T_k/4\}).$$

In particular,

$$\sum_{k=1}^{\infty} \beta_k < \infty$$

so the Borel-Cantelli lemma implies that

$$\Big| \sum_{\nu \in H_{\varphi(k)} \setminus \Delta_{\varphi(k)}} \xi_\nu^* / h_{\varphi(k)}^{1/2} - y_k \Big| \leqq \text{const} \cdot \beta_k \quad \text{a.s.}$$

Finally, by passing to a richer probability space (if necessary) there exists (see Lemma A1, Berkes and Philipp (1979)) a q-parameter Wiener process $\{W(\tau),\ \tau \in [0, \infty)^q\}$ in R^N with covariance matrix Γ and a random field $\{\xi'_\nu,\ \nu \in Z_+^q\}$ having the same distribution as the field $\{\xi_\nu,\ \nu \in Z_+^q\}$ such that

$$h_{\varphi(k)}^{-1/2} \Big| \sum_{\nu \in H_{\varphi(k)} \setminus \Delta_{\varphi(k)}} \xi'_\nu - W(H_{\varphi(k)} \setminus \Delta_{\varphi(k)}) \Big| \leqq \text{const} \cdot \beta_k \quad \text{a.s.} \tag{6.2}$$

But, using the definitions of the maximum terms D and M in (4.4) and (5.2) respectively, we have for any $\nu \leqq n$ with $n \in G_d$ and $[n] \geqq \text{const}$,

$$\begin{aligned} \Big| \sum_{\mu \leqq \nu} \xi'_\mu - W(\nu) \Big| \leqq & \sum_{\mu \in L,\ \mu \leqq \mu_n} \Big| \sum_{\nu \in H_\mu \setminus \Delta_\mu} \xi'_\nu - W(H_\mu \setminus \Delta_\mu) \Big| \\ & + q \sum_{p=1}^{q} (D_p(n) + \hat{D}_p(n)) \\ & + \sum_{\mu \in L,\ \mu \leqq \mu_n} \Big| \sum_{\nu \in \Delta_\mu} \xi'_\nu - W(\Delta_\mu) \Big| \\ & + \sum_{J \subset \{1, 2, \ldots, q\},\ J \neq \phi} (M_{\mu_n}^{(J)} + \hat{M}_{\mu_n}^{(J)}). \end{aligned} \tag{6.3}$$

Therefore, from Lemmas 6, 8 and 9 and (6.2) the expression in the first line of (6.3) is almost surely bounded by

$$\begin{aligned} & \text{const}(\omega) \left(\left(\sum_{k=1}^{\infty} \beta_k \right) [\mu_n]^{\beta/2} + [n]^{(1-\lambda)/2} + \sum_{\mu \in L,\ \mu \leqq \mu_n} [\mu]^{\beta/2 - 1} + [t_{\mu_n}]^{(1-\lambda)/2} \right) \\ & \leqq \text{const}(\omega) \left([\mu_n]^{\beta/2} + [n]^{(1-\lambda)/2} \right) \\ & \leqq \text{const}(\omega) \left([n]^{\beta/2(\beta+1)} + [n]^{(1-\lambda)/2} \right) \\ & \leqq \text{const}(\omega) \, [n]^{(1-\lambda)/2} \end{aligned}$$

for a constant $\lambda > 0$. (In applying Lemma 9 we need the fact that $n \in G_d$, $[n] \geqq \text{const.}$ imply $t_{\mu_n} \in G_\rho$.) Hence Theorem 1 is proved.

To get the explicit value of λ stated after Theorem 1 we note that by a lemma of Sotres and Malay Ghosh (1977) the value of α in Lemma (2.5) of Kuelbs and Philipp (1979) can be chosen as $\alpha = \varepsilon\delta/8$. Applying this to the proof of Lemma 4 we get that the value of α in Lemma 4 can be chosen as $\alpha = \varepsilon^q \delta / 8^q$. The explicit value of α in Lemma 4 yields automatically an explicit value of t in Lemma 5 and continuing we can make all the constants in the proof of Theorem 1 explicit and we arrive at the given value $\lambda = \varepsilon^{6q} \delta^5 d^4 / 8^{q+9}$.

7. Proof of Theorem 3

Let ζ_k, $-\infty<k<\infty$ be independent $N(0,1)$ r.v.'s and set $\eta_k=(\zeta_{k-1}+\zeta_k)/2$. Consider independent copies $\{\eta_k^{(l)}, -\infty<k<\infty\}$ $(-\infty<l<\infty)$ of the sequence $\{\eta_k, -\infty<k<\infty\}$ and put

$$\xi_v=\eta_{v_2}^{(v_1)}, \quad v=(v_1,v_2)\in Z^2.$$

Evidently $\{\xi_v, v\in Z^2\}$ is a stationary 2-dependent Gaussian field with $E\xi_v=0$ and $\gamma_0=\sum_{v\in Z^2} E\xi_0\xi_v=1$; we show that it satisfies the requirements of Theorem 3.

Suppose (1.12) holds for a standard Wiener process $\{W(t), t\in[0,\infty)^2\}$ and a positive nondecreasing function f satisfying (1.10), (1.11); we assume also $f(\infty)=\infty$ (the case of bounded f will be treated later) and, without loss of generality, that f is integer valued. Then we have, setting $t_k=\sum_{i\leq k} f(i)$, $T_k=k f(k)$,

$$\Big|\sum_{\substack{1\leq v_1\leq k\\ 1\leq v_2\leq f(k)}} \xi_v - W(k,f(k))\Big| = o((T_k \log\log T_k)^{1/2}/f(T_k)) \quad \text{a.s.} \tag{7.1}$$

Notice also

$$E\Big(\sum_{1\leq i\leq m} \eta_i\Big)^2=m-1/2. \tag{7.2}$$

Let $n_k=\min\{j: f(j)\geq k\}$ and set, for $k=1,2,\ldots$

$$x_k=\sum_{\substack{v_1=k\\ 1\leq v_2\leq f(k)}} \xi_v$$

$$y_k=\sum_{\substack{1\leq v_1<n_k\\ v_2=k}} \xi_v.$$

Since the r.v.'s x_k are independent and normally distributed we can apply the upper-lower class form of the law of the iterated logarithm (see Feller (1943) p. 399) to get

$$\sum_{1\leq i\leq k} x_i<\sqrt{s_k}(2\log\log s_k+4\log_3 s_k)^{1/2}$$
$$\leq\sqrt{s_k}(\sqrt{2\log\log s_k}+1) \quad \text{a.s. for } k\geq k_0.$$

where

$$s_k=\sum_{i\leq k} Ex_i^2=\sum_{i\leq k}(f(i)-1/2)=t_k-k/2 \tag{7.3}$$

by (7.2). On the other hand, y_k is normally distributed with $Ey_k=0$, $Ey_k^2\leq n_k$ and thus the Borel-Cantelli lemma gives

$$|y_k|\leq\sqrt{n_k}\log k \quad \text{a.s. for } k\geq k_0.$$

From (1.11) it follows that there is an integer $d\geq 1$ such that $n_{k\,d}\geq 2(n_k-1)\geq\sqrt{2}\,n_k$ for $k\geq k_0$ and thus $\sqrt{n_1}+\ldots+\sqrt{n_M}\leq\text{const.}\ \sqrt{n_M}$ for $M\geq 1$.

Hence

$$|\sum_{1\leqq i\leqq M} y_i|\leqq \text{const}\sqrt{n_M}\log M \quad \text{a.s. for } M\geqq M_0.$$

But we have clearly $n_{f(k)}\leqq k$ and consequently

$$\begin{aligned}\sum_{\substack{1\leqq v_1\leqq k\\ 1\leqq v_2\leqq f(k)}} \xi_v &= \sum_{1\leqq i\leqq k} x_i + \sum_{1\leqq i\leqq f(k)} y_i \\ &\leqq (2s_k \log\log s_k)^{1/2} + s_k^{1/2} + \text{const}\sqrt{k}\log f(k) \\ &\leqq (2s_k \log\log s_k)^{1/2} + 2T_k^{1/2} \quad \text{a.s. for } k\geqq k_0. \qquad (7.4)\end{aligned}$$

On the other hand, there exist independent $N(0,1)$ r.v.'s ξ_v^*, $v\in Z_+^2$ such that $W(n)=\sum_{e\leqq v\leqq n}\xi_v^*$ for $n\in Z_+^2$. Repeating the above argument for the ξ_v^* instead of ξ_v using the lower class function $\varphi(t)=(2\log\log t)^{1/2}$ in place of $\varphi(t)=(2\log\log t + 4\log_3 t)^{1/2}$ we get

$$P\{W(k,f(k))\geqq (2t_k\log\log t_k)^{1/2} - T_k^{1/2} \text{ i.o.}\}=1. \qquad (7.5)$$

To deduce a contradiction from (7.1), (7.4), (7.5) it sufficies to show that for sufficiently small c we have

$$\begin{aligned}&(2s_k\log\log s_k)^{1/2} + 2T_k^{1/2} + c(T_k\log\log T_k)^{1/2}/f(T_k) \\ &\quad < (2t_k\log\log t_k)^{1/2} - T_k^{1/2} \qquad \text{for } k\geqq k_0 \qquad (7.6)\end{aligned}$$

which follows by a simple calculation using the assumptions made on f, $\text{const}\cdot T_k\leqq t_k\leqq T_k$ (which is evident from (1.11)) and noticing that, by (7.3),

$$\sqrt{s_k}\leqq\sqrt{t_k}\,(1-1/2f(k))^{1/2}\leqq\sqrt{t_k}\,(1-1/4f(k))$$

and consequently

$$(2s_k\log\log s_k)^{1/2}\leqq(2t_k\log\log t_k)^{1/2}(1-1/4f(k)).$$

Hence Theorem 3 is proved in the case $f(\infty)=\infty$.

Assume now (1.12) holds for a bounded f; let m denote an integer such that $f(\infty)<m$. Then (7.1) holds with $f(k)=m$; on the other hand, using (7.2) and the ordinary law of the iterated logarithm we get

$$\limsup_{k\to\infty}((2m-1)k\log\log k)^{-1/2}\sum_{\substack{1\leqq v_1\leqq k\\ 1\leqq v_2\leqq m}}\xi_v=1 \quad \text{a.s.}$$

$$\limsup_{k\to\infty}(2mk\log\log k)^{-1/2}\,W(k,m)=1 \quad \text{a.s.}$$

The latter two relations together evidently contradict to (7.1).

To conclude this chapter we prove a remark made in the introduction, namely we show that if ξ_v, $v\in Z_+^q$ are independent random vectors in R^N satisfying (1.1) and having a common nonzero covariance matrix Γ then there exists a q-parameter Wiener process $\{W(\tau),\ \tau\in[0,\infty)^q\}$ in R^N with covariance matrix Γ such that (1.13) holds. This result is closely related to Theorem 1′ of Major (1976) where a somewhat restricted form of (1.13) is proved for i.i.d. ξ_ν with finite variances. For a proof of the statement above we first note that by adapting the "patching-together" argument of Major (1976) pp. 223–224 to the multiparameter case it sufficies to show that under the given conditions there exists, for every $\varepsilon>0$, a q-parameter Wiener process $\{W(\tau),\ \tau\in[0,\infty)^q\}$ in R^N with covariance matrix Γ such that

$$\limsup_{[n]\to\infty}([n]\operatorname{loglog}[n])^{-1/2}\Big|\sum_{\nu\le n}\xi_\nu - W(n)\Big|\le\varepsilon \quad \text{a.s.} \tag{7.7}$$

holds. Let now $\gamma\in(0,1/64q^2)$ be given, define $\zeta_\mu=(\zeta_\mu(1),\ldots,\zeta_\mu(q))$ by

$$\zeta_\mu(k)=[(1+\gamma)^{\mu_k+a_0}]-[(1+\gamma)^{a_0}]$$

for $\mu=(\mu_1,\ldots,\mu_q)\geqq 0$ where a_0 is an integer satisfying

$$(1+\gamma)^{a_0}\geqq 4/\gamma.$$

Set, for $\mu\in Z^q_+$

$$\bar{H}_\mu=\{\nu\in Z^q_+ : \zeta_{\mu-e}<\nu\leqq\zeta_\mu\}$$

$$Y_\mu=\Big(\sum_{\nu\in\bar{H}_\mu}\xi_\nu\Big)/(\operatorname{card}\bar{H}_\mu)^{1/2}.$$

Define also, for any nonempty subset J of $\{1,2,\ldots,q\}$ and any $\mu\in Z^q_+$

$$\tilde{M}^{(J)}_\mu=\max_{\substack{\zeta_{\mu-e}(k)+1\leqq\omega_k\leqq\zeta_\mu(k)\\ k\in J}}\Big|\sum_{\substack{1\leqq\nu_k\leqq\zeta_{\mu-e}(k),\ k\notin J\\ \zeta_{\mu-e}(k)+1\leqq\nu_k\leqq\omega_k,\ k\in J}}\xi_\nu\Big|.$$

Now, if $\psi(k)$ is a one-to-one mapping of the set of positive integers onto Z^q_+ then Theorem 1 of Berkes and Philipp (1979) applies to the sequence $\{Y_{\psi(k)}, k\geqq 1\}$ with $T_k=\text{const.}\ [\psi(k)]^4$ and one gets (using the second half of Lemma 5) that there exists a sequence $\{y_k,\ k\geqq 1\}$ of independent $N(0,\Gamma)$ random vectors such that

$$P\{|Y_{\psi(k)}-y_k|\geqq[\psi(k)]^{-2}\}\leqq[\psi(k)]^{-2} \qquad (k\geqq 1).$$

Also, using Theorem 1 of Wichura (1969) together with the central limit theorem with remainder and the Borel-Cantelli lemma we get

$$\limsup_{[\mu]\to\infty}([\zeta_\mu]\operatorname{loglog}[\zeta_\mu])^{-1/2}\ \tilde{M}^{(J)}_\mu\leqq 2^q N\gamma^{1/4} \quad \text{a.s.}$$

for any fixed J. Arguing as in the proof of Theorem 1, we easily get from these two statements that relation (7.7) holds with $\varepsilon=2^{3q}N\gamma^{1/4}$. Since γ can be chosen arbitrary small, the proof is completed.

8. Proof of Theorems 2 and 4

As we mentioned in the introduction, we shall only sketch the proof of Theorems 2 and 4 since minor changes to the proof of Theorem 1 are all that we require. We show these changes first in the case of Theorem 2. In what follows, we formulate modified versions of the lemmas in Sects. 2–5 which are needed in the proof of Theorem 2.

Lemma 3 (*). *Suppose $n \in G_\beta^*$ where G_β^* is defined by* (1.8). *Then*

$$E\Big(\sum_{e \leqq \mu \leqq n} \sum_{e \leqq \nu \leqq n} \xi_\mu^{(i)} \xi_\nu^{(j)}\Big) = [n](\gamma_{i,j} + \text{const} \cdot \theta \cdot \log^{-\varepsilon\beta/2}[n])$$

where $\gamma_{i,j}$ is defined in Theorem 1. *Moreover, $\Gamma = (\gamma_{i,j})^{N \times N}$ is nonnegative definite.*

Proof. The lemma follows immediately from the proof of Lemma 3 and the observation (playing the role of (2.4)) that $n \in G_\beta^*$ implies $n_k \geqq \text{const} \cdot \log^{\beta/2}[n]$ for $1 \leqq k \leqq q$.

Lemma 5 (*). *Under the hypotheses of Theorem* 1 *there exists a constant $t^* > 0$ such that*

$$\sup_{|u| \leqq \log^{t^*}[n]} |f_{n,\tau}(u) - \exp(-\langle u, \Gamma u \rangle/2)| \leqq \text{const} \cdot \log^{-t^*}[n] \tag{8.1}$$

uniformly for $\tau \geqq 0$ and $n \in G_\beta^*$. The constant t^* can be made as large as desired by choosing β large.

Proof. In the proof of Lemma 5 we modify the definition of L_ν, L and K as follows:

$$L_\nu = \{\mu \in Z_+^q : (\nu_k - 1) n_k \log^{-\gamma}[n] < \mu_k \leqq \nu_k n_k \log^{-\gamma}[n],\ k = 1, \ldots, q\}$$

$$L = \bigcup_{1 \leqq \nu_k \leqq \log^\gamma[n],\ k=1,\ldots,q} L_\nu$$

$$K_\nu = \bigcup_{k=1}^{q} \{\mu \in L_\nu : \nu_k n_k \log^{-\gamma}[n] - \mu_k \leqq n_k^{1/2}\}$$

where $\gamma > 0$ is a suitable constant. Then (3.3), (3.4) get replaced by

$$\sum_{1 \leqq \nu_k \leqq \log^\gamma[n],\ k=1,\ldots,q} (\text{card } K_\nu)^{1/2} \leqq \text{const } [n]^{1/2} \log^{-\gamma_1}[n]$$

$$\rho(L_\mu \setminus K_\mu, L_\nu \setminus K_\nu) \leqq \text{const} \cdot \log^{-\gamma_1}[n]$$

with a constant $\gamma_1 > 0$. From this point on, we can follow the proof of Lemma 5 with evident changes to get (8.1).

We introduce the blocks H_μ as in Sect. 4 but the edges of the blocks will grow now at a subexponential rate (instead of polynomial rate). Specifically, we set for any $\mu = (\mu_1, \ldots, \mu_q) \geqq 0$

$$t_\mu(k) = \sum_{l=1}^{\mu_k} \exp(l^\gamma)$$

$$t_\mu = (t_\mu(1), \ldots, t_\mu(q))$$

$$H_\mu = \{\nu \in Z_+^q : t_\mu + e \leqq \nu \leqq t_{\mu+e}\}$$

$$\Delta_\mu = \bigcup_{k=1}^{q} \{\nu \in H_\mu : t_{\mu+e}(k) - \exp(\tfrac{1}{2}\mu_k^\gamma) < \nu_k \leqq t_{\mu+e}(k)\}$$

for a suitable $\gamma\in(0,1)$. Next, put G_β^* (resp. $G_{\beta/8}^*$) in place of G_d (resp. G_ρ) throughout Sects. 4, 5. Then Lemmas 6, 8, 9 remain valid if we replace the norming factors

$$[n]^{(\lambda_0-1)/2}, \qquad [\mu]^{1-\beta/2}, \qquad [t_\mu]^{(\lambda-1)/2}$$

by

$$[n]^{-1/2}\log^\tau[n], \quad [t_\mu]^{-1/2}\log^\tau[t_\mu]\cdot[\mu]^2, \quad [t_\mu]^{-1/2}(\log\log[t_\mu])^\tau$$

respectively, where $\tau>0$ is a suitable constant. (To prove the analogue of Lemma 9 we need an exponential version of Lemma 7, namely

$$P\{M(m,n)\geqq x[n]^{1/2}\}\leqq \mathrm{const}\cdot\exp(-c_1x^2) \quad \text{for } |x|\leqq c_1\log^{1/2}[n]$$

for some constant $c_1>0$. Such an estimate can be proved in a standard way.) To conclude the proof of Theorem 2 we proceed just as for the proof of Theorem 1 in Sect. 6.

We turn now to the proof of Theorem 4. We observe first that by (1.16), (1.17)

$$E\Big(\sum_{m+e\leqq v\leqq m+n}\xi_v\Big)^2\leqq \mathrm{const}\cdot[n] \qquad m\geqq 0,\ n\in Z_+^q. \tag{8.2}$$

Hence Lemmas 4 and 5 are trivial in this case by (8.2), (1.18) and the fact that $\{\xi_v,\ v\in Z_+^q\}$ is a centered Gaussian field. Next we notice that in the proof of Theorem 1 Lemmas 6–9 were deduced from Lemmas 4, 5 and thus they remain valid also in the case of Theorem 4. The only point in the rest of the proof of Theorem 4 where change is required is the estimate for I in the proof of Lemma 10 (which was deduced from mixing condition (1.2) in the case of Theorem 1). Here we use the method of Morrow (1980) where a one-parameter version of Theorem 4 is proved. One readily verifies that the estimates for the quantity $\lambda_k(u)$ in (3.1) of the just mentioned paper carry over for the case $q\geqq 2$ (just order the blocks H_μ contained in G_d) and yield the estimate $|I|\leqq \mathrm{const}\cdot[\varphi(k)]^{-9N}$ for $u\in R^N$. The conclusion of the proof is again the same as in §6.

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