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First-Passage Percolation, Network Flows and Electrical Resistances

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Summary. We show that the first-passage times of first-passage percolation on $\mathbb{Z}^2$ are such that $P(\theta_{0n}<n(\mu-\varepsilon))$ and $P(\theta_{0n}>n(\mu+\varepsilon))$ decay geometrically as $n\to\infty$, where θ may represent any of the four usual first-passage-time processes. The former estimate requires no moment condition on the time coordinates, but there exists a geometrically-decaying estimate for the latter quantity if and only if the time coordinate distribution has finite moment generating function near the origin. Here, μ is the time constant and $\varepsilon>0$. We study the line-to-line first-passage times and describe applications to the maximum network flow through a randomly-capacitated subsection of $\mathbb{Z}^2$, and to the asymptotic behaviour of the electrical resistance of a subsection of $\mathbb{Z}^2$ when the edges of the subsection are wires in an electrical network with random resistances. In the latter case we show, for example, that if each edge-resistance equals 1 or ∞ ohms with probabilities p and $1-p$ respectively, then the effective resistance R_n across opposite faces of an n by n box satisfies the following:

(a) if $p<\frac{1}{2}$ then $P(R_n=\infty)\to 1$ as $n\to\infty$,

(b) if $p>\frac{1}{2}$ then there exists $\nu(p)<\infty$ such that

$$P(p^{-1}\leqq \liminf_{n\to\infty} R_n \leqq \limsup_{n\to\infty} R_n \leqq \nu(p))=1.$$

There are some corresponding results for certain other two-dimensional lattices, and for higher dimensions.

1. Introduction

Let L be a lattice (such as the square lattice $\mathbb{Z}^2$), and suppose that to each edge e of L there corresponds a random variable $t(e)$, called the *time coordinate* of e, where $\{t(e): e\in L\}$ is a family of non-negative independent identically distribut-

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ed variables with some common distribution function F. If $r=(\boldsymbol{v}_0, e_1, \ldots, e_n, \boldsymbol{v}_n)$ is a *path* in L, comprising edges e_i joining vertices $\boldsymbol{v}_{i-1}$ and $\boldsymbol{v}_i$, then the *passage time* of r is defined to be $t(r)=\sum_i t(e_i)$. If A and B are disjoint sets of vertices of L, then the passage time from A to B is defined to be $t(A,B)=\inf\{t(r)$: r joins some vertex in A to some vertex in $B\}$. *First-passage percolation* studies the random variables $t(A,B)$ for certain sets A and B; the origins of the subject lie in Hammersley and Welsh (1965), and more recent results are given in Smythe and Wierman (1978) and Cox and Kesten (1981). For historical and mathematical reasons, L is usually taken to be the square lattice $\mathbb{Z}^2$ with vertices $\{(x,y)$: x, $y=0, \pm 1, \ldots\}$ and edges joining pairs of vertices which are unit distance apart. We state most of our results for this case also, and henceforth suppose that L is this lattice. Many corresponding results are valid for other two-dimensional lattices, and for $\mathbb{Z}^d$ with $d \geqq 3$, and we indicate these when appropriate. *All our remarks pertain to* $\mathbb{Z}^2$ *only, unless otherwise indicated.*

Passage times of particular interest are the *absolute point-to-point time* a_{mn} and *absolute point-to-line time* $b_{mn}(m<n)$, given by

$$a_{mn}=t((m,0),(n,0)), \qquad b_{mn}=t((m,0),H_n)$$

where H_n is the vertical line $x=n$ in L. We shall also be interested in cylinder times. Let $C_{mn}=\{(x,y): m<x<n\}$. *The cylinder point-to-point time* t_{mn} and *cylinder point-to-line time* s_{mn} are defined to be

$$\begin{aligned} t_{mn}=\inf\{t(r)\text{: } & r \text{ joins } (m,0) \text{ to } (n,0), \text{ and } r \text{ is contained} \\ & \text{in } C_{mn} \text{ except for its endpoints}\}, \\ s_{mn}=\inf\{t(r)\text{: } & r \text{ joins } (m,0) \text{ to } H_n, \text{ and } r \text{ is contained} \\ & \text{in } C_{mn} \text{ except for its endpoints}\}. \end{aligned}$$

It is known that there exists a *time constant* $\mu=\mu(F)$ with the following properties:

(1.1) if $F_n \to F$, in the sense of weak convergence, then $\mu(F_n)\to\mu(F)$;

(1.2) $\frac{1}{n}a_{0n}\to\mu$ in probability;

$\frac{1}{n}a_{0n}\to\mu$ a.s. if and only if $\int(1-F(x))^4\,dx<\infty$,

(1.3) $\frac{1}{n}b_{0n}\to\mu$ a.s.

See Smythe and Wierman (1978, Chap. 5), Cox and Durrett (1981) and Cox and Kesten (1981) for proofs of these and other relevant facts. Here, we are interested in the passage times a, b, s and t, and their rates of convergence, when normalized, and show that, if $\varepsilon>0$, there exists a constant $B=B(\varepsilon)$ such that $0<B(\varepsilon)\leqq\infty$ and

$$P(\theta_{0n}<n(\mu-\varepsilon))\leqq\exp(-n(B(\varepsilon)+o(1))) \quad \text{as } n\to\infty, \tag{1.4}$$

without any moment assumption on the time coordinates, and

(1.5) $$P(\theta_{0n} > n(\mu+\varepsilon)) \leqq \exp(-n(B(\varepsilon)+o(1))) \quad \text{as } n\to\infty,$$

if the time coordinates have a moment generating function, in that

(1.6) $$\int e^{\gamma x} dF(x) < \infty \quad \text{for some } \gamma > 0;$$

in cases (1.4) and (1.5) above, θ may represent a, b, s or t.

These results have applications to the question of ascertaining the maximum network flow through a randomly capacitated subsection of L. Suppose that each edge e of L is assigned a random capacity $c(e)$, where $\{c(e)\colon e\in L\}$ is a family of non-negative independent identically distributed random variables with common distribution function F, and let B_{nm} denote the rectangle $\{(x,y)\colon 0\leqq x\leqq n,\ 0\leqq y\leqq m\}$. By the max-flow min-cut theorem of Ford and Fulkerson (see Bollobás (1979, p. 47)), the maximum flow ϕ_{nm} across B_{nm} from left to right equals the shortest passage time from top to bottom in the dual network, in which the passage time of each edge e^* is set equal to the capacity of the unique edge e of the primal network which intersects e^*. Hence (see Theorem 5.1)

(1.7) $$\frac{1}{n}\phi_{nn} \to \mu(F) \text{ a.s.} \quad \text{if } \int (1-F(x))^4 dx < \infty;$$

we shall see that, if $n(m)=e^{cm}$, then the value of the limit as $m\to\infty$ of the sequence $\{m^{-1}\phi_{n(m),m}\}$ depends on the numerical value of c.

A closely related physical problem is to determine the electrical resistance of B_{nm} when the edges of L are wires in an electrical network and a typical edge e has some random resistence $r(e)$ ohms, where $\{r(e)\colon e\in L\}$ is a family of non-negative independent identically distributed variables (possibly taking the value ∞ ohms); see Kirkpatrick (1978) and Stauffer (1979) for surveys of this area. We are able to make some progress with this problem. For example, consider the simplest nontrivial case in which the edge-resistances have the Bernoulli distribution

(1.8) $$P(r(e)=1)=p, \qquad P(r(e)=\infty)=1-p$$

where $0\leqq p\leqq 1$; this corresponds exactly to the case of Bernoulli percolation, in which each edge of L is deleted with probability $1-p$ (and therefore can transmit no electricity) or remains with probability p (and has some standard resistance of, say, 1 ohm). We introduce two supplementary vertices, labelled 0 and ∞, in the following way. We join each vertex on the left (respectively right) edge of B_{nm} to the new vertex 0 (respectively ∞) by wires with zero resistance. Let R_{nm} be the resistance of the ensuing electrical network between the terminals labelled 0 and ∞. We shall show the following:

(1.9) if $p<\frac{1}{2}$, $\quad p(R_{nn}=\infty)\to 1 \quad$ as $\quad n\to\infty$,

(1.10) if $p>\frac{1}{2}$, there exists $\nu(p)<\infty$ such that

$$\frac{1}{p} \leqq \liminf_{n\to\infty} R_{nn} \leqq \limsup_{n\to\infty} R_{nn} \leqq \nu(p) \text{ a.s.}$$

In the latter case, we conjecture that $\lim R_{nn}$ exists. Other partial results are available for more general distributions, resistances of rectangles, and certain other lattices in two or more dimensions.

Papanicolaou and Varadhan (1979) and Golden and Papanicolaou (1983) have studied the closely related situation in which the conducting medium is the sheet $\mathbb{R}^2$, the conductivity of which is a stationary random process. For this case, Golden and Papanicolaou (1983) have shown that R_{nn} converges in mean square so long as the resistances are concentrated a.s. on an interval $[a,b]$, where $0<a<b<\infty$. Künnemann (1983) has studied similar results for the case of $\mathbb{Z}^2$. See Sect. 6 for further discussion.

Each of the sections of this paper begins with a statement of the results of that section; the proofs are collected together at the ends of the sections.

Here is some notation. For any vector $\boldsymbol{k}=(k_1,\dots,k_d)$ and $x\in\mathbb{R}$, we write $x\boldsymbol{k}$ and $\boldsymbol{k}+x$ for the vectors

$$x\boldsymbol{k}=(xk_1,\dots,xk_d), \qquad \boldsymbol{k}+x=(k_1+x,\dots,k_d+x).$$

For vectors $\boldsymbol{k}$, $\boldsymbol{l}\in\mathbb{R}^d$, we write $\boldsymbol{k}\leqq\boldsymbol{l}$ (respectively $\boldsymbol{k}<\boldsymbol{l}$) if and only if $k_i\leqq l_i$ (respectively $k_i<l_i$) for all $i=1,2,\dots,d$. We denote the *origin* of $\mathbb{Z}^d$ by $\boldsymbol{0}$, and write $\boldsymbol{1}=(1,1,\dots,1)$ for the vector of one's. We denote by λ the mean time coordinate:

$$\lambda=\int_0^\infty x\,dF(x). \tag{1.11}$$

The integer part of x is written as $\lfloor x\rfloor$.

The *complement* of the event E is denoted by $\bar{E}$; the *cardinality* of the set A is denoted by $|A|$.

2. Line-to-line Passage Times

Let B_{nm} be the subgraph of L with vertex set $\{(x,y)\colon 0\leqq x\leqq n,\ 0\leqq y\leqq m\}$, and define the *line-to-line passage time* l_{nm} by

(2.1) $l_{nm}=\inf\{t(r)\colon r$ joins $(0,a)$ to (n,b) for some $0\leqq a,\ b\leqq m$, and r is contained in $B_{nm}\}$.

In this section we investigate the asymptotic behaviour of l_{nm} for large m and n. The most interesting case is when $m=n$.

Theorem 2.1. *Suppose that $m=m(n)\to\infty$.*

(a) *If $m(n)=O(n)$ then $\liminf_{n\to\infty}\frac{1}{n}l_{nm}\geqq\mu$ a.s.*

(b) *If $\int(1-F(x))^4\,dx<\infty$ then $\limsup_{n\to\infty}\frac{1}{n}l_{nm}\leqq\mu$ a.s., and if, further, $m(n)=O(n)$ then, as $n\to\infty$,*

$$\frac{1}{n}l_{nm}\to\mu \text{ a.s. and in } L^1.$$

We shall see later (see Corollary 4.2) that similar conclusions are valid subject to the weaker condition that $n^{-1}\log m(n)\to 0$ as $n\to\infty$ in place of the condition that $m(n)=O(n)$. The referee has pointed out that, for the case $m(n)=n$ at least, the almost sure convergence of $n^{-1}l_{nn}$ is valid without any assumption on F. We do not explore this here, but note that this is a consequence of the argument of Cox and Durrett (1981, pp. 589–590) if $F(0)>\frac{1}{2}$, and follows more generally by the truncation argument of Cox and Kesten (1981, Lemma 2).

The method of proof of Theorem 2.1 is easily adapted to provide a corresponding result for first-passage percolation through $\mathbb{Z}^d$ where $d\geqq 3$. Let $B_n(d)$ be the box $\{\boldsymbol{x}\in\mathbb{Z}^d\colon \boldsymbol{0}\leqq\boldsymbol{x}\leqq n\boldsymbol{1}\}$ and let $l_n(d)$ be the shortest passage time of the paths within $B_n(d)$ which join two vertices of the form $(0,x_2,\dots,x_d)$ and $(n,y_2,\dots,y_d)$ for some $0\leqq x_2,\dots,x_d,y_2,\dots,y_d\leqq n$.

Theorem 2.2. *If $d\geqq 3$ and $\int x^2\,dF(x)<\infty$ then, as $n\to\infty$,*

$$\frac{1}{n}l_n(d)\to\mu(d)\ \text{a.s. and in } L^1,$$

where $\mu(d)=\mu(d,F)$ is the time constant of $\mathbb{Z}^d$.

Proof of Theorem 2.1. We consider the case $m(n)=n$ only; the more general case proceeds in exactly the same way.

(a) First we show that, if $\varepsilon>0$,

$$P(l_{nn}<n(\mu-\varepsilon))\to 0 \quad\text{as}\quad n\to\infty. \tag{2.2}$$

We shall truncate the time coordinate distribution. Let $M>0$, and replace each time coordinate t by

$$t^M=\begin{cases} t & \text{if } t\leqq M\\ M & \text{if } t>M,\end{cases}$$

with distribution function F^M, associated passage times θ^M $(\theta=a,b,s,t,l)$ and time constant μ^M. Let $\eta>0$ and let $P_n(\eta)$ be the vertices on the left edge of B_{nn} whose coordinates are of the form $(0,\lfloor k\eta n\rfloor)$ for $k=0,1,\dots,\lfloor\eta^{-1}\rfloor$. Note that $|P_n(\eta)|\leqq 1+\eta^{-1}$. Choose M and η such that

$$\mu\leqq\mu^M+\varepsilon,\qquad \eta M<\varepsilon; \tag{2.3}$$

this is possible by (1.1). Now,

$$\begin{aligned} P(l_{nn}<n(\mu-3\varepsilon)) &\leqq P(l^M_{nn}<n(\mu^M-2\varepsilon))\\ &\leqq(1+\eta^{-1})P(b^M_{0n}<n(\mu^M-2\varepsilon)+M\eta n)\\ &\leqq(1+\eta^{-1})P(b^M_{0n}<n(\mu^M-\varepsilon)),\end{aligned} \tag{2.4}$$

since if $l^M_{nn}<n(\mu^M-2\varepsilon)$ then some point $Q=(0,q)$ on the left edge of B_{nn} is joined to the line $x=n$ by a path with passage time less than $n(\mu^M-2\varepsilon)$ in the truncated system; the vertex in $P_n(\eta)$ which is nearest to Q may be joined to Q by a straight segment with at most ηn edges and passage time not exceeding

$M\eta n$, and therefore to the line $x=n$ by a path with passage time less than $n(\mu^M-2\varepsilon)+M\eta n$. Now (2.2) follows immediately from (2.4) by using (1.3). In the next section we shall see that (2.2) implies that $P(b_{0n}^M<n(\mu^M-\varepsilon))$ decays geometrically in n, and part (a) of the theorem follows by summing (2.4) over n and applying the Borel-Cantelli lemma.

(b) Let $a_{0n}(k)$ be the infimum of the lengths of paths joining $(0,0)$ to $(n,0)$ which lie entirely within the horizontal cylinder $\{(x,y)\colon |y|\leqq k\}$. By the usual subadditivity argument,

$$\frac{1}{n}a_{0n}(k)\to\mu(k) \text{ a.s. and in } L^1, \tag{2.5}$$

where $\mu(k)=\mu(F,k)$ is a constant; the ergodic theorem for subadditive processes is applicable since, if $k\geqq 2$,

$$E a_{01}(k)<\infty,$$

by virtue of the assumption that $\int(1-F(x))^4\,dx<\infty$ (see Smythe and Wierman (1978, p. 68)). An easy adaptation of the argument of Smythe and Wierman (1978, p. 88) yields that

$$\mu(k)\to\mu \quad \text{as} \quad k\to\infty.$$

For all $n>\frac{1}{2}k$ we have that

$$\frac{1}{n}l_{nn}\leqq\frac{1}{n}a_n(k) \tag{2.6}$$

where $a_n(k)$ is the absolute passage time between $(0,k)$ and (n,k) over paths within the cylinder $\{(x,y)\colon |y-k|\leqq k\}$; $\{a_n(k)\colon n\geqq 1\}$ has the same joint distributions as $\{a_{0n}(k)\colon n\geqq 1\}$. Thus

$$\begin{aligned}\limsup_{n\to\infty}\frac{1}{n}l_{nn}&\leqq\lim_{n\to\infty}\frac{1}{n}a_n(k)\\&=\mu(k) \text{ a.s.}\\&\to\mu \quad\text{as}\quad k\to\infty,\end{aligned}$$

and a.s. convergence follows. L^1 convergence follows from (2.5), (2.6) and uniform integrability.

The proof of Theorem 2.2 follows the proof of Theorem 2.1, the only substantial difference being that we are unable to use truncation and the continuity of the time constant. Before presenting this proof, we recall some facts about first-passage percolation on $\mathbb{Z}^d$. Let

$$C_{0n}=\{\boldsymbol{x}\in\mathbb{Z}^d\colon 0<x_1<n\} \quad\text{and}\quad H_n=\{\boldsymbol{x}\in\mathbb{Z}^d\colon x_1=n\},$$

and define the usual passage times

$$a_{0n}=t(\boldsymbol{0},(n,0,\dots,0)),\qquad b_{0n}=t(\boldsymbol{0},H_n),$$

$$s_{0n}=\inf\{t(r)\colon r \text{ joins } \boldsymbol{0} \text{ to } H_n \text{ and } r \text{ is contained in } C_{0n} \text{ except for its endpoints}\},$$

$$t_{0n}=\inf\{t(r)\colon r \text{ joins } \boldsymbol{0} \text{ to } (n,0,\dots,0) \text{ and } r \text{ is contained in } C_{0n} \text{ except for its endpoints}\}.$$

Lemma 2.3. *If the time coordinates have finite variance then, as $n\to\infty$,*

$$\frac{1}{n}\theta_{0n}\to\mu(d) \text{ a.s. and in } L^1,$$

where $\mu(d)=\mu(d,F)$ is the time constant of $\mathbb{Z}^d$, and θ may represent a, b, s or t.

Proof. By the ergodic theorem for subadditive processes, there exist constants μ_1 and μ_2 such that, as $n\to\infty$,

$$\frac{1}{n}a_{0n}\to\mu_1,\quad \frac{1}{n}t_{0n}\to\mu_2 \text{ a.s. and in } L^1. \tag{2.7}$$

By the ergodic theorem for "superconvolutive sequences" (see Theorem 2.9 of Smythe and Wierman (1978) or Theorem 3.3 of Kingman (1976)), there exists a constant μ_3 such that, as $n\to\infty$,

$$\frac{1}{n}s_{0n}\to\mu_3 \text{ a.s. and in } L^1.$$

By Theorem 6 of Cox and Durrett (1981),

$$\frac{1}{n}b_{0n}\to\mu_1 \text{ a.s.}\quad \text{as}\quad n\to\infty. \tag{2.8}$$

Cox and Durrett proved this for the case $d=2$, but their proof applies also to higher dimensions. Theorem 3 of Cox and Durrett (1981) imposes a less stringent condition on F than that it have finite variance; it seems clear that, in this lemma also, a rather weaker condition would suffice. Next we note that $\mu_1=\mu_2$; this is proved by Smythe and Wierman (1978, Theorem 5.1) for the case $d=2$, and their proof is valid in higher dimensions even though it is not known whether or not routes exist in general. Finally, the inequality

$$0\leqq b_{0n}\leqq s_{0n}\leqq t_{0n} \tag{2.9}$$

implies that $\mu_1=\mu_2=\mu_3$, and this shared value is the time constant $\mu(d)$. The L^1 convergence of $n^{-1}b_{0n}$ follows from (2.7), (2.8) and (2.9).

Proof of Theorem 2.2. Let L_n be the face

$$\{(0,x_2,\dots,x_d)\colon 0\leqq x_2,\dots,x_d\leqq n\}\qquad \text{of } B_n(d).$$

Let $\eta>0$ and let $P_n(\eta)$ be the vertices of L_n which have the form $(0,\lfloor k_2\eta n\rfloor,\dots,\lfloor k_d\eta n\rfloor)$ where $0\leqq k_2,\dots,k_d\leqq\lfloor\eta^{-1}\rfloor$; we call $P_n(\eta)$ the set of *special vertices.*

Note that

$$|P_n(\eta)| \leqq (1+\eta^{-1})^{d-1}. \tag{2.10}$$

Let $\boldsymbol{v}$ be a typical vertex in L_n, and let $E(\boldsymbol{v})$ be the event that there exists a path in L_n joining $\boldsymbol{v}$ to some special vertex with passage time not exceeding $2(n\eta+1)(d-1)\lambda$, where λ is the mean time coordinate (see (1.11)). L_n has $2(d-1)$ bounding hyperplanes of the form $H(k,m)=\{\boldsymbol{x}\in\mathbb{Z}^d\colon x_k=m\}$ for $m=0,n$ and $k=2,\ldots,d$. If $\boldsymbol{v}$ belongs to exactly h of these hyperplanes, then there are at least $2(d-1)-h$ edge-disjoint paths joining $\boldsymbol{v}$ to some special vertex, having at most $e(n)=\lfloor(\eta n+1)(d-1)\rfloor$ edges. Thus, by Čebyšev's inequality, the complement of $E(\boldsymbol{v})$ satisfies

$$\begin{aligned} P(\overline{E(\boldsymbol{v})}) &\leqq \{P(t_1+\ldots+t_{e(n)}>2e(n)\lambda)\}^{2(d-1)-h} \\ &\leqq \left(\frac{\sigma^2}{\lambda^2\eta(d-1)n}\right)^{2(d-1)-h} = p_h, \quad \text{say,} \end{aligned}$$

where $t_1,\ldots,t_{e(n)}$ are independent time coordinates and $\sigma^2=\mathrm{var}(t_1)<\infty$. The event E_n, that every $\boldsymbol{v}\in L_n$ is joined to some special vertex of L_n by a path whose passage time is "not too large", satisfies

$$P(\bar{E}_n) \leqq \sum_{h=0}^{d-1} N_h p_h,$$

where $N_h=O(n^{d-1-h})$ is the number of vertices of L_n which are in exactly h of the bounding hyperplanes of L_n. Thus

$$P(\bar{E}_n) \leqq \sum_{h=0}^{d-1} O(n^{d-1-h})\, O\left(\frac{1}{n^{2(d-1)-h}}\right) = O\left(\frac{1}{n^{d-1}}\right). \tag{2.11}$$

Let $\varepsilon>0$ and pick η such that

$$3\eta\lambda(d-1)<\varepsilon. \tag{2.12}$$

Then, as in (2.4), for all large n,

$$\{l_n(d)<n(\mu(d)-2\varepsilon)\} \subseteq \bar{E}_n \cup \Big\{ \bigcup_{\boldsymbol{v}\in P_n(\eta)} \{b(\boldsymbol{v},n)<n(\mu(d)-\varepsilon)\}\Big\}$$

where $b(\boldsymbol{v},n)$ $(=t(\boldsymbol{v},H_n))$ is the point-to-hyperplane time between $\boldsymbol{v}$ (in L_n) and the hyperplane $H_n=\{\boldsymbol{x}\in\mathbb{Z}^d\colon x_1=n\}$. By (2.11) and Lemma 2.3, we have that

$$\begin{aligned} &P(l_n(d)<n(\mu(d)-2\varepsilon)) \\ &\quad \leqq O\left(\frac{1}{n^{d-1}}\right)+(1+\eta^{-1})^{d-1}P(b_{0n}<n(\mu(d)-\varepsilon))\to 0 \quad \text{as} \quad n\to\infty. \end{aligned} \tag{2.13}$$

We shall see in the next section (see Theorem 3.4) that (2.13) implies that $P(b_{0n}<n(\mu(d)-\varepsilon))$ decays geometrically in n, and the Borel-Cantelli lemma

then gives that

$$P\left(\liminf_{n\to\infty}\frac{1}{n}l_n(d)\geqq\mu(d)\right)=1$$

as required.

To show the appropriate upper bound for $\limsup_{n\to\infty}\frac{1}{n}l_n(d)$, let k be a positive integer and define $t_{0n}(k)$ to be the infimum of the passage times of paths between $\boldsymbol{0}$ and $(n,0,\ldots,0)$ which are entirely contained in the box $\{\boldsymbol{x}\in\mathbb{Z}^d\colon 0<x_1<n,\ |x_i|\leqq k \text{ for } i=2,\ldots,d\}$ except for their endpoints. The argument of the proof of Theorem 2.1 is easily adapted to yield, for $n>\frac{1}{2}k$,

$$\frac{1}{n}l_n(d)\leqq\frac{1}{n}t_n(k)\to\mu(d,k) \text{ a.s. and in } L^1$$

as $n\to\infty$, where $\{t_n(k)\colon n\geqq 1\}$ has the same joint distributions as $\{t_{0n}(k)\colon n\geqq 1\}$ and $\{\mu(d,k)\colon k=1,2,\ldots\}$ is a sequence of constants satisfying

$$\mu(d,k)\to\mu(d) \quad \text{as} \quad k\to\infty.$$

3. The Rate of Convergence to the Time Constant

In this section we prove that the random variables $n^{-1}\theta_{0n}$ converge to the time constant at a geometric rate, where θ may represent a, b, s, or t. First, we deal with first-passage times on the lattice $\mathbb{Z}^2$.

Theorem 3.1. *If* $\varepsilon>0$, *there exist constants* $A(\varepsilon)$, $B(\varepsilon)>0$ *such that*

$$P(\theta_{0n}<n(\mu-\varepsilon))\leqq A(\varepsilon)\exp(-nB(\varepsilon)) \quad \textit{for all } n,$$

where $\theta=a$, b, s *or* t.

Theorem 3.2. *If* $\varepsilon>0$ *and there exists* $\gamma>0$ *such that*

$$\int_0^\infty \exp(\gamma x)\,dF(x)<\infty$$

then there exist constants $C(\varepsilon)$, $D(\varepsilon)>0$ *such that*

$$P(\theta_{0n}>n(\mu+\varepsilon))\leqq C(\varepsilon)\exp(-nD(\varepsilon)) \quad \textit{for all } n,$$

where $\theta=a$, b, s *or* t.

It is clear that the finiteness of the moment generating function of F is also necessary for the conclusion of Theorem 3.2 to hold for any θ; this follows from the observation that θ_{0n} is at least as big as the minimum of the time coordinates of the four edges incident with the origin.

We shall prove Theorems 3.1 and 3.2 for the cases $\theta=b$ and $\theta=t$ respectively; the other cases follow from the obvious inequalities $b_{0n}\leqq a_{0n}$, $s_{0n}\leqq t_{0n}$.

It will appear as a direct consequence of Theorem 3.1 that $P(\theta_{0n}<n(\mu-\varepsilon))$ decays approximately geometrically as $n\to\infty$, where θ may represent a, s or t.

Corollary 3.3. *Suppose that* $\mu > 0$. *For each* ε, *there exists* $B(\varepsilon)$ *such that* $0 \leqq B(\varepsilon) \leqq \infty$ *and*

$$P(\theta_{0n} < n(\mu - \varepsilon)) = \exp(-n(B(\varepsilon) + o(1))) \quad \text{as} \quad n \to \infty,$$

where $\theta = a$, s *or* t. *Furthermore*

(a) $B(\varepsilon) = 0$ *if* $\varepsilon < 0$ *and* $B(\varepsilon) > 0$ *if* $\varepsilon > 0$,

(b) $B(\varepsilon) < \infty$ *if* $\varepsilon < \beta$ *and* $B(\varepsilon) = \infty$ *if* $\varepsilon \geqq \beta$, *where* $\beta = \sup\{\varepsilon : F(\mu - \varepsilon) > 0\} \geqq 0$,

(c) *if* $\beta > 0$ *then* B *is convex and continuous on* $(-\infty, \beta)$ *and strictly increasing on* $[0, \beta)$.

Results similar to Theorems 3.1, 3.2 and Corollary 3.3 hold for lattices other than $\mathbb{Z}^2$. For example, the proofs of Theorems 3.1 and 3.2 may be adapted readily to deal with a large family of infinite graphs in two and higher dimensions; see Kesten (1982, Chap. 2) for a description of such a family. Rather than explore such generalizations in depth, we restrict ourselves to a statement of a corresponding result for percolation in $\mathbb{Z}^d$.

Theorem 3.4. *Let* a, b, s, t *denote the passage times of first-passage percolation in* $\mathbb{Z}^d$, *and suppose* $\varepsilon > 0$.

(a) *If the time coordinates have finite variance, then there exist constants* $A(\varepsilon)$, $B(\varepsilon) > 0$ *such that*

$$P(\theta_{0n} < n(\mu - \varepsilon)) \leqq A(\varepsilon) \exp(-nB(\varepsilon)) \quad \text{for all } n,$$

and

(b) *if there exists* $\gamma > 0$ *such that*

$$\int_0^\infty \exp(\gamma x)\, dF(x) < \infty,$$

then there exist constants $C(\varepsilon)$, $D(\varepsilon) > 0$ *such that*

$$P(\theta_{0n} > n(\mu + \varepsilon)) \leqq C(\varepsilon) \exp(-nD(\varepsilon)) \quad \text{for all } n,$$

where θ *may represent* a, b, s *or* t.

Proof of Theorem 3.1. We may suppose that $\mu > 0$, and shall show that, for $0 < \varepsilon < \mu/5$,

$$P(b_{0n} < n(\mu - 5\varepsilon)) \leqq A e^{-nB} \tag{3.1}$$

for positive constants A and B depending on ε. The idea of the proof is based on a "block approach". We shall suppose that r is a path joining $\boldsymbol{0}$ to the line $H_n = \{(n, y) \colon y = 0, \pm 1, \ldots\}$ with passage time smaller than $n(\mu - 5\varepsilon)$, and shall divide r into segments, each of whose two endpoints are fairly distant from each other. Almost all of the passage times of these segments will be close to that predicted by the limit theorem for first-passage times, and it is correspondingly unlikely that such a path r exists.

First we need some notation. For any $\boldsymbol{k}=(k_1,k_2)\in\mathbb{Z}^2$ and integers M, N satisfying $M>N>1$,

$$S(\boldsymbol{k})=\{\boldsymbol{v}\in L: N\boldsymbol{k}\leqq \boldsymbol{v}<N(\boldsymbol{k}+1)\}$$

$$T(\boldsymbol{k})=\{\boldsymbol{v}\in L: N\boldsymbol{k}-M\leqq \boldsymbol{v}<N(\boldsymbol{k}+1)+M\}.$$

$S(\boldsymbol{k})$ is a square box, side length $N-1$, with bottom left hand corner at $N\boldsymbol{k}$; $T(\boldsymbol{k})$ is a larger square box, containing $S(\boldsymbol{k})$ at its centre, with side length $2M+N-1$. Thus $\{S(\boldsymbol{k})\colon \boldsymbol{k}\in\mathbb{Z}^2\}$ is a partition of L; each $\boldsymbol{v}\in L$ belongs to exactly one of the S's. We think of M as being much larger than N, but much smaller than n.

Suppose that $r=(\boldsymbol{v}(0), e(1), \ldots, e(v), \boldsymbol{v}(v))$ is a path in L joining $\boldsymbol{v}(0)=\boldsymbol{0}$ to some vertex $\boldsymbol{v}(v)$ on the line H_n. We divide r into segments by following its progress amongst the S's and T's. Let $\boldsymbol{k}(0)=\boldsymbol{0}$ and $a(0)=0$. Let $\boldsymbol{v}(a(1))$ be the earliest vertex $\boldsymbol{v}(a)$ in r with the property that $\boldsymbol{v}(a)\notin T(\boldsymbol{k}(0))$; let $\boldsymbol{k}(1)$ be the unique $\boldsymbol{k}$ such that $\boldsymbol{v}(a(1))\in S(\boldsymbol{k})$. Continue recursively to find sequences $a(0)$, $a(1), \ldots, a(\tau)$, $\boldsymbol{k}(0), \boldsymbol{k}(1), \ldots, \boldsymbol{k}(\tau)$ such that

(a) $0=a(0)<a(1)<\ldots<a(\tau)\leqq v$

(b) $\boldsymbol{v}(a(i))\in S(\boldsymbol{k}(i))$

(c) $a(i+1)$ is the smallest integer a, exceeding $a(i)$, such that $\boldsymbol{v}(a)\notin T(\boldsymbol{k}(i))$.

The final terms $a(\tau)$ and $\boldsymbol{k}(\tau)$ satisfy

$$\boldsymbol{v}(j)\in T(\boldsymbol{k}(\tau)) \quad \text{if } a(\tau)\leqq j\leqq v. \tag{3.2}$$

The sequence $\sigma=(\boldsymbol{k}(0),\ldots,\boldsymbol{k}(\tau))$ may contain double points, since r may visit some S twice. We remove all such double points by the process of "loop-removal". Let u_1 be the largest index for which there exists p_1 $(<u_1)$ with the property that $\boldsymbol{k}(p_1)=\boldsymbol{k}(u_1)$. From the collection of possible such values of p_1, choose the smallest. Form the subsequence $\sigma_1=(\boldsymbol{k}(0), \ldots, \boldsymbol{k}(p_1), \boldsymbol{k}(u_1+1), \ldots, \boldsymbol{k}(\tau))$ of σ by removing the loop $(\boldsymbol{k}(p_1+1), \ldots, \boldsymbol{k}(u_1))$. The section $(\boldsymbol{k}(u_1), \ldots, \boldsymbol{k}(\tau))$ of the original sequence is free of double points, and, by the minimality of p_1, it has no points in common with the earlier portion $(\boldsymbol{k}(0), \ldots, \boldsymbol{k}(p_1-1))$. Thus, if the new sequence σ_1 has a double point, then there exist p_2, u_2 such that $p_2<u_2<p_1$ and $\boldsymbol{k}(p_2)=\boldsymbol{k}(u_2)$. Again we choose the largest such u_2 and then the smallest such p_2 and remove the section $(\boldsymbol{k}(p_2+1), \ldots, \boldsymbol{k}(u_2))$ to obtain a subsequence σ_2 of σ_1. We continue in this way until we arrive at a subsequence

$$\tilde{\sigma}=(\boldsymbol{k}(0),\ldots,\boldsymbol{k}(p_2),\boldsymbol{k}(u_2+1),\ldots,\boldsymbol{k}(p_1),\boldsymbol{k}(u_1+1),\ldots,\boldsymbol{k}(\tau)) \tag{3.3}$$

without double points. We re-label the terms in (3.3) to obtain

$$\tilde{\sigma}=(\boldsymbol{l}(0),\ldots,\boldsymbol{l}(\rho))$$

where $\boldsymbol{l}(a)=\boldsymbol{k}(j_a)$ and $0=j_0<j_1<\ldots<j_\rho\leqq\tau$; it is mostly with this loopless sequence that we shall work. Note that j_ρ and τ may not be equal, since u_1 may equal τ; however, it is easily seen that $\boldsymbol{k}(j_\rho)=\boldsymbol{k}(\tau)$. Now,

$$|k_m(j+1)-k_m(j)|\leqq\frac{M}{N}+1 \quad \text{for } m=1,2,\ j=0,1,\ldots,\tau-1,$$

where $\boldsymbol{k}(j)=(k_1(j),k_2(j))$, since the first exit from $T(\boldsymbol{k}(j))$ takes r to a vertex distance 1 from some vertex in $T(\boldsymbol{k}(j))$. This property is not changed by loop-removal, in that

$$(3.4) \qquad |l_m(j+1)-l_m(j)| \leqq \frac{M}{N}+1 \qquad \text{for } m=1,2,\ j=0,1,\dots,\rho-1.$$

To see this, note that if $i<j$ and $\boldsymbol{k}(i)=\boldsymbol{k}(j)$, then $S(\boldsymbol{k}(j+1))$ contains a vertex distance 1 from some vertex in $T(\boldsymbol{k}(i))$.

Consider the portion $r(i)=(\boldsymbol{v}(a(i-1)),\dots,\boldsymbol{v}(a(i)))$ of r which stretches between $S(\boldsymbol{k}(i-1))$ and $S(\boldsymbol{k}(i))$, and define

$$L(i)=\max_{m=1,2} |v_m(a(i))-v_m(a(i-1))|$$

to be the maximum of the two coordinate differences between its endpoints. It is clear from the construction of the a's and $\boldsymbol{k}$'s that

$$(3.5) \qquad M \leqq L(i) \leqq M+N \qquad \text{for } 1\leqq i \leqq \tau.$$

Until now, we have merely specified those segments of r in which we are interested. Next, we proceed to *colour* the $\boldsymbol{l}$'s in a way which depends on the random time coordinates of L. For $1\leqq i\leqq \rho$, consider the vector $\boldsymbol{l}(i)=\boldsymbol{k}(j_i)\in\mathbb{Z}^2$. Examine the segment $r(j_i)$, stretching between $S(\boldsymbol{k}(j_i-1))$ and $S(\boldsymbol{k}(j_i))$, and let $t(r(j_i))$ be its passage time. Let $\varepsilon>0$. We colour $\boldsymbol{l}(i)$ *white* if

$$(3.6) \qquad t(r(j_i)) \leqq (\mu-2\varepsilon)L(j_i);$$

otherwise we colour $\boldsymbol{l}(i)$ *black*. Thus each $\boldsymbol{l}(i)$ is coloured white if and only if the segment of r which leads to $S(\boldsymbol{k}(j_i))$ has a passage time which is small compared with $\mu L(j_i)$. Let w be the number of white points in the sequence of $\boldsymbol{l}$'s. We need to estimate ρ and w in terms of ε, n, M, N, and do this in the next lemma.

Lemma 3.5. *Suppose that ε, n, M, N and r satisfy the following:*

(i) *ε is small, in that $0<\varepsilon<\mu/5$,*
(ii) *M/N is large, in that $M(\mu-3\varepsilon)\geqq(M+N)(\mu-4\varepsilon)$,*
(iii) *n is large, in that $n\varepsilon\geqq(M+2N)(\mu-4\varepsilon)$,*
(iv) *the passage time of r is small, in that $t(r)\leqq n(\mu-5\varepsilon)$.*

Then $w\geqq\dfrac{\varepsilon\rho}{2\mu}$ and $\rho\geqq\dfrac{n}{M+N}-1$.

Proof. Suppose that (i)–(iv) hold. By the definition of r, the x coordinates of the vertices of r are such that

$$(3.7) \qquad n=v_1(v)-v_1(0)=(v_1(v)-v_1(a(\tau)))+(v_1(a(\tau))-v_1(a(j_\rho))) + \sum_{i=1}^{\rho}(v_1(a(j_i))-v_1(a(j_{i-1}))).$$

Now

$$\begin{aligned}
|v_1(v)-v_1(a(\tau))| &\leqq M+N, \quad \text{by (3.2)},\\
|v_1(a(\tau))-v_1(a(j_\rho))| &\leqq N, \quad \text{since } \boldsymbol{k}(\tau)=\boldsymbol{k}(j_\rho),\\
|v_1(a(j_i))-v_1(a(j_{i-1}))| &\leqq |v_1(a(j_i))-v_1(a(j_i-1))|\\
&\quad +|v_1(a(j_i-1))-v_1(a(j_{i-1}))|\\
&\leqq L(j_i)+N,
\end{aligned}$$

since, by the definition of loop-removal, $\boldsymbol{v}(a(j_{i-1}))$ and $\boldsymbol{v}(a(j_i-1))$ belong to the same $S(\boldsymbol{k})$, for some $\boldsymbol{k}$. Thus, from (3.7),

$$n \leqq M+2N+\sum_{i=1}^{\rho}(L(j_i)+N). \tag{3.8}$$

However, by (3.5),

$$N=\frac{N}{M}M \leqq \frac{N}{M}L(j) \quad \text{for all } j,$$

and so (3.8) yields

$$\sum_{i=1}^{\rho} L(j_i) \geqq \frac{M}{M+N}(n-(M+2N)) \geqq n\left(\frac{\mu-5\varepsilon}{\mu-3\varepsilon}\right) \quad \text{by (ii) and (iii)}.$$

Hence

$$\begin{aligned}
(\mu-3\varepsilon)\sum_1^{\rho} L(j_i) &\geqq n(\mu-5\varepsilon) \geqq t(r) \quad \text{by (iv)}\\
&\geqq \sum_{i=1}^{\rho} t(r(j_i))\\
&\geqq \sum_{i:\boldsymbol{I}(i)\,\text{black}} L(j_i)(\mu-2\varepsilon) \quad \text{by condition (3.6)}\\
&=(\mu-2\varepsilon)\left\{\sum_1^{\rho} L(j_i)-\sum_{i:\boldsymbol{I}(i)\,\text{white}} L(j_i)\right\}.
\end{aligned} \tag{3.9}$$

Therefore, by (3.5),

$$w(M+N) \geqq \sum_{i:\boldsymbol{I}(i)\,\text{white}} L(j_i) \geqq \frac{\varepsilon}{\mu-2\varepsilon}\sum_1^{\rho} L(j_i) \geqq \frac{\varepsilon}{\mu-2\varepsilon}\rho M$$

giving that

$$w \geqq \frac{\varepsilon\rho}{2\mu} \quad \text{by (i) and (ii)}. \tag{3.10}$$

Also, it is a consequence of (3.2) and the fact that $v_1(v)=n$ that

$$l_1(\rho)-l_1(0) \geqq \frac{1}{N}(n-M-N),$$

and hence, by (3.4),

$$\rho \geqq \frac{n-M-N}{M+N},$$

completing the proof of the lemma.

Lemma 3.6. *The event* $\{l(i)$ is white$\}$ *is contained in the event* $\mathscr{E}(i)=\{$some vertex in $S(\boldsymbol{k}(j_i))$ is joined to some vertex outside the square $\{v\varepsilon L: N\boldsymbol{k}(j_i)-(M-N)\leqq v<N\boldsymbol{k}(j_i)+M\}$ by a path with passage time less than $(M+N)(\mu-2\varepsilon)\}$. *For any subcollection* $\Lambda=\{\boldsymbol{l}(b(i)): 1\leqq i\leqq\sigma\}$ *of the* $\boldsymbol{l}$*'s* one has

(3.12) $P(\text{all } \boldsymbol{l}(b(i)) \text{ in } \Lambda \text{ are white})$

$$\leqq P(\mathscr{E}(b(i)) \text{ occurs for each } 1\leqq i\leqq\sigma)\leqq p^{\delta\sigma},$$

where $p=p(M,N,\varepsilon)$ *is the probability that some vertex in* $S(\boldsymbol{0})$ *is joined to some vertex outside the square* $[-M+N,M)^2$ *by a path with passage time less than* $(M+N)(\mu-2\varepsilon)$, *and* $\delta=\delta(M,N)>0$.

Proof. The first statement follows from the observation that $r(j_i)$ joins some vertex inside $S(\boldsymbol{k}(j_i))$ to some vertex outside

$$\{\boldsymbol{v}\varepsilon L: N\boldsymbol{k}(j_i)-(M-N)\leqq\boldsymbol{v}<N\boldsymbol{k}(j_i)+M\}.$$

To prove (3.12) note first that $P(\mathscr{E}(i))=p(M,N,\varepsilon)$. Next note that the event $\mathscr{E}(i)$ depends only on the time coordinates in the square $\{\boldsymbol{v}\varepsilon L: N\boldsymbol{k}(j_i)-M\leqq v\leqq N\boldsymbol{k}(j_i)+M\}$, and thus the events $\mathscr{E}(a)$ and $\mathscr{E}(b)$ are independent if

$$|l_m(a)-l_m(b)|\geqq\frac{4M+3N}{N} \quad \text{for either } m=1 \text{ or } m=2. \tag{3.13}$$

But for any collection Λ, containing σ distinct $\boldsymbol{l}$'s, we can select a subset of the $\boldsymbol{l}$'s, each pair of which satisfies (3.13), in the following way. Pick $\boldsymbol{l}(1)$ arbitrarily in Λ. Assume $\boldsymbol{l}(1)$, $\boldsymbol{l}(2)$, ..., $\boldsymbol{l}(j)$ have already been picked and denote by $C(i)$ the square $\{\boldsymbol{l}=(l_1,l_2): |l_m-l_m(i)|\leqq(4M+3N)N^{-1}$ for $m=1$ and $m=2\}$, for $1\leqq i\leqq j$. Then pick $\boldsymbol{l}(j+1)$ arbitrarily from $\Lambda\setminus(C(1)\cup\ldots\cup C(j))$, so long as the last set is nonempty. Since each $C(i)$ contains at most $((8M+6N)N^{-1}+1)^2$ points of $\mathbb{Z}^2$, one can select at least $\delta\sigma$ $\boldsymbol{l}$'s in this way, where $\delta=\delta(M,N)$ is given by

$$\delta(M,N)=\left(\frac{N}{8(M+N)}\right)^2. \tag{3.14}$$

Let $\Lambda'=\{\boldsymbol{l}(c(i)): 1\leqq i\leqq\delta\sigma\}$ be such a subcollection of Λ. Then

$$P(\Lambda \text{ is white})\leqq P(\Lambda' \text{ is white})\leqq\prod_{i\leqq\delta\sigma}P(\mathscr{E}(c(i)) \text{ occurs})\leqq p(M,N,\varepsilon)^{\delta\sigma}.$$

Here is a final lemma before we complete the proof of the theorem.

Lemma 3.7. *If* $0<\varepsilon<\mu/5$, *then*

$$p=p(M,N,\varepsilon)\to 0, \quad \text{as} \quad M,N\to\infty,$$

so long as $M\geqq N(2\mu/\varepsilon)$.

Proof. Let B_1, B_2, B_3, B_4 denote the following rectangles:

$$B_1=[-M+N,M]\times[N,M], \qquad B_2=[N,M]\times[-M+N,M],$$
$$B_3=[-M+N,M]\times[-M+N,0], \quad B_4=[-M+N,0]\times[-M+N,M].$$

If a vertex in $S(\boldsymbol{0})$ is joined to a vertex outside the square $[-M+N, M)^2$ by a path with passage time not exceeding $(M+N)(\mu-2\varepsilon)$, then one of the B's is crossed between its longer sides by a path with passage time not exceeding $(M+N)(\mu-2\varepsilon)$. Thus

$$p(M,N,\varepsilon) \leqq 4P(l_{M-N,2M-N} \leqq (M+N)(\mu-2\varepsilon))$$

where l is the line-to-line passage time of Sect. 2. If $M \geqq N(2\mu/\varepsilon)$ and $0<\varepsilon<\mu/5$ then $2M-N \leqq 3(M-N)$ and $(M+N)(\mu-2\varepsilon) \leqq (M-N)(\mu-\varepsilon)$, giving that

$$\begin{aligned} p(M,N,\varepsilon) &\leqq 4P(l_{M-N,3(M-N)} \leqq (M-N)(\mu-\varepsilon)) \\ &\to 0 \quad \text{as} \quad M-N \to \infty \end{aligned}$$

by Eq. (2.2), suitably generalized as necessary for the proof of the general version of Theorem 2.1.

We may now complete the *proof of Theorem 3.1.* Each path r from $\boldsymbol{0}$ to H_n has an associated sequence $\boldsymbol{l}(0)=\boldsymbol{0}, \ldots, \boldsymbol{l}(\rho)$. The number of possible choices for this sequence of $\boldsymbol{l}$'s is at most $\left(8\left(\frac{M}{N}+1\right)\right)^{\rho}$, since $S(\boldsymbol{l}(i+1))$ contains a vertex which is adjacent to some boundary vertex of $T(\boldsymbol{l}(i))$, and there are at most $8(M+N)$ such vertices. Given the sequence of $\boldsymbol{l}$'s, there are $\binom{\rho}{w}$ ways of choosing a set of cardinality w (as the white points). Now, let η be a constant $(0<\eta<1)$ to be chosen shortly, and suppose that $0<\varepsilon<\mu/5$. Choose M, N such that

$$(3.15) \qquad 2\mu N \leqq M\varepsilon \leqq (3\mu-\varepsilon)N, \qquad p(M,N,\varepsilon)<\eta;$$

this is possible by Lemma 3.7. The first condition of (3.15) implies that (ii) of Lemma 3.5 holds. Suppose also that n is large enough for (iii) of Lemma 3.5 to hold; by (i) and (iii), we have that $n \geqq M+2N$, giving by Lemma 3.5 that $\rho \geqq Kn$ where

$$K=K(M,N)=\frac{N}{(M+N)(M+2N)}.$$

By Lemmas 3.5, 3.6 and 3.7,

$$(3.16) \qquad P(b_{0n}<n(\mu-5\varepsilon)) \leqq \sum_{\rho \geqq Kn} \left(8\left(\frac{M}{N}+1\right)\right)^{\rho} \sum_{w \geqq \varepsilon\rho/(2\mu)} \binom{\rho}{w} p^{\delta w}$$

where $\delta=\delta(M,N)=\left(\frac{N}{8(M+N)}\right)^2>0$.

By (3.15),

$$8\left(\frac{M}{N}+1\right) \leqq \frac{24\mu}{\varepsilon} \quad \text{and} \quad \delta \geqq \frac{\varepsilon^2}{576\mu^2}.$$

Choose η small enough so that

$$(3.17) \qquad 48\mu\eta^{\varepsilon^3/(1152\mu^3)}<\varepsilon.$$

Then

$$P(b_{0n}<n(\mu-5\varepsilon))\leqq \sum_{\rho\geqq Kn}\left(\frac{24\mu}{\varepsilon}\right)^{\rho}\sum_{w\geqq \varepsilon\rho/(2\mu)} 2^{\rho}\eta^{w\varepsilon^2/(576\mu^2)}. \tag{3.18}$$

The final series in (3.18) is geometric with ratio smaller than one, and so

$$P(b_{0n}<n(\mu-5\varepsilon))\leqq \sum_{\rho\geqq Kn} K_2\left(\frac{48\mu}{\varepsilon}\eta^{\varepsilon^3/(1152\mu^3)}\right)^{\rho}\leqq K_3\alpha^n$$

by (3.17), for some constants K_2, K_3, α, where

$$0<\alpha=\left(\frac{48\mu}{\varepsilon}\eta^{\varepsilon^3/(1152\mu^3)}\right)^{K}<1.$$

This completes the proof of the theorem.

Proof of Theorem 3.2. This is much simpler than the previous proof, and proceeds by the subadditive inequality and standard estimates for the tails of sums of independent, identically distributed random variables. Let $\varepsilon>0$ and let N be chosen such that

$$g_N=\frac{1}{N}E(t_{0N})\leqq\mu+\varepsilon. \tag{3.19}$$

If $n=rN$ for some integer r, then by the subadditive inequality

$$P(t_{0n}>n(\mu+2\varepsilon))\leqq P(Y_1+\ldots+Y_r>n(\mu+2\varepsilon))$$

where $Y_i=t_{(i-1)N,iN}$. The Y's are independent and identically distributed; furthermore, Y_1 is no greater than the sum of the time coordinates of the edges in the straight segment joining the origin to $(N,0)$, implying that

$$E(e^{\xi Y_1})\leqq\left\{\int_0^\infty e^{\xi x}\,dF(x)\right\}^N<\infty \tag{3.20}$$

for small positive values of ξ. Writing $Z_i=Y_i-E(Y_i)$, we deduce that

$$\begin{aligned}P(t_{0n}>n(\mu+2\varepsilon))&\leqq P(Z_1+\ldots+Z_r>n(\mu+2\varepsilon)-ng_N)\\&\leqq P(Z_1+\ldots+Z_r>rN\varepsilon)\quad\text{by (3.19)}\\&\leqq\left(\frac{E(e^{\xi Z_1})}{e^{N\varepsilon\xi}}\right)^r\quad\text{for }\xi\geqq 0.\end{aligned}$$

Recall that $E(Z_1)=0$, so that $E(e^{\xi Z_1})=1+o(\xi)$. Hence

$$P(t_{0n}>n(\mu+2\varepsilon))\leqq\beta(\xi)^n \tag{3.21}$$

where $\beta(\xi)=(1-\xi N\varepsilon+o(\xi))^{1/N}$. Choose $\xi_0>0$ such that $0<\beta(\xi_0)<1$ to deduce the appropriate estimate in the case when n is a multiple of N.

For general values of n, write $n=rN+s$ where $0\leqq s<N$, and note that

$$t_{0n}\leqq t_{0,rN}+t_{rN,n}$$

where the latter two variables are independent. Thus

$$P(t_{0n}>n(\mu+3\varepsilon))\leqq P(t_{0,rN}>n(\mu+2\varepsilon))+P(t_{rN,n}>n\varepsilon). \tag{3.22}$$

Bound $t_{rN,n}$ above by the sum $t_1+\ldots+t_s$ of the time coordinates of the segment of the x-axis between $(rN,0)$ and $(n,0)$ to obtain, by the generalization of Markov's inequality, that

$$P(t_{rN,n}>n\varepsilon)\leqq E(e^{\xi t_1})^s e^{-n\varepsilon\xi}$$

for $\xi>0$, and the proof is complete.

Proof of Corollary 3.3. The following inclusion is obvious:

$$\{\theta_{0n}<n(\mu-\varepsilon)\}\cap\{\theta_{n,n+m}<m(\mu-\varepsilon)\}\subseteq\{\theta_{0,n+m}<(n+m)(\mu-\varepsilon)\} \tag{3.23}$$

for m, $n\geqq 0$ and $\theta=a$, t. Thus

$$P(\theta_{0n}<n(\mu-\varepsilon))P(\theta_{0m}<m(\mu-\varepsilon))\leqq P(\theta_{0,n+m}<(n+m)(\mu-\varepsilon)); \tag{3.24}$$

to see this for $\theta=a$, use the *FKG* inequality (see Smythe and Wierman (1978, p. 12)). By subadditivity,

$$B_\theta(\varepsilon)=\lim_{n\to\infty}\left(-\frac{1}{n}\log P(\theta_{0n}<n(\mu-\varepsilon))\right) \tag{3.25}$$

exists for $\theta=a$, t, and satisfies $0\leqq B_\theta(\varepsilon)\leqq\infty$ for all ε. Equation (3.23) does not hold for the cylinder point-to-line passage times s_{0n}, but a similar relation holds by considering which point on the line $x=n$ is the endpoint of the shortest cylinder path from $(0,0)$. For any route r of s_{0n}, let $(n,Y(r))$ be the endpoint of r which lies on the line $x=n$. From the collection of all such vertices $(n,Y(r))$, let (n,Y) be the vertex for which $|Y(r)|$ is a minimum (if there are two such vertices, let (n,Y) be the one in the *upper* half-plane). Now, the events $\{Y=y\}$ partition the sample space as y ranges over the integers, and so

$$\bigcup_y(\{s_{0n}<n(\mu-\varepsilon),Y=y\}\cap\{s^y_{n,n+m}<m(\mu-\varepsilon)''\})$$
$$\subseteq\{s_{0,n+m}<(n+m)(\mu-\varepsilon)\}$$

where $s^y_{n,n+m}$ is the infimum of the passage times of paths joining (n,y) to H_{n+m} which lie in the cylinder $C_{n,n+m}$ except for their endpoints. Thus

$$\begin{aligned} P(s_{0,n+m}<(n+m)(\mu-\varepsilon)) &\geqq \sum_y P(s_{0n}<n(\mu-\varepsilon),Y=y)\\ &\qquad\cdot P(s_{0m}<m(\mu-\varepsilon)) \qquad \text{by independence}\\ &=P(s_{0n}<n(\mu-\varepsilon))P(s_{0m}<m(\mu-\varepsilon)),\end{aligned}$$

and the existence of

$$B_s(\varepsilon) = \lim_{n\to\infty} \left(-\frac{1}{n} \log P(s_{0n} < n(\mu-\varepsilon)) \right)$$

follows from (3.24) for all ε as before. Also, $0 \leqq B_s(\varepsilon) \leqq \infty$. It remains to show that

$$B_a(\varepsilon) = B_s(\varepsilon) = B_t(\varepsilon) \quad \text{for all } \varepsilon \tag{3.26}$$

and that this common value, $B(\varepsilon)$ say, satisfies conclusions (a), (b) and (c) of the corollary. Note first that Eq. (1.2) and Theorem 3.1 imply that

$$B_a(\varepsilon) = 0 \quad \text{if } \varepsilon < 0, \tag{3.27}$$

$$B_\theta(\varepsilon) > 0 \quad \text{if } \varepsilon > 0, \text{ where } \theta = a,\ s \text{ or } t. \tag{3.28}$$

Note also that

$$B_\theta(\varepsilon) = \infty \quad \text{if } \varepsilon \geqq \beta, \text{ where } \theta = a,\ s \text{ or } t, \tag{3.29}$$

since any path joining $\boldsymbol{0}$ to H_n with passage time strictly smaller than $n(\mu-\varepsilon)$ contains some edge with time coordinate strictly smaller than $\mu-\varepsilon$; this is impossible if $\varepsilon \geqq \beta$, by the definition of β. Similarly

$$B_\theta(\varepsilon) < \infty \quad \text{if } \varepsilon < \beta, \text{ where } \theta = a,\ s \text{ or } t, \tag{3.30}$$

since we have that

$$P(\theta_{0n} < n(\mu-\varepsilon)) \geqq F((\mu-\varepsilon)-)^n, \tag{3.31}$$

by considering the segment of the x-axis which joins $\boldsymbol{0}$ to $(n,0)$.

Next we show that (3.26) holds; it suffices, by (3.29), to restrict our attention to values of ε satisfying $\varepsilon < \beta$. To show that $B_a(\varepsilon) = B_t(\varepsilon)$ for $\varepsilon < \beta$, note first that $a_{0n} \leqq t_{0n}$, and so $B_a(\varepsilon) \leqq B_t(\varepsilon)$. To show that equality holds here, suppose that $\varepsilon < \beta$ and let $\delta > 0$; now choose k such that

$$P(a_{0k} < k(\mu-\varepsilon)) \geqq \exp(-k(B_a(\varepsilon)+\delta)).$$

Let $a_{0k}(m)$ be the infimum of the passage times of paths joining $\boldsymbol{0}$ to $(k,0)$ which lie entirely within the cylinder $\{(x,y)\colon -m < x < k+m\}$. Then $a_{0k}(m) \downarrow a_{0k}$ as $m \to \infty$, and so we may find a value of m such that

$$P(a_{0k}(m) < k(\mu-\varepsilon)) \geqq \exp(-k(B_a(\varepsilon)+2\delta)). \tag{3.32}$$

With this value of m, let n be a positive integer and let t_1 (respectively t_2) be the sum of the time coordinates in the straight segment joining $(-m,0)$ to $\boldsymbol{0}$ (respectively $(nk,0)$ to $(nk+m,0)$). Then

$$\begin{aligned}\{t_{-m,nk+m} < (nk+2m)(\mu-\varepsilon)\} \supseteq \{t_1 < m(\mu-\varepsilon)\} &\cap \{t_2 < m(\mu-\varepsilon)\} \\ &\cap \{a_{0,nk}(m) < nk(\mu-\varepsilon)\},\end{aligned}$$

giving, by the FKG inequality, that

$$P(t_{0,nk+2m}<(nk+2m)(\mu-\varepsilon))\geqq F((\mu-\varepsilon)-)^{2m}P(a_{0k}(m)<k(\mu-\varepsilon))^n.$$

Take logarithms, divide by nk and let $n\to\infty$ to obtain from (3.32) that

$$B_t(\varepsilon)\leqq B_a(\varepsilon)+2\delta$$

whenever ε is such that $F((\mu-\varepsilon)-)>0$. Now let $\delta\downarrow 0$ to obtain

$$B_a(\varepsilon)=B_t(\varepsilon) \qquad \text{if } \varepsilon<\beta.$$

Next we show that $B_s(\varepsilon)=B_t(\varepsilon)$ for $\varepsilon<\beta$. As before, $s_{0n}\leqq t_{0n}$ and so $B_s(\varepsilon)\leqq B_t(\varepsilon)$ for all ε. Suppose that $\varepsilon<\beta$ and $\delta>0$, and choose k such that

$$P(s_{0k}<k(\mu-\varepsilon))\geqq \exp(-k(B_s(\varepsilon)+\delta)).$$

Next choose m such that the event $E(k,m)$, that $\boldsymbol{0}$ is joined to H_k by a path which is contained within $\{(x,y)\colon\ 0<x<k,\ -m\leqq y\leqq m\}$, except for its endpoints, and which has passage time strictly less than $k(\mu-\varepsilon)$, satisfies

$$P(E(k,m))\geqq \exp(-k(B_s(\varepsilon)+2\delta)); \tag{3.33}$$

there exists such a value for m by virtue of the fact that

$$\{s_{0k}<k(\mu-\varepsilon)\}=\bigcup_m E(k,m).$$

For each $\boldsymbol{v}=(v_1,v_2)\in\mathbb{Z}^2$, we define $F(\boldsymbol{v})$ to be the event that $\boldsymbol{v}$ is joined to H_{v_1+k} by a path which is contained within

$$C(\boldsymbol{v})=\{(x,y)\colon v_1<x<v_1+k,\ -m\leqq y-v_2\leqq m\},$$

except for its endpoints, and which has passage time strictly less than $k(\mu-\varepsilon)$; thus $E(k,m)=F(\boldsymbol{0})$, and $P(F(\boldsymbol{v}))=P(F(\boldsymbol{0}))$ for all $\boldsymbol{v}\in\mathbb{Z}^2$. For each $\boldsymbol{v}\in\mathbb{Z}^2$ we define the path $r(\boldsymbol{v})$ as follows. If $F(\boldsymbol{v})$ does not occur then $r(\boldsymbol{v})$ is empty. If $F(\boldsymbol{v})$ occurs then let V be the set of vertices $\boldsymbol{w}$ in $\{(v_1+k,y)\colon\ -m\leqq y-v_2\leqq m\}$ such that $\boldsymbol{v}$ is joined to $\boldsymbol{w}$ by a path, lying entirely in $C(\boldsymbol{v})$ except for its endpoints, which has passage time strictly less than $k(\mu-\varepsilon)$, and let $\boldsymbol{w}'$ be the vertex in V whose vertical displacement from (v_1+k,v_2) has a minimum absolute value (if there are two such vertices, we flip a fair coin to choose between them with equal probability). Let $r(\boldsymbol{v})$ be a path in $C(\boldsymbol{v})$ which joins $\boldsymbol{v}$ to $\boldsymbol{w}'$ and which has passage time strictly smaller than $k(\mu-\varepsilon)$. Now we define the path r as follows. Let $\boldsymbol{w}_0=\boldsymbol{0}$. If $F(\boldsymbol{w}_0)$ does not occur, then r is empty. If $F(\boldsymbol{w}_0)$ occurs, we let $r(\boldsymbol{0})$ be the first section of r, and write $\boldsymbol{w}_1$ for the right hand endpoint of $r(\boldsymbol{0})$. If $F(\boldsymbol{w}_1)$ does not occur, then we stop the construction of r; otherwise we take the union of $r(\boldsymbol{0})$ and $r(\boldsymbol{w}_1)$ to be the first section of r. Once $r(\boldsymbol{w}_0)$, $r(\boldsymbol{w}_1)$, …, $r(\boldsymbol{w}_{j-1})$ have been chosen, we stop the construction of r if $F(\boldsymbol{w}_j)$ does not occur (where $\boldsymbol{w}_j$ is the right hand endpoint of $r(\boldsymbol{w}_{j-1})$); we append $r(\boldsymbol{w}_j)$ otherwise. Now for any positive integer n,

$$P(t_{0,nk}<nk(\mu-\varepsilon))\geqq P\left(\left\{\bigcap_{j=0}^{n-1}F(w_j)\right\}\cap\{w_n=(nk,0)\}\right)\tag{3.34}$$
$$=P(F(0))^n P(Z_1+\ldots+Z_n=0\,|\,F)$$

where $F=\bigcap_{j=0}^{n-1}F(w_j)$ and, conditional on F, Z_j is the vertical displacement of w_j from $w_{j-1}+(k,0)$, including the sign of this displacement. Thus, conditional on F, the Z's are independent identically distributed symmetric random variables, satisfying $|Z_j|\leqq m$, and it is a consequence of the local central limit theorem (see Gnedenko and Kolmogorov (1954, Sect. 49)) that

$$P(Z_1+\ldots+Z_n=0\,|\,F)\geqq\alpha n^{-\frac{1}{2}}\quad\text{for all } n\tag{3.35}$$

and for some constant $\alpha=\alpha(k,m)>0$. We combine (3.33), (3.34) and (3.35) to find that

$$P(t_{0,nk}<nk(\mu-\varepsilon))\geqq\exp(-nk(B_s(\varepsilon)+2\delta))\,\alpha n^{-\frac{1}{2}}.$$

Take logarithms, divide by nk and let $n\to\infty$ to obtain

$$B_t(\varepsilon)\leqq B_s(\varepsilon)+2\delta.$$

But $\delta>0$ was arbitrary and so $B_t(\varepsilon)=B_s(\varepsilon)$ if $\varepsilon<\beta$, as required; the proof of (3.26) is complete and we write $B(\varepsilon)=B_\theta(\varepsilon)$ for $\theta=a$, s or t. Conclusions (a) and (b) follow immediately from (3.27), (3.28), (3.29) and (3.30), and it remains only to show that (c) holds also.

The remaining part of the corollary relies upon the convexity of B on $(-\infty,\beta)$. Let $\varepsilon_1<\varepsilon_2<\varepsilon_3<\beta$ and let $k=k(n)$ be a sequence of positive integers with the property that

$$\frac{k(n)}{n}\leqq\frac{\varepsilon_3-\varepsilon_2}{\varepsilon_3-\varepsilon_1}\quad\text{for all } n,\qquad \frac{k(n)}{n}\to\frac{\varepsilon_3-\varepsilon_2}{\varepsilon_3-\varepsilon_1}\quad\text{as } n\to\infty.\tag{3.36}$$

Divide the interval $[0,n]$ into the two subintervals $[0,k]$ and $[k,n]$ in the usual way to obtain

$$P(t_{0n}<n(\mu-\varepsilon_2))\geqq P(t_{0k}<k(\mu-\varepsilon_1))\,P(t_{0,n-k}<(n-k)(\mu-\varepsilon_3));\tag{3.37}$$

this holds since, by (3.36),

$$k(\mu-\varepsilon_1)+(n-k)(\mu-\varepsilon_3)\leqq n(\mu-\varepsilon_2).$$

Take logarithms of (3.37), divide by n and let $n\to\infty$ to obtain, by (3.36),

$$B(\varepsilon_2)\leqq\frac{\varepsilon_3-\varepsilon_2}{\varepsilon_3-\varepsilon_1}B(\varepsilon_1)+\frac{\varepsilon_2-\varepsilon_1}{\varepsilon_3-\varepsilon_1}B(\varepsilon_3),\tag{3.38}$$

giving that B is convex on $(-\infty,\beta)$. Hence B is continuous on $(-\infty,\beta)$, and (3.27) implies that $B(0)=0$ if $\beta>0$. From its definition, B is a nondecreasing function, and (3.28) and (3.38) imply that B is strictly increasing on $[0, \beta)$ if $\beta > 0$. This completes the proof of part (c).

Proof of Theorem 3.4. This follows exactly the scheme of the previous proofs. The corresponding version of Lemma 3.7 is proved by using Eq. (2.13) in place of Eq. (2.2).

4. Crossings of Rectangles

Let $B_{nm} = \{(x, y): 0 \leqq x \leqq n,\ 0 \leqq y \leqq m\}$, and let l_{nm} be the minimum passage time of the set of paths within B_{nm} which join some vertex on the left hand side of B_{nm} to some vertex on the right hand side of B_{nm}, as before. We saw in Sect. 2 that $n^{-1} l_{nn} \to \mu$ a.s.; it is the purpose of this section to find out how big $m = m(n)$ need be in order that $n^{-1} l_{n,m(n)}$ converges as $n \to \infty$ to some limit which is strictly smaller than μ; since this is impossible if $\mu = 0$, *we suppose henceforth that* $\mu > 0$. It is not surprising that $m(n)$ needs to grow exponentially in n. Note first that, by Corollary 3.3,

(4.1)
$$C(\delta) = B(\mu - \delta) = \lim_{n \to \infty} \left(-\frac{1}{n} \log P(s_{0n} < n\delta) \right)$$

exists for $0 \leqq \delta \leqq \mu$ and has the following properties. Let

(4.2)
$$\nu = \inf\{v: F(v) > 0\}.$$

Then

(a) C: $[0, \mu] \to [0, \infty]$ and is a nonincreasing function,

(b) $C(\delta) = \infty$ if $\delta \leqq \nu$, and $C(\delta) < \infty$ if $\delta > \nu$,

(c) $C(\mu) = 0$ if $\nu < \mu$,

(d) C is continuous and strictly decreasing on $(\nu, \mu]$.

The next theorem explores the asymptotic behaviour of $n^{-1} l_{n,m(n)}$ when $m(n) = e^{cn}$ for some $c > 0$. Similar techniques may be used to provide corresponding results for other functions $m(n)$ which grow beyond bounds as $n \to \infty$.

Theorem 4.1. *Let* $m(n) = e^{cn}$ *and suppose that* $\nu < \mu$.

(a) *If* $0 < c < C(\nu+)$ *then* $c = C(\delta)$ *for a unique value of* δ, *and*

$$\frac{1}{n} l_{n,m(n)} \to \delta \ \textit{a.s.} \quad \textit{as} \quad n \to \infty.$$

(b) *If* $c \geqq C(\nu+)$ *then*

$$\frac{1}{n} l_{n,m(n)} \to \nu \ \textit{a.s.} \quad \textit{as} \quad n \to \infty.$$

Corollary 4.2. *Suppose that* $m(n) \to \infty$ *but* $\frac{1}{n} \log m(n) \to 0$ *as* $n \to \infty$. *If* $\nu < \mu$ *then*

$$\liminf_{n \to \infty} \frac{1}{n} l_{n,m(n)} \geqq \mu \ \textit{a.s.}$$

If, further, $\int(1-F(x))^4\,dx<\infty$ *then*

$$\frac{1}{n}l_{n,m(n)}\to\mu \text{ a.s.} \quad \text{as} \quad n\to\infty.$$

This extends the results of Theorem 2.1.

Proof of Theorem 4.1. (a) Suppose $c=C(\delta)$ where $\nu<\delta<\mu$ and let ε be such that $\nu<\delta-\varepsilon<\delta<\delta+\varepsilon<\mu$. Any route for l_{nm} must start at some vertex in $\{(0,y)\colon 0\leqq y\leqq m\}$, and so

$$\begin{aligned}\sum_n P(l_{nm}<n(\delta-\varepsilon)) &\leqq \sum_n (m(n)+1)P(s_{0n}<n(\delta-\varepsilon))\\ &=\sum_n \exp(-n(C(\delta-\varepsilon)-c+o(1)))\\ &<\infty,\end{aligned}$$

since $C(\delta-\varepsilon)-c=C(\delta-\varepsilon)-C(\delta)>0$. Thus

$$\liminf_{n\to\infty}\frac{1}{n}l_{n,m(n)}\geqq\delta \text{ a.s.} \quad \text{as} \quad n\to\infty. \tag{4.3}$$

For the reverse inequality, divide B_{nm} into $k(n)=m(n)/(2n^3)$ rectangles with dimensions n by $2n^3$; for ease of notation we shall neglect the possibility that these, and forthcoming, quantities may not be integer-valued. Each such rectangle is a copy of the rectangle $T_n=\{(x,y)\colon 0\leqq x\leqq n,\ |y|\leqq n^3\}$. Let $s_{0n}(\alpha)$ be the minimum point-to-line passage time within T_n of paths joining $\boldsymbol{0}$ to the line $x=n$. Then

$$\begin{aligned}P(s_{0n}(\alpha)<n(\delta+\varepsilon)) &\geqq P(s_{0n}<n(\delta+\varepsilon) \text{ but } \boldsymbol{0} \text{ is joined to neither}\\ &\qquad \text{of the lines } y=\pm n^3 \text{ by a path with}\\ &\qquad \text{passage time less than } n(\delta+\varepsilon))\\ &\geqq P(s_{0n}<n(\delta+\varepsilon))-2P(b_{0,n^3}<n(\delta+\varepsilon))\\ &=\exp(-n(C(\delta+\varepsilon)+o(1)))-o(\exp(-n^2))\end{aligned}$$

by Theorem 3.1. Therefore

$$\begin{aligned}\sum_n P(l_{n,m(n)}>n(\delta+\varepsilon)) &\leqq \sum_n \{P(s_{0n}(\alpha)>n(\delta+\varepsilon)\}^{k(n)}\\ &\leqq \sum_n \exp(-k(n)\exp(-n(C(\delta+\varepsilon)+o(1))))\\ &=\sum_n \exp(-\exp(n(c-C(\delta+\varepsilon))+o(n)))\\ &<\infty,\end{aligned}$$

since $c-C(\delta+\varepsilon)=C(\delta)-C(\delta+\varepsilon)>0$. Thus

$$\limsup_{n\to\infty}\frac{1}{n}l_{n,m(n)}\leqq\delta \text{ a.s.} \quad \text{as} \quad n\to\infty$$

as required.

(b) If $c \geqq C(\nu+)$ then, by monotonicity and part (a),

$$\limsup_{n\to\infty} \frac{1}{n} l_{n,m(n)} \leqq \delta \text{ a.s.} \quad \text{as} \quad n\to\infty$$

for all $\delta > \nu$. By the definition of ν,

$$\frac{1}{n} l_{n,m(n)} \geqq \nu \text{ a.s.,} \qquad \text{for all } n,$$

and the conclusion follows.

Proof of Corollary 4.2. From Theorem 4.1,

$$\liminf_{n\to\infty} \frac{1}{n} l_{n,m(n)} \geqq \delta \text{ a.s.}$$

for all $\delta < \mu$, and the first conclusion follows. The final part follows from part (b) of Theorem 2.1.

5. Network Flows

The foregoing results have applications to the problem of ascertaining the value of the maximum network flow through rectangles of $\mathbb{Z}^2$ whose edges have random capacities. See Bollobás (1979) for an account of the general problem and the max-flow min-cut theorem.

To each edge e of L we assign a *random capacity* $c(e)$, where $\{c(e): e \in L\}$ is a family of non-negative independent identically distributed random variables with distribution function F. Let B_{nm} be the n by m rectangle $\{(x,y): 0 \leqq x \leqq n, 0 \leqq y \leqq m\}$ as usual. We introduce two new vertices, labelled 0 and ∞, called the *source* and the *sink* respectively, so that 0 (respectively ∞) is joined to each vertex on the left (respectively right) hand side of B_{nm} by edges with infinite capacity; we call this augmented graph B_{nm}^{+}. See Fig. 1. Do not confuse the source 0 with the origin $\boldsymbol{0}$. Let ϕ_{nm} be the maximum flow through B_{nm}^{+} from 0 to ∞, subject to the random edge capacities. We are interested also in the related problem of first-passage percolation through $\mathbb{Z}^2$ when the time coordinates have distribution function F, and denote by μ the time constant of this process.

Theorem 5.1. *We have that* $\liminf_{m\to\infty} \frac{1}{m} \phi_{mm} \geqq \mu$ *a.s. If* $\int (1-F(x))^4 dx < \infty$ *then* $\frac{1}{m} \phi_{mm} \to \mu$ *a.s. and in* L^1 *as* $m \to \infty$.

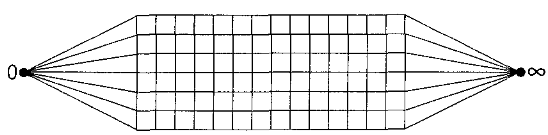

Fig. 1. The graph B_{nm}^{+}

As noted after Theorem 2.1, the almost sure convergence of $m^{-1}\phi_{mm}$ is actually valid without any assumption on the distribution function F.

Let ν and $C(\delta)$ be given by (4.2) and (4.1) respectively.

Theorem 5.2. *Suppose that $\nu<\mu$ and $n(m)=e^{cm}$ where $c>0$.*

(a) *If $0<c<C(\nu+)$ then $c=C(\delta)$ for a unique $\delta\in(\nu,\mu)$ and*

$$\frac{1}{m}\phi_{n(m),m}\to\delta \text{ a.s.} \quad \text{as} \quad m\to\infty.$$

(b) *If $c\geqq C(\nu+)$ then*

$$\frac{1}{m}\phi_{n(m),m}\to\nu \text{ a.s.} \quad \text{as} \quad m\to\infty.$$

As noted before Theorem 4.1, similar asymptotic results may be obtained for $m^{-1}\phi_{n(m),m}$ for other functions $n(m)$ tending to infinity as $n\to\infty$.

A case of special interest is when the capacities have the Bernoulli distribution,

$$P(c(e)=0)=1-p, \qquad P(c(e)=1)=p, \tag{5.1}$$

where $0\leqq p\leqq 1$. For this case, the following is an immediate corollary of Kesten (1980a, b).

Corollary 5.3. *If the edge capacities have distribution* (5.1) *then, as $m\to\infty$,*

$$\frac{1}{m}\phi_{mm}\to\mu(p)\begin{cases}=0 \text{ if } p\leqq\frac{1}{2}\\ >0 \text{ if } p>\frac{1}{2}\end{cases} \text{ a.s.}$$

Related problems of flows through randomly-capacitated networks are considered by Grimmett and Welsh (1982); they were concerned with the asymptotic properties of the maximum flows through large complete graphs and infinite branching trees.

Proofs of Theorems 5.1 and 5.2. We shall use the max-flow min-cut theorem (see Bollobás (1979, p. 47)). The *dual lattice* L^D of L is the graph constructed as follows: place a vertex in the centre of each face of L and join two vertices in L^D if and only if the corresponding faces of L have an edge in common. Note that L^D is isomorphic to L. Consider the rectangle

$$B^D_{nm}=\{(x+\tfrac{1}{2},y+\tfrac{1}{2}):0\leqq x\leqq n-1,\ -1\leqq y\leqq m\}$$

in L^D, together with all its interior edges, except for those in the top and bottom sides. To each edge e of L^D we assign a time coordinate equal in value to the random capacity of the unique edge of L which e crosses. Each non-self-intersecting path from the top side to the bottom side of B^D_{nm} corresponds to a unique cutset of B^+_{nm}, and the passage time of such a path equals the size of the corresponding cutset. By the max-flow min-cut theorem, ϕ_{nm} equals the minimum passage time of the paths joining the top and bottom edges of B^D_{nm}. Thus

ϕ_{nm} has the same distribution as $l_{m+1,n-1}$, where l is the line-to-line passage time of Sect. 2. Theorems 5.1 and 5.2 follow from Theorems 2.1 and 4.1, respectively.

6. Electrical Resistances of Random Networks

The electrical resistances of random subgraphs of $\mathbb{Z}^d$ have been studied by various authors in the physics literature (see, for example, Kirkpatrick (1978) and Stauffer (1979)). The basic problem is as follows. Consider the graph $\mathbb{Z}^d$ and suppose that each edge e of $\mathbb{Z}^d$ is a wire with some electrical resistance $r(e)$, where $\{r(e)\colon e \in \mathbb{Z}^d\}$ is a collection of non-negative independent identically distributed (possibly infinite-valued) random variables with distribution function G. Let $B_n(d)$ be the cube $\{\boldsymbol{x} \in \mathbb{Z}^d\colon 0 \leqq x_1, x_2, \ldots, x_d \leqq n\}$. What is the effective resistance of $B_n(d)$ across opposite hyperplanes? More precisely, introduce two new vertices, labelled 0 and ∞, such that 0 (respectively ∞) is joined to every vertex which is in both $B_n(d)$ and the hyperplane $x_1 = 0$ (respectively $x_1 = n$) by edges made from a substance with zero resistance (see Figure 1 for the case $d = 2$). We connect a battery across the two terminals 0 and ∞ so that 0 is at potential 0 volts and ∞ is at potential 1 volt. This potential difference induces a random *potential function* Φ_n, mapping the vertex set of $B_n(d)$ into $[0, 1]$ such that $\Phi_n(0) = 0$ and $\Phi_n(\infty) = 1$. It is clear that

$$\Phi_n((0, x_2, \ldots, x_d)) = 0 \quad \text{and} \quad \Phi_n((n, x_2, \ldots, x_d)) = 1$$

for all $0 \leqq x_2, \ldots, x_d \leqq n$. Also, currents flow along the edges of $B_n(d)$, and the potentials and currents satisfy well-known laws, called Kirchhoff's Laws and Ohm's Law. For any realization of the set of resistance there is a corresponding set of potentials and currents which satisfies these laws. Immediate consequences of these laws are the rules for combining resistances in series and in parallel (see Fig. 2).

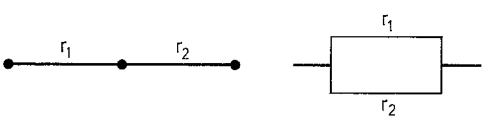

Fig. 2. Resistances r_1 and r_2 in series and parallel

Two resistances r_1 and r_2 in series have combined resistance $r = r_1 + r_2$; two resistances r_1 and r_2 in parallel have combined resistance r such that $r^{-1} = r_1^{-1} + r_2^{-1}$. See Bollobás (1979, Chap. 2), Doyle and Snell (1982) and Kesten (1982, Sect. 11.3) for general discussions of electrical network theory.

Let $R_n(d)$ denote the effective resistance of $B_n(d)$ between the vertices 0 and ∞; it is our purpose to study the asymptotic behaviour of $R_n(d)$ for large n. Of great importance is the probability

$$P(r(e) = \infty) = 1 - G(\infty -)$$

that an edge-resistance is infinite. We suppose that

$$p = P(r(e) < \infty), \qquad 1 - p = P(r(e) = \infty) \tag{6.1}$$

and, for $x < \infty$,

$$P(r(e) \leqq x) = pJ(x) \tag{6.2}$$

where J corresponds to a probability measure concentrated on $[0, \infty)$. We call an edge *insulating* if it has infinite resistance, and *conducting* otherwise; we call a path *insulating* (respectively *conducting*) if all its edges are insulating (respectively conducting).

First, we state our result for the case $d = 2$.

Theorem 6.1. *If each edge-resistance has distribution given by* (6.1) *and* (6.2), *and* $d = 2$, *then*

(a) *if* $p < \frac{1}{2}$ *then* $P(R_n(2) = \infty) \to 1$ *as* $n \to \infty$,

(b) *if* $p > \frac{1}{2}$ *then there exists* $v(p) < \infty$ *such that*

$$\begin{aligned}\{p \textstyle\int x^{-1}\, dJ(x)\}^{-1} &\leqq \liminf_{n \to \infty} R_n(2) \\ &\leqq \limsup_{n \to \infty} R_n(2) \leqq v(p) \textstyle\int x\, dJ(x) \text{ a.s.}\end{aligned}$$

The dependence on whether $p < \frac{1}{2}$ or $p > \frac{1}{2}$ is not surprising in the light of the fact (Kesten (1980b)) that the critical probability for bond percolation on $\mathbb{Z}^2$ equals $\frac{1}{2}$, since, if $p < \frac{1}{2}$ and n is large, it is unlikely that there exists a conducting path between 0 and ∞.

The above theorem contains no information about the case $p = \frac{1}{2}$. It may be shown that the event $\{R_n(2) \to \infty\}$ is a tail event of the collection of independent edge-resistances, and so $P(R_n(2) \to \infty)$ equals either 0 or 1. A result of Kesten (1982, Theorem 11.2) implies that $P(R_n(2) \to \infty) = 1$ for $p = \frac{1}{2}$ so long as $J(0) < 1$.

If $p \int x^{-1}\, dJ(x) = \infty$, then the left hand side of the conclusion of part (b) of the theorem is interpreted as zero. In the interesting case when $p > \frac{1}{2}$ and J has finite mean and finite harmonic mean, we see that the sequence $\{R_n(2)\}$ is a.s. bounded away from zero and infinity; it is tempting to conjecture that the sequence converges as $n \to \infty$.

Results similar to Theorem 6.1 hold for other two-dimensional lattices too, so long as they are one of a matching pair. We have a partial result for higher dimensions. Consider Bernoulli percolation on $\mathbb{Z}^d$, in which each edge is declared *open* (respectively *closed*) with probability p (respectively $1 - p$), and let $W(\mathbf{v})$ be the number of edges in the open cluster which contains the vertex $\mathbf{v}$. Define the *critical probability*

$$p_T(d) = \sup\{p : E(W(\mathbf{0})) < \infty\}.$$

Theorem 6.2. *If* $d > 2$ *and the edge-resistance distribution is given by* (6.1) *and* (6.2) *then*

(a) *if* $p < p_T(d)$ *then* $P(R_n(d) = \infty) \to 1$ *as* $n \to \infty$,

(b) *if* $p>\frac{1}{2}$ *then there exists* $\nu(p)<\infty$ *such that*

$$\{p\int x^{-1}dJ(x)\}^{-1}\leqq\liminf_{n\to\infty}(n^{d-2}R_n(d))$$
$$\leqq\limsup_{n\to\infty}(n^{d-2}R_n(d))\leqq\nu(p)\int x\,dJ(x)\ \text{a.s.}$$

Again, we are tempted to conjecture that the sequence $\{n^{d-2}R_n(d)\}$ converges as $n\to\infty$, for large enough values of p.

The constant $\nu(p)$ in Theorems 6.1 and 6.2 is related to the time constant $\mu(p)$ of the first-passage percolation process on $\mathbb{Z}^2$ with time coordinate distribution given by

$$P(t(e)=0)=1-p,\qquad P(t(e)=1)=p; \tag{6.3}$$

more specifically

$$\nu(p)=2p/\mu(p)^2. \tag{6.4}$$

In the case when $p>\frac{1}{2}$, Kesten (1982) gives more useful estimates for $R_n(2)$; he explores upper and lower bounds involving powers $(p-\frac{1}{2})^{-\gamma}$.

The methods of Sects. 4 and 5 may be applied to yield partial results about the resistance across the opposite hyperplanes of rectangular parallelepipeds in $\mathbb{Z}^d$. The following is an example of such a result in two dimensions. Let B_{nm}^+ be the rectangle $\{(x,y)\colon 0\leqq x\leqq n,\ 0\leqq y\leqq m\}$ together with the two vertices 0 and ∞, as usual, and suppose for simplicity that each edge-resistance has the Bernoulli distribution

$$P(r(e)=1)=p,\qquad P(r(e)=\infty)=1-p$$

where $0\leqq p\leqq 1$. Let R_{nm} be the resistance of the ensuing electrical network between the terminals 0 and ∞. The corresponding first-passage percolation process has time coordinates $t(e)=r(e)^{-1}$, with distribution given by (6.3). In the latter process, note that the constant ν, defined in (4.2), satisfies $\nu=0$ if $p<1$. Let $\mu(p)$ be the time constant and let $C(\delta)$ be given by (4.1), for $0<\delta\leqq\mu(p)$.

Theorem 6.3. (a) *If* $p<\frac{1}{2}$, *there exists a constant* $\beta_p\in(0,\infty)$ *such that, if* $n=n(m)=\beta\log m$ *then, as* $m\to\infty$,

$$P(R_{nm}=\infty)\to\begin{cases}0 & \text{if } \beta<\beta_p\\ 1 & \text{if } \beta>\beta_p.\end{cases}$$

(b) *Suppose* $\frac{1}{2}<p<1$, *and let* c *be such that* $0<c<C(0+)$. *Then* $c=C(\delta)$ *for a unique* $\delta\in(0,\mu(p))$. *If* $n=n(m)\to\infty$ *as* $m\to\infty$, *but* $n(m)\leqq e^{cm}$ *for all large* m, *then*

$$\frac{1}{p}\leqq\liminf_{m\to\infty}\left(\frac{m}{n}R_{nm}\right)\leqq\limsup_{m\to\infty}\left(\frac{m}{n}R_{nm}\right)\leqq\frac{2p}{\delta^2}\ \text{a.s.}$$

Thus, if $\frac{1}{2}<p<1$, then $R_{n(m),m}$ grows at about the rate $n(m)/m$ so long as $n(m)\leqq e^{cm}$ and c is not too large. Of course, there exists $c_0=c_0(p)$ such that, as $m\to\infty$,

$$P(R_{nm}=\infty)\to 1\qquad\text{if } n=e^{cm} \text{ and } c>c_0(p);$$

this holds because, for large enough c, B^D_{nm} is crossed vertically by an insulating path with probability tending to one (see Grimmett (1981)).

A continuous-space version of this problem has been studied by Papanicolaou and Varadhan (1979) and Golden and Papanicolaou (1983); they studied a continuous sheet $\mathbb{R}^2$ whose conductivity varies about the sheet in the manner of a stationary random process. Under the assumption that each resistance satisfies

$$0 < a \leqq r(e) \leqq b < \infty \text{ a.s.}$$

for constants a and b, they showed that the effective resistance R'_n between opposite sides of the square $\{(x,y) \in \mathbb{R}^2 \colon 0 \leqq x,\ y \leqq n\}$ converges in mean square as $n \to \infty$. Their proof deals with the case of $\mathbb{R}^2$, but appears to apply in the case of $\mathbb{Z}^2$ also; see Künnemann (1983) for an explicit discussion of the lattice case. In conjunction with a result of Straley (1977), this implies that if

$$P(r(e)=a) = P(r(e)=b) = \tfrac{1}{2}$$

for some constants a, b satisfying $0 < a < b < \infty$, then

$$R_n(2) \to (ab)^{\frac{1}{2}} \quad \text{in mean square.}$$

As remarked earlier, the question of the convergence of $\{R_n(2)\}$ remains unresolved if the edge-resistances may take the values 0 and ∞.

Proof of Theorem 6.1. We write B_n and R_n for $B_n(2)$ and $R_n(2)$.

(a) Let E_n be the event that there exists a conducting path in B_n from 0 to ∞. Then, if $p < \frac{1}{2}$,

$$P(E_n) \to 0 \quad \text{as} \quad n \to \infty$$

by the result of Kesten (1980b).

(b) The lower bound is easy to obtain. Define $H_k = \{(k,y)\colon -\infty < y < \infty\}$ and for each $k = 0, 1, 2, \ldots$ in turn, join together all the vertices in H_k by connections with zero resistance. The effect of this on B_n is to replace it by a network of the form of Fig. 3, with $n+1$ vertices, and such that vertices k and

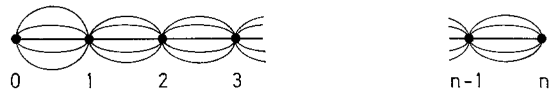

Fig. 3. The electrical network obtained by connecting together the vertices in each vertical line

$k+1$ $(0 \leqq k < n)$ are joined by $n+1$ edges, each of which has a random resistance; the resistance R'_n of this network is such that

$$R'_n \leqq R_n. \tag{6.5}$$

By the series and parallel rules,

$$R'_n = \sum_{i=0}^{n-1} \left(\sum_{j=0}^{n} \frac{1}{r_{ij}} \right)^{-1}$$

where r_{ij} is the resistance of the edge of L joining (i,j) to $(i+1,j)$. But "harmonic means to not exceed arithmetic means", and so

$$R'_n \geqq n^2 \left(\sum_{i=0}^{n-1} \sum_{j=0}^{n} \frac{1}{r_{ij}} \right)^{-1} \to \left(E\left(\frac{1}{r_{00}} \right) \right)^{-1} = (p \int x^{-1} dJ(x))^{-1} \text{ a.s.}$$

by the law of large numbers and (6.2).

Combined with (6.5) this yields the lower bound. See Kesten (1982) for a better lower bound which expresses R'_n in terms of powers of the difference $(p - \frac{1}{2})$.

Next we show an upper bound for R_n. For each edge e of L define

$$c(e) = \begin{cases} 1 & \text{if } e \text{ is conducting} \\ 0 & \text{if } e \text{ is insulating.} \end{cases}$$

Let ψ_n be the maximum number of edge-disjoint paths from 0 to ∞ in B_n which contain only conducting edges, and let ϕ_n be the size of a maximal flow in B_n between 0 and ∞ when each edge e has capacity $c(e)$. By a standard result of integer programming (see, for example, Bollobás (1979, p. 48)) there is a maximal flow which sends either 0 or 1 units along each edge, and thus $\psi_n = \phi_n$. It follows from Corollary 5.3 that

$$\frac{1}{n} \psi_n \to \mu(p) \text{ a.s.} \quad \text{as} \quad n \to \infty \tag{6.6}$$

where $\mu(p)$ is the time constant of Bernoulli first-passage percolation on $\mathbb{Z}^2$. From B_n we obtain a new electrical network by removing all connections which are not in one of these paths, and by separating the paths at any vertex where they cross; this network is a set of $\psi = \psi_n$ paths in parallel (see Fig. 4). It is easy to see that the resistance R''_n of the new network is such that

$$R_n \leqq R''_n, \tag{6.7}$$

since we have deleted connections in its construction (think of the separation of two paths which cross at some vertex $\mathbf{v}$ as the cutting of an imaginary link with zero resistance between them at $\mathbf{v}$). Denote the resulting paths by

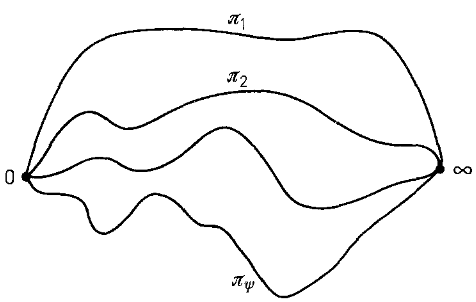

Fig. 4. The electrical network obtained by separating the ψ edge-disjoint conducting paths joining 0 to ∞, and neglecting all other conducting edges

$\pi_1, \ldots, \pi_\psi$, and let $r(\pi_i)$ be the sum of the resistances of the edges of π_i. By the series and parallel rules, we have that

$$\frac{1}{R_n} \geqq \frac{1}{R_n''} \geqq \sum_{i=1}^{\psi} \frac{1}{r(\pi_i)}$$

$$\geqq \psi^2 \Big/ \sum_{i=1}^{\psi} r(\pi_i)$$

by the harmonic-arithmetic mean inequality

$$\geqq \psi^2 \Big/ \sum_e r(e)\, c(e)$$

where the last sum is over all edges in B_n, with the convention that $0.\ \infty = 0$; thus, this summation is the sum of the resistances of the conducting edges of B_n. Now apply (6.6) and the law of large numbers to find that

$$\frac{1}{R_n} \geqq \frac{\psi^2}{n^2} \frac{n^2}{\sum r(e)\, c(e)} \to \mu(p)^2 \{2p \int x\, dJ(x)\}^{-1} \text{ a.s.}$$

as required. If $p \int x\, dJ(x) = \infty$, then we interpret the last term as zero.

Proof of Theorem 6.2. (a) Let $H = \{\boldsymbol{x} \in B_n(d)\colon x_1 = 0\}$. If $p < p_T(d)$, then

$$P(R_n(d) < \infty) \leqq \sum_{\boldsymbol{v} \in H} P(\boldsymbol{v} \text{ is in a conducting path of length } n)$$

$$\leqq \sum_{\boldsymbol{v} \in H} P(W(\boldsymbol{v}) \geqq n)$$

$$\leqq (n+1)^{d-1} P(W(\boldsymbol{0}) \geqq n)$$

$$\leqq (n+1)^{d-1} c_1 \exp(-c_2 n)$$

for positive constants $c_1(p)$, $c_2(p)$, by the results of Chap. 5 of Kesten (1982). Thus

$$P(R_n(d) = \infty) \to 1 \quad \text{as} \quad n \to \infty \quad \text{if } p < p_T(d).$$

(b) The lower bound is obtained, just as in the previous proof, by shorting out all the vertices of $B_n(d)$ in hyperplanes of the form $\{\boldsymbol{x} \in \mathbb{Z}^d\colon x_1 = k\}$ for $k = 0, \ldots, n$. Note that the lower bound is valid for all values of p. To obtain the upper bound, decompose $B_n(d)$ into the disjoint union $P(1) \cup \ldots \cup P((n+1)^{d-2})$ of two-dimensional subsets of $B_n(d)$: each P is a set of the form $\{\boldsymbol{x} \in B_n(d)\colon x_3 = j_3, x_4 = j_4, \ldots, x_d = j_d\}$, as $(j_3, \ldots, j_d)$ ranges over $\{0, 1, \ldots, n\}^{d-2}$. Let $R_{n,k}$ be the resistance of $P(k)$ between 0 and ∞, neglecting all other edges in $B_n(d)$. Then

$$\frac{1}{R_n(d)} \geqq \sum_{k=1}^{(n+1)^{d-2}} \frac{1}{R_{n,k}}.$$

Fix $\varepsilon > 0$ and let

$$N(n, \varepsilon) = |\{k\colon R_{n,k} < (1+\varepsilon)\, v(p) \int x\, dJ(x)\}|,$$

where $v(p)$ is the constant of Theorem 6.1. Then

$$\frac{1}{n^{d-2}R_n(d)} \geqq \frac{N(n,\varepsilon)}{(n+1)^{d-2}} \frac{1}{v(p)(1+\varepsilon)} \{\int x\,dJ(x)\}^{-1}. \tag{6.8}$$

But $N(n,\varepsilon)$ has the Bernoulli distribution with parameters $(n+1)^{d-2}$ and p_n, where

$$p_n = P(R_n(2) < (1+\varepsilon)\,v(p)\int x\,dJ(x)).$$

We claim that

$$\frac{N(n,\varepsilon)}{(n+1)^{d-2}} \to 1 \text{ a.s.} \quad \text{as} \quad n \to \infty. \tag{6.9}$$

To see this, note from Theorem 6.1(b) that $p_n \to 1$ as $n \to \infty$, and, for $0<\delta<1$, pick N such that $p_n > 1-\delta$ for all $n \geqq N$. Then, for $n \geqq N$,

$$P(N(n,\varepsilon) \leqq (n+1)^{d-2}(1-2\delta)) \leqq P(S_n \leqq (n+1)^{d-2}(1-2\delta)) \tag{6.10}$$

where S_n is Bernoulli with parameters $(n+1)^{d-2}$ and $1-\delta$. In the usual way, there exists $c(\delta)>0$ such that

$$P(S_n \leqq (n+1)^{d-2}(1-2\delta)) \leqq \exp(-(n+1)^{d-2}c(\delta)) \qquad \text{for all } n,$$

giving from (6.10) and the Borel-Cantelli lemma that

$$\liminf_{n\to\infty} \frac{N(n,\varepsilon)}{(n+1)^{d-2}} \geqq 1-2\delta \text{ a.s.} \qquad \text{for all } \delta>0,$$

and (6.9) is proved. Let $n \to \infty$ in (6.8) to obtain

$$\liminf_{n\to\infty}(n^{d-2}R_n(d)) \leqq (1+\varepsilon)\,v(p)\int x\,dJ(x) \text{ a.s.}$$

This holds for all positive ε, and so the proof is complete.

Proof of Theorem 6.3. We sketch this.

(a) $P(R_{nm}<\infty) = P(B_{nm}$ is crossed from left to right by a conducting path), and the result follows immediately from Theorem 2 of Grimmett (1981) with, in the notation of that paper, β_p set equal to α_p^{-1}.

(b) The proof of the lower bound in Theorem 6.1 is easily adapted to deal with this more general case. To show the upper bound, the proof of Theorem 6.1 may be followed to the point when

$$R_{nm} \leqq \frac{nm}{\phi_{nm}^2} \frac{\sum r(e)c(e)}{nm}.$$

But $\phi_{nm} \geqq \phi_{e^{cm},m}$ for all large m, and so

$$\liminf_{m\to\infty} \frac{1}{m}\phi_{nm} \geqq \lim_{m\to\infty} \frac{1}{m}\phi_{e^{cm},m} = \delta \text{ a.s.}$$

by Theorem 5.2.

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