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Some Sojourn Time Problems for Strongly Dependent Gaussian Processes*

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1. Introduction

Let $X(t)$, $t \geqq 0$, be a real stationary Gaussian process with $EX(t)=0$, $EX^2(t)=1$ and with continuous covariance function $r(t)$ such that $\lim_{t\to\infty} r(t)=0$. For $t>0$ and $a=a(t)>0$ with $\lim_{t\to\infty} a(t)=\infty$, let

$$M_t(a(t))=\int_0^t I[|X(s)|>a(t)]\,ds,$$

where $I[\cdot]$ is the indicator function. Recently Berman [1] has shown that under suitable normalization and under suitable assumptions on $r(t)$ and $a(t)$, the distribution of $M_t(a(t))$ converges to the Rosenblatt distribution.

This paper deals with similar sojourn time problems. For $t>0$ and $b=b(t)>0$ with $\lim_{t\to\infty} b(t)=0$, let

$$N_t(b(t))=\int_0^t I[|X(s)|<b(t)]\,ds,$$

and let

$$R_t(a(t), b(t))=M_t(a(t))+N_t(b(t)).$$

We shall first show that $N_t(b(t))$ has the same limiting distribution as that of $M_t(a(t))$ under suitable normalization and under suitable assumptions on $r(t)$ and $b(t)$. Next, we shall show that $R_t(a(t), b(t))$ has also some limiting distribution, under suitable normalization. But, it will be shown that the limiting distribution of $R_t(a(t), b(t))$ depends on choosing $a(t)$ and $b(t)$. In fact, for many $a(t)$ and $b(t)$, $R_t(a(t), b(t))$ has the same limiting distribution as that of $M_t(a(t))$, but for some specifically chosen $a(t)$ and $b(t)$, $R_t(a(t), b(t))$ has a different limiting distribution.

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2. Results

We start with treating the problem in a general situation.

Let $\phi(x)$ be the standard normal density and let

$$\mathscr{G}=\left\{G_t(x)\,\middle|\,\int_{-\infty}^{\infty} G_t^2(x)\,\phi(x)\,dx<\infty \text{ and } \int_{-\infty}^{\infty} G_t(x)\,\phi(x)\,dx=0\right\}.$$

Let $H_n(x)$, $n=0,1,2,\ldots$ denote the n-th Hermite polynomial defined by

$$H_n(x)=(-1)^n\,\phi^{-1}(x)\left(\frac{d}{dx}\right)^n \phi(x).$$

As is well known, any $G_t(x)$ in $\mathscr{G}$ has the expansion as

$$G_t(x)=\sum_{n=0}^{\infty}\frac{J_t(n)}{n!}H_n(x) \tag{2.1}$$

in the sense that $\lim_{N\to\infty}\int\left|G_t(x)-\sum_{n=0}^{N}\frac{J_t(n)}{n!}H_n(x)\right|^2\phi(x)dx=0$. Here $J_t(n)=\int G_t(x)\cdot H_n(x)\,\phi(x)\,dx$, and $J_t(0)=0$ because of condition $\int G_t(x)\,\phi(x)\,dx=0$. It is noted that if $X(t)$ is a stationary Gaussian process with $EX(t)=0$, $EX^2(t)=1$ and with continuous covariance function $r(t)$, then

$$E\left\{\int_0^t H_n(X(s))\,ds\right\}=0, \qquad n\geqq 1$$

and

$$E\left\{\int_0^t H_n(X(s))\,ds\int_0^t H_m(X(s))\,ds\right\}$$
$$=2(n!)\,\delta_{nm}\int_0^t (t-s)\,r^n(s)\,ds, \qquad n,m\geqq 1,$$

where $\delta_{nm}=1$ for $n=m$ and $=0$ for $n\neq m$. We shall use these facts implicitly below.

We state the following lemma including Lemmas 2.1 and 2.2 in [1], although it can be shown by the same argument as in the proofs in [1].

Lemma 1. *Let $X(t)$, $t\geqq 0$, be a stationary Gaussian process with $EX(t)=0$, $EX^2(t)=1$ and with continuous covariance function $r(t)$, and let $G_t(x)$ be a function in $\mathscr{G}$. If there exists an integer $m\geqq 1$ such that*

$$\limsup_{t\to\infty}\frac{\operatorname{Var}\int_0^t G_t(X(s))\,ds}{\frac{2}{m!}J_t^2(m)\int_0^t (t-s)\,r^m(s)\,ds}\leqq 1, \tag{2.2}$$

then

$$\frac{\int_0^t G_t(X(s))\,ds}{|J_t(m)|\left[\frac{2}{m!}\int_0^t (t-s)\,r^m(s)\,ds\right]^{1/2}} \tag{2.3}$$

has the same limiting distribution as

$$\frac{\int_0^t H_m(X(s))\,ds}{\left[2m!\int_0^t (t-s)\,r^m(s)\,ds\right]^{1/2}} \tag{2.4}$$

in the sense that if one exists then so does the other and the two are equal.

The condition (2.2) may be written in more usable form by the computation for $\operatorname{Var}\int_0^t G_t(X(s))\,ds$. Suppose that $G_t(x)$ is differentiable with respect to x and that $G_t'(x)$ is integrable over $(-\infty,\infty)$. Denote by $\phi(u,v;\rho)$ the standard bivariate normal density with correlation ρ, that is,

$$\phi(u,v;\rho)=\frac{1}{2\pi(1-\rho^2)^{1/2}}\exp\left\{-\frac{1}{2(1-\rho^2)}(u^2-2uv\rho+v^2)\right\}.$$

Then we have, by using the relation

$$\frac{\partial}{\partial\rho}\phi(u,v;\rho)=\frac{\partial^2}{\partial u\,\partial v}\phi(u,v;\rho),$$

that

$$\begin{aligned}
&\operatorname{Var}\int_0^t G_t(X(s))\,ds\\
&=E\int_0^t\int_0^t G_t(X(x))\,G_t(X(y))\,dx\,dy\\
&=\int_0^t\int_0^t dx\,dy\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G_t(u)\,G_t(v)\,\phi(u,v;r(x-y))\,du\,dv\\
&=2\int_0^t (t-s)\,ds\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G_t(u)\,G_t(v)\,\phi(u,v;r(s))\,du\,dv\\
&=2\int_0^t (t-s)\,ds\int_0^{r(s)} d\rho\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G_t(u)\,G_t(v)\frac{\partial}{\partial\rho}\phi(u,v;\rho)\,du\,dv\\
&=2\int_0^t (t-s)\,ds\int_0^{r(s)} d\rho\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G_t(u)\,G_t(v)\frac{\partial^2}{\partial u\,\partial v}\phi(u,v;\rho)\,du\,dv\\
&=2\int_0^t (t-s)\,ds\int_0^{r(s)} d\rho\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G_t'(u)\,G_t'(v)\,\phi(u,v;\rho)\,du\,dv.
\end{aligned} \tag{2.5}$$

In the case of the function $G_t(x)$ being piecewise constant as a function of x, $G_t'(x)$ may be replaced by a linear combination of delta functions. In [1], $G_t(x)=I[x>a(t)]-[1-\Phi(a(t))]$ and $G_t(x)=I[|x|>a(t)]-2[1-\Phi(a(t))]$ are considered, where $\Phi(x)=\int_{-\infty}^{x}\phi(u)\,du$.

Now, our main theorems are the following.

Theorem 1. *Let $X(t)$, $t \geqq 0$, be a stationary Gaussian process with $EX(t)=0$, $EX^2(t)=1$ and with continuous covariance function $r(t)$. Assume that $r(t) \geqq 0$ for all sufficiently large $t>0$ and $\lim_{t\to\infty} r(t)=0$, and let $b=b(t)$ be a positive function of $t>0$ with $\lim_{t\to\infty} b(t)=0$. Furthermore, suppose that there exists δ, $0<\delta<1$, such that*

$$\lim_{t\to\infty} t^{-\delta} \int_0^t r^2(s)\, ds = \infty, \tag{2.6}$$

and that there exist some positive constants C and γ with $0<\gamma<\delta$ such that

$$b^2(t) \geqq C t^{-\gamma} \quad \text{for all large } t. \tag{2.7}$$

Then

$$\frac{N_t(b(t)) - 2t[\Phi(b(t)) - \frac{1}{2}]}{2b(t)\,\phi(b(t)) \left[\int_0^t (t-s) r^2(s)\, ds\right]^{1/2}}$$

and

$$\frac{\int_0^t H_2(X(s))\, ds}{2\left[\int_0^t (t-s) r^2(s)\, ds\right]^{1/2}} \tag{2.8}$$

have the same limiting distributions in the sense that if one exists then so does the other and the two are equal.

Theorem 2. *Under the same assumptions as in Theorem 1, let $a=a(t)$ be a positive increasing function of $t>0$ with $\lim_{t\to\infty} a(t)=\infty$ such that*

$$0<b(t)<a(t)<\infty \quad \text{for all } t>0, \tag{2.9}$$

$$a^2(t)=o(\log t) \quad \text{for } t\to\infty \tag{2.10}$$

and

$$a^2(t)=o\left(\frac{1}{r(t^\beta)}\right) \quad \text{for } t\to\infty, \tag{2.11}$$

where $\beta=\delta-\gamma$. Furthermore, suppose that there exists a constant $c>0$ such that

$$\left|\frac{a(t)\,\phi(a(t))}{b(t)\,\phi(b(t))} - 1\right| > c > 0 \quad \text{for all large } t. \tag{2.12}$$

Then

$$\frac{R_t(a(t), b(t)) - 2t[\frac{1}{2} - \Phi(a(t)) + \Phi(b(t))]}{2|a(t)\,\phi(a(t)) - b(t)\,\phi(b(t))| \left[\int_0^t (t-s) r^2(s)\, ds\right]^{1/2}}$$

has the same limiting distribution as that of (2.8) in the sense that if one exists then so does the other and the two are equal.

Theorem 3. *In Theorem 2, replace condition (2.6) by*

$$\lim_{t\to\infty} t^{-\delta} \int_0^t r^4(s)\, ds = \infty, \tag{2.13}$$

condition (2.9) *by*

$$0<b(t)<1<a(t)<\infty \quad \text{for all } t>0$$

and condition (2.12) *by*

$$\lim_{t\to\infty} \frac{a(t)\,\phi(a(t))}{b(t)\,\phi(b(t))}=1. \tag{2.14}$$

Then

$$\frac{R_t(a(t),b(t))-2t[\tfrac{1}{2}-\Phi(a(t))+\Phi(b(t))]}{(\sqrt{3})^{-1}\,[a^2(t)-b^2(t)]\,a(t)\,\phi(a(t))\left[\int_0^t (t-s)\,r^4(s)\,ds\right]^{1/2}}$$

and

$$\frac{\int_0^t H_4(X(s))\,ds}{4\sqrt{3}\left[\int_0^t (t-s)\,r^4(s)\,ds\right]^{1/2}} \tag{2.15}$$

have the same limiting distributions in the sense that if one exists then so does the other and the two are equal. The limiting distribution of (2.15) *may be different from that of* (2.8).

Remark 1. Under condition (2.14), (2.10) implies (2.7). In fact, we have

$$\begin{aligned} b(t)\,e^{-b^2(t)/2} &\sim a(t)\,e^{-a^2(t)/2} \\ &\geqq a(t)\,e^{-\varepsilon\log t} \qquad \text{(for any } \varepsilon>0) \\ &= a(t)\,t^{-\varepsilon}, \end{aligned}$$

so that

$$b(t)\geqq a(t)\,e^{b^2(t)/2}\,t^{-\varepsilon}\geqq t^{-\varepsilon},$$

because $a(t)>1$. Therefore, in Theorem 3, we need not assume (2.7) explicitly.

3. Proof of Theorem 1

In what follows, we may drop t in $a(t)$ and $b(t)$, and may write simply a for $a(t)$ and b for $b(t)$.

Let

$$G_t(x)=I[|x|<b]-2[\Phi(b)-\tfrac{1}{2}].$$

Then, by (2.5),

$$\operatorname{Var}\int_0^t G_t(X(s))\,ds=4\int_0^t (t-s)\,ds \int_0^{r(s)} [\phi(b,b;y)-\phi(b,b;-y)]\,dy,$$

and

$$J_t(2)=-2\phi(b)\,H_1(b)=-2b\,\phi(b).$$

By Lemma 1, it is sufficient to show that

$$\limsup_{t\to\infty} \frac{\int_0^t (t-s)\,ds \int_0^{r(s)} [\phi(b,b;y)-\phi(b,b;-y)]\,dy}{b^2\phi^2(b)\int_0^t (t-s)r^2(s)\,ds} \leqq 1. \tag{3.1}$$

Let $\beta=\delta-\gamma$. It is shown by the same way as in [1] that

$$\frac{\int_{t^\beta}^t (t-s)\,ds \int_0^{r(s)} [\phi(b,b;y)-\phi(b,b;-y)]\,dy}{b^2\phi^2(b)\int_0^t (t-s)r^2(s)\,ds}$$

is asymptotically less than or equal to 1. So, it suffices to estimate the following ratio. By the argument similar to that in [1], we have

$$\frac{\int_0^{t^\beta} (t-s)\,ds \int_0^{r(s)} [\phi(b,b;y)-\phi(b,b;-y)]\,dy}{b^2\phi^2(b)\int_0^t (t-s)r^2(s)\,ds} \leqq \frac{2e^{b^2}}{b^2} \times \frac{t^{1+\beta}}{\int_0^t (t-s)r^2(s)\,ds},$$

and by condition (2.7), this is less than or equal to

$$\frac{Ct^{1+\delta}}{\int_0^t (t-s)r^2(s)\,ds} \quad \text{(for some constant } C>0\text{)},$$

which tends to 0 as $t\to\infty$ by (2.6). The proof is thus completed.

4. Proof of Theorem 2

Let

$$G_t(x)=I[|x|>a]+I[|x|<b]-2[\tfrac{1}{2}-\Phi(a)+\Phi(b)].$$

Then, by (2.5)

$$\operatorname{Var}\int_0^t G_t(X(s))\,ds = 4\int_0^t (t-s)\,ds \int_0^{r(s)} \{[\phi(a,a;y)-\phi(a,a;-y)]$$
$$-2[\phi(a,b;y)-\phi(a,b;-y)]+[\phi(b,b;y)-\phi(b,b;-y)]\}\,dy, \tag{4.1}$$

and

$$J_t(2)=2[\phi(a)H_1(a)-\phi(b)H_1(b)]=2[a\phi(a)-b\phi(b)].$$

In order to show that (2.2) holds for $m=2$, we first consider the integral (4.1) over $t^\beta \leqq s \leqq t$. Then we have

$$I_1 \equiv \frac{1}{[a\phi(a)-b\phi(b)]^2 \int_0^t (t-s) r^2(s)\,ds} \int_{t^\beta}^t (t-s)\,ds$$

$$\cdot \int_0^{r(s)} \{[\phi(a,a;y)-\phi(a,a;-y)]-2[\phi(a,b;y)-\phi(a,b;-y)]$$

$$+[\phi(b,b;y)-\phi(b,b;-y)]\}\,dy$$

$$=\frac{1}{[a\phi(a)-b\phi(b)]^2 \int_0^t (t-s) r^2(s)\,ds} \int_{t^\beta}^t (t-s)\,ds$$

$$\cdot \int_0^{r(s)} (1-y^2)^{-1/2} \left\{\phi^2(a)\left[\exp\left(\frac{a^2 y}{1+y}\right)-\exp\left(-\frac{a^2 y}{1-y}\right)\right]\right.$$

$$-2\phi(a)\phi(b)\exp\left(-\frac{(a^2+b^2)y^2}{2(1-y^2)}\right)\left[\exp\left(\frac{aby}{1-y^2}\right)-\exp\left(-\frac{aby}{1-y^2}\right)\right]$$

$$\left.+\phi^2(b)\left[\exp\left(\frac{b^2 y}{1+y}\right)-\exp\left(-\frac{b^2 y}{1-y}\right)\right]\right\} dy. \tag{4.2}$$

Note that

$$\exp\left(\frac{zy}{1+y}\right)-\exp\left(-\frac{zy}{1-y}\right)\sim 2zy \tag{4.3}$$

and

$$\exp\left(\frac{zy}{1-y^2}\right)-\exp\left(-\frac{zy}{1-y^2}\right)\sim 2zy \tag{4.4}$$

for small y and zy. But, when t is large, y is sufficiently small because $\lim_{t\to\infty} r(t)=0$ and $a^2(t)y$ is small because $a^2(t)r(t^\beta)$ is small under condition (2.11), by the same reasoning as in [1]. $a(t)b(t)y$ and $b^2(t)y$ are also small. Therefore we can apply (4.3) and (4.4) to (4.2), and we have

$$I_1 \sim \frac{1}{[a\phi(a)-b\phi(b)]^2 \int_0^t (t-s) r^2(s)\,ds} \int_{t^\beta}^t (t-s)\,ds$$

$$\cdot \int_0^{r(s)} 2[a^2\phi^2(a)-2ab\phi(a)\phi(b)+b^2\phi^2(b)]\,y\,dy$$

$$=\frac{\int_{t^\beta}^t (t-s) r^2(s)\,ds}{\int_0^t (t-s) r^2(s)\,ds} \leqq 1.$$

Next we estimate the integral (4.1) over $0 \leqq s \leqq t^\beta$. We have, by exactly the same way as in [1],

$$\int_0^{t^\beta}(t-s)\,ds\int_0^{r(s)}[\phi(a,a;y)-\phi(a,a;-y)]\,dy$$

$$=\phi^2(a)\int_0^{t^\beta}(t-s)\,ds\int_0^{r(s)}\frac{\phi(a,a;y)-\phi(a,a;-y)}{\phi^2(a)}\,dy$$

$$\leqq 2\phi^2(a)\,t^{1+\delta}$$

so that

$$I_2\equiv\frac{1}{[a\phi(a)-b\phi(b)]^2\int_0^t(t-s)r^2(s)\,ds}\int_0^{t^\beta}(t-s)\,ds$$

$$\cdot\int_0^{r(s)}[\phi(a,a;y)-\phi(a,a;-y)]\,dy$$

$$\leqq\frac{2\phi^2(a)}{[a\phi(a)-b\phi(b)]^2}\times\frac{t^{1+\delta}}{\int_0^t(t-s)r^2(s)\,ds}.$$

By condition (2.6),

$$\frac{t^{1+\delta}}{\int_0^t(t-s)r^2(s)\,ds}\to 0\quad\text{as } t\to\infty.$$

On the other hand,

$$\frac{\phi^2(a)}{[a\phi(a)-b\phi(b)]^2}=\frac{1}{a^2}\times\frac{1}{\left[1-\dfrac{b\phi(b)}{a\phi(a)}\right]^2}\to 0\quad\text{as } t\to\infty$$

because of condition (2.12). $\left(\text{Note that (2.12) implies that } \left|\dfrac{b\phi(b)}{a\phi(a)}-1\right|>\dfrac{c}{1+c}>0,\right.$ $\left.\text{if we choose } c<1.\right)$ Therefore, $I_2\to 0$ as $t\to\infty$.

Similarly, but noting that $\exp(b^2(t))\to 1$ as $t\to\infty$, we have

$$\frac{1}{[a\phi(a)-b\phi(b)]^2\int_0^t(t-s)r^2(s)\,ds}\int_0^{t^\beta}(t-s)\,ds$$

$$\cdot\int_0^{r(s)}[\phi(b,b;y)-\phi(b,b;-y)]\,dy$$

$$\leqq\frac{2\phi^2(b)}{[a\phi(a)-b\phi(b)]^2}\times\frac{t^{1+\beta}}{\int_0^t(t-s)r^2(s)\,ds}$$

$$=\frac{2}{\left[\dfrac{a\phi(a)}{b\phi(b)}-1\right]^2}\times\frac{1}{b^2}\times\frac{t^{1+\beta}}{\int_0^t(t-s)r^2(s)\,ds}$$

$$\leqq \frac{2}{\left[\frac{a\phi(a)}{b\phi(b)}-1\right]^2} \times \frac{t^{1+\delta}}{C\int_0^t (t-s)r^2(s)\,ds}$$

(by (2.7)), which tends to 0 as $t \to \infty$.

Finally, we have to estimate

$$I_3 \equiv \frac{1}{[a\phi(a)-b\phi(b)]^2 \int_0^t (t-s)r^2(s)\,ds} \int_0^{t^\beta} (t-s)\,ds \cdot \int_0^{r(s)} [\phi(a,b;y)-\phi(a,b;-y)]\,dy.$$

Note that for $0 \leqq |y| \leqq 1$,

$$\begin{aligned}\frac{\phi(a,b;y)}{\phi(a)\phi(b)} &= (1-y^2)^{-1/2} \exp\left\{\frac{1}{2(1-y^2)}(-a^2y^2-b^2y^2+2aby)\right\}\\ &= (1-y^2)^{-1/2} \exp\left\{-\frac{(a-b)^2y^2}{2(1-y^2)}+\frac{aby}{(1+y)}\right\}\\ &\leqq (1-y^2)^{-1/2} \exp\left(\frac{aby}{1+y}\right)\\ &\leqq (1-y^2)^{-1/2} e^{ab}.\end{aligned}$$

Then, by the same argument as in [1], we have

$$\int_0^{t^\beta} (t-s)\,ds \int_0^{r(s)} [\phi(a,b;y)-\phi(a,b;-y)]\,dy \leqq 2\phi(a)\phi(b)t^{1+\beta}e^{ab}.$$

Hence we have

$$\begin{aligned}I_3 &= \frac{1}{[a\phi(a)-b\phi(b)]^2 \int_0^t (t-s)r^2(s)\,ds} \cdot \int_0^{t^\beta} (t-s)\,ds \int_0^{t(s)} [\phi(a,b;y)-\phi(a,b;-y)]\,dy\\ &\leqq \frac{2\phi(a)\phi(b)e^{ab}t^{1+\beta}}{[a\phi(a)-b\phi(b)]^2 \int_0^t (t-s)r^2(s)\,ds}\\ &= \frac{2}{\left[1-\frac{b\phi(b)}{a\phi(a)}\right]\left[\frac{a\phi(a)}{b\phi(b)}-1\right]} \times \frac{e^{ab}}{ab} \times \frac{t^{1+\beta}}{\int_0^t (t-s)r^2(s)\,ds}.\end{aligned}$$

If we could show

$$\frac{e^{2a(t)b(t)}}{a(t)b(t)} = O(t^\gamma) \qquad \text{for } t \to \infty, \tag{4.5}$$

I_3 tends to 0. However, we have

$$\frac{e^{2a(t)b(t)}}{a(t)b(t)}=\frac{(e^{2a(t)})^{b(t)}}{a(t)b(t)}\leqq\frac{t^{\varepsilon b(t)}}{a(t)b(t)}$$

for any $\varepsilon>0$ because of (2.10), and by (2.7), this is asymptotically less than or equal to

$$\frac{t^{\varepsilon}t^{\gamma/2}}{C^{1/2}a(t)}$$

which has the order of $o(t^{\varepsilon+\gamma/2})$. Thus we have obtained (4.5), and the proof of the theorem is completed.

5. Proof of Theorem 3

We shall show that (2.2) holds for $m=4$. Note that

$$\begin{aligned}J_t(4)&=2[\phi(a)H_3(a)-\phi(b)H_3(b)]\\&=2[a(a^2-3)\phi(a)-b(b^2-3)\phi(b)]\\&\sim 2(a^2-b^2)a\phi(a)\end{aligned}$$

by (2.14). We first consider the integral (4.1) over $t^{\beta}\leqq s\leqq t$. Then we have

$$\begin{aligned}J\equiv&\frac{12}{(a^2-b^2)^2\int_0^t(t-s)r^4(s)ds}\int_{t^{\beta}}^{t}(t-s)ds\int_0^{r(s)}\left\{\frac{\phi(a,a;y)-\phi(a,a;-y)}{a^2\phi^2(a)}\right.\\&\left.-\frac{\phi(a,b;y)-\phi(a,b;-y)}{ab\phi(a)\phi(b)}+\frac{\phi(b,b;y)-\phi(b,b;-y)}{b^2\phi^2(b)}\right\}dy\\=&\frac{12}{(a^2-b^2)^2\int_0^t(t-s)r^4(s)ds}\int_{t^{\beta}}^{t}(t-s)ds\\&\cdot\int_0^{r(s)}(1-y^2)^{-1/2}\left\{\frac{1}{a^2}\left[\exp\left(\frac{a^2y}{1+y}\right)-\exp\left(-\frac{a^2y}{1-y}\right)\right]\right.\\&-\frac{2}{ab}\exp\left\{-\frac{(a^2+b^2)y^2}{2(1-y^2)}\right\}\left[\exp\left(\frac{aby}{1-y^2}\right)-\exp\left(-\frac{aby}{1-y^2}\right)\right]\\&\left.+\frac{1}{b^2}\left[\exp\left(\frac{b^2y}{1+y}\right)-\exp\left(-\frac{b^2y}{1-y}\right)\right]\right\}dy.\end{aligned}\tag{5.1}$$

When zy and y are small, we have

$$\exp\left(\frac{zy}{1+y}\right)-\exp\left(-\frac{zy}{1-y}\right)\sim 2zy+\frac{zy^3}{3}(z^2-6z+6)\tag{5.2}$$

and

$$\exp\left(\frac{zy}{1-y^2}\right)-\exp\left(-\frac{zy}{1-y^2}\right)\sim 2zy+\frac{zy^3}{3}(z^2+6).\tag{5.3}$$

Recall that $a^2 y$, aby and $b^2 y$ are small for large t. Thus, if we apply (5.2) and (5.3) to the integrand in (5.1), then we have

$$J \sim \frac{12}{(a^2-b^2)^2 \int_0^t (t-s) r^4(s)\, ds} \int_{t^\beta}^t (t-s)\, ds$$

$$\cdot \int_0^{r(s)} \frac{y^3}{3} \{(a^4-6a^2+6) - 2(a^2 b^2+6) + 6(a^2+b^2) + (b^4-6b^2+6)\}\, dy$$

$$= \frac{\int_{t^\beta}^t (t-s) r^4(s)\, ds}{\int_0^t (t-s) r^4(s)\, ds} \leqq 1.$$

It remains to show that

$$\frac{1}{\frac{1}{3}(a^2-b^2)^2 a^2 \phi^2(a) \int_0^t (t-s) r^4(s)\, ds}$$

$$\cdot 4 \int_0^{t^\beta} (t-s)\, ds \int_0^{r(s)} \{[\phi(a,a;y) - \phi(a,a;-y)]$$

$$-2[\phi(a,b;y) - \phi(a,b;-y)] + [\phi(b,b;y) - \phi(b,b;-y)]\}\, dy$$

tends to 0. But it is carried out in the same way as in the proof of Theorem 2, if we use (2.13) in place of (2.6). The proof of Theorem 3 is thus completed.

6. Further Discussions

6.1. When $G_t(x)$ does not depend on t ($G(x)$, say), the limiting distribution of (2.3) has been studied by Dobrushin and Major [2] and Taqqu [3, 4]. In the case $G_t(x) = G(x)$, we call $m = \min\{q \mid J_t(q) \neq 0\}$ the Hermite rank of $G(x)$. The underlying Gaussian process $X(t)$ considered in [2-4] is assumed to have the covariance function $r(t)$ which is regularly varying of inder $-\alpha$ for $t \to \infty$ for some α with $0 < \alpha < 1/m$. However, we can show a statement similar to Lemma 1 in Section 2 under relatively weaker conditions on $r(t)$ as in Theorems 1-3.

Theorem 4. *Let $X(t)$, $t \geqq 0$, be a stationary Gaussian process with $EX(t) = 0$, $EX^2(t) = 1$ and with continuous covariance function $r(t)$. Assume that $r(t) \geqq 0$ for all sufficiently large $t > 0$ and $\lim_{t \to \infty} r(t) = 0$. Furthermore suppose that $G(x)$ has the Hermite rank m and that there exists δ, $0 < \delta < 1$, such that*

$$\lim_{t \to \infty} t^{-\delta} \int_0^t r^m(s)\, ds = \infty. \tag{6.1}$$

Then

$$\frac{\int_0^t G(X(s))\,ds}{|J(m)|\left[\frac{2}{m!}\int_0^t (t-s)\,r^m(s)\,ds\right]^{1/2}}$$

has the same limiting distribution as (2.4) *in the sense that if one exists then so does the other and the two are equal, where* $J(m)=\int G(x)\,H_m(x)\,\phi(x)\,dx$.

Proof. We have

$$\int_0^t G(X(s))\,ds=\frac{J(m)}{m!}\int_0^t H_m(X(s))\,ds+\sum_{n=m+1}^{\infty}\frac{J(n)}{n!}\int_0^t H_n(X(s))\,ds.$$

Hence it suffices to show that

$$E|Z(t)|^2\equiv E\left(\sum_{n=m+1}^{\infty}\frac{J(n)}{n!}\int_0^t H_n(X(s))\,ds\right)^2=o\left(\int_0^t (t-s)\,r^m(s)\,ds\right).$$

We have

$$E|Z(t)|^2=2\sum_{n=m+1}^{\infty}\frac{J^2(n)}{n!}\int_0^t (t-s)\,r^n(s)\,ds \leqq 2\max_{n\geqq m+1}\int_0^t (t-s)\,r^n(s)\,ds\times\sum_{n=m+1}^{\infty}\frac{J^2(n)}{n!}.$$

Note that

$$\sum_{n=0}^{\infty}\frac{J^2(n)}{n!}=\int_{-\infty}^{\infty} G^2(x)\,\phi(x)\,dx<\infty.$$

Therefore, for some constant $C>0$,

$$\begin{aligned}E|Z(t)|^2&\leqq C\max_{n\geqq m+1}\int_0^t (t-s)\,r^n(s)\,ds\\&=C\max_{n\geqq m+1}\left(\int_0^{t^\delta}+\int_{t^\delta}^t\right)(t-s)\,r^n(s)\,ds\\&\leqq C\left(t^{1+\delta}+\int_{t^\delta}^t (t-s)\,r^{m+1}(s)\,ds\right)\end{aligned}$$

for large t, since $r(t)\geqq 0$ for all large t, where δ is the one in (6.1). It follows from (6.1) that

$$\lim_{t\to\infty}\frac{t^{1+\delta}}{\int_0^t (t-s)\,r^m(s)\,ds}=0.$$

We also have

$$\frac{\int_{t^\delta}^{t}(t-s)\,r^{m+1}(s)\,ds}{\int_0^t (t-s)\,r^m(s)\,ds} \geqq \frac{\sup_{s>t^\delta} r(s)\int_{t^\delta}^{t}(t-s)\,r^m(s)\,ds}{\int_0^t (t-s)\,r^m(s)\,ds}$$

$$\leqq \sup_{s>t^\delta} r(s) \quad \text{(for large } t\text{)}$$

$$\to 0 \quad \text{as } t\to\infty.$$

The proof is thus completed.

Remark 2. It follows from Theorem 4 that

$$\lim_{t\to\infty}\frac{\operatorname{Var}\int_0^t G(X(s))\,ds}{J^2(m)\dfrac{2}{m!}\int_0^t (t-s)\,r^m(s)\,ds}=1.$$

Therefore, we see that the m in condition (2.2) coincides with the Hermite rank of $G(x)$ in the case where $G_t(x)$ does not depend on t. However, in the case where $G_t(x)$ may depend on t, it is not necessary that $J_t(n)=0$ for $n<m$.

Remark 3. For the case of the stationary Gaussian process the covariance function of which $r(t)$ is regularly varying of index $-\alpha$ for $t\to\infty$, for some α with $0<\alpha<1/m$, where m is the one in (2.4), the existence of limiting distributions of (2.4) is known. (See [2, 3, 4].) Berman [1] also gave the limiting distribution of (2.8), which is known as the Rosenblatt distribution.

6.2. Finally, we give a result about the joint limiting behavior of $M_t(a(t))$ and $N_t(b(t))$, being motivated by Theorem 4.2 in [1].

Theorem 5. *In each case of Theorems* 2 *and* 3, *we have*

$$\lim_{t\to\infty}\operatorname{Corr}(M_t(a(t)),N_t(b(t)))=-1. \tag{6.2}$$

Proof. Berman [1] showed that

$$\operatorname{Var} M_t(a)\sim 4a^2\phi^2(a)\int_0^t (t-s)\,r^2(s)\,ds. \tag{6.3}$$

On the other hand, it follows from Theorems 1–3 that

$$\operatorname{Var} N_t(b)\sim 4b^2\phi^2(b)\int_0^t (t-s)\,r^2(s)\,ds \tag{6.4}$$

and

$$\operatorname{Var} R_t(a,b)\sim\begin{cases} 4(a\phi(a)-b\phi(b))^2\int_0^t (t-s)\,r^2(s)\,ds & \\ \qquad\qquad \text{for } a \text{ and } b \text{ in Theorem 2} & \\ \frac{1}{3}(a^2-b^2)^2a^2\phi^2(a)\int_0^t (t-s)\,r^4(s)\,ds & \\ \qquad\qquad \text{for } a \text{ and } b \text{ in Theorem 3.} & \end{cases} \tag{6.5}$$

In the case of Theorem 3, (6.2) can be obtained by the same argument as in the proof of Theorem 4.2 in [1]. As to the case of Theorem 2, we have, from (6.3)–(6.5), that

$$\operatorname{Var} R_t(a,b) \sim \operatorname{Var} M_t(a) + \operatorname{Var} N_t(b) - 2(\operatorname{Var} M_t(a))^{1/2} (\operatorname{Var} N_t(b))^{1/2},$$

which implies (6.2).

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References

1. Berman, S.M.: High level sojourns for strongly dependent Gaussian processes. Z. Wahrscheinlichkeitstheorie verw. Gebiete **50**, 223–236 (1979)
2. Dobrushin, R.L., Major, P.: Non-central limit theorems for non-linear functionals of Gaussian fields. Z. Wahrscheinlichkeitstheorie verw. Gebiete **50**, 27–52 (1979)
3. Taqqu, M.S.: Convergence of integrated processes of arbitrary Hermite rank. Z. Wahrscheinlichkeitstheorie verw. Gebiete **50**, 53–83 (1979)
4. Taqqu, M.S.: Self-similar processes and related ultraviolet and infrared catastrophes. Technical Report No. 423, School of Operations Research and Industrial Engineering, Cornell University (1979)

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