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Existence and Uniqueness of DLR Measures for Unbounded Spin Systems

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Summary. A system of random variables (spins) S_x, $x \in \mathbb{Z}^\nu$, taking on values in $\mathbb{R}$ is considered. Conditional probabilities for the joint distributions of a finite number of spins are prescribed; a DLR measure is then a process on the random variables which is consistent with the assigned conditional probabilities [1, 2]. A case of physical interest both in Statistical Mechanics and in the lattice approximation to Quantum Field Theory is considered for which the spins interact pairwise via a potential $J_{xy} S_x S_y$, $J_{xy} \in \mathbb{R}$ and via a self-interaction $F(S_x)$, which, as $|S_x| \to \infty$, diverges at least quadratically [3].

By use of a technique introduced in [2] it is proven that the set

$$\mathfrak{C} = \{\nu \text{ is DLR} \mid \exists\, c(\nu), \sup_{x \in \mathbb{Z}^\nu} \int \nu(dS_x)\, |S_x| < c(\nu)\}$$

is a compact (in the local weak topology, Def. 1.1) non-void Choquet simplex [4]. Sufficient conditions are then given in order to obtain the measures in $\mathfrak{C}$ as limits of Gibbs measures for finitely many spins in a wide class of boundary conditions, Theorem 1.2. Uniqueness in $\mathfrak{C}$ is then discussed by means of a theorem by Dobrušin [2] and a sufficient condition for unicity is obtained which can be physically interpreted as a mean field condition [5]. Therefore the mean field temperature is rigorously proven to be an upper bound for the critical temperature.

0. Introduction

In 1970 Dobrušin [2] published a paper where the problem of existence and uniqueness of a random field with a given system of conditional distributions was investigated.

In our paper we consider a particular class of random fields, the continuous unbounded spin systems on a lattice $\mathbb{Z}^\nu$ and, exploiting Dobrušin's results, we study the existence and uniqueness conditions of the so called DLR measures [1]. These

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measures are those associated with a particular parametrization of the conditional probabilities, suggested by Statistical Mechanics (see Section 1).

The study of these fields has been receiving increasing attention recently especially in connection with the Euclidean quantum field theory and the renormalization group approach to critical phenomena (automodels).

In the sequel we will briefly outline the results of this paper in the language of Statistical Mechanics. The reader not familiar with this language is kindly asked to refer to the definitions given in Section 1, before reading following paragraphs.

We assume superstable potentials [1, 3] and specify the interaction to be given by a pairwise potential $J_{xy}S_xS_y$ and by a self-interaction $F(S_x)$ which diverges at least quadrically as $|S_x| \to \infty$, see Definition 1.2. Following a technique due to Dobrušin [2] we prove that the set

$$\mathfrak{E} = \{\nu \mathrm{DLR} \,|\, \exists c(\nu), \sup_{x \in \mathbb{Z}^\nu} \int \nu(\mathrm{d}S_x)\, |S_x| \leqq c(\nu)\}$$

is a compact [in the local weak topology, see Def. 1.1] non void set which is also a Choquet simplex [4]. We also show how to obtain the measures in $\mathfrak{E}$ as thermodynamic limits of finite volume Gibbs measures with external boundary conditions [3b]. This requires some limitation on the external boundary conditions which are given in Theorem 1.2. We then study uniqueness in the class $\mathfrak{E}$. By a Dobrušin theorem [2] a sufficient condition can be obtained in terms of the Vaseršteín distance between conditional probabilities of a single spin w.r.t. different boundary conditions. The Vaseršteín distance between Borel probability measures on $\mathbb{R}$ can be computed quite explicitely [6] so that a condition on uniqueness can be derived. This turns out to be the mean field theory estimate for the critical temperature which is therefore rigorously proven to be an upper bound for the critical temperature.

In Section 1 we give the notation used throughout the paper and then we prove existence. The more technical lemmas are left to Appendix A. In Section 2 we discuss the uniqueness problem; Appendix B is related to this section.

After the first draft of this paper was finished we were told that the problem considered here had also been studied by Royer [9]. He considered nearest neighbour ferromagnetic interactions. He proved existence with different techniques than those used in Section 1. A uniqueness theorem was obtained in [9] by use of an approach somewhat similar to that of Lanford [7] and of Israel [8].

Section 1. Existence of DLR Measures

Definition 1.1 (Phase space). We consider the phase space $\mathfrak{X}$ of unbounded spin configurations on the lattice $\mathbb{Z}^\nu$

$$\mathfrak{X} = \{\mathbf{S} \colon \mathbb{Z}^\nu \to \mathbb{R}\}. \tag{1.1}$$

S_x denotes the value of $\mathbf{S}$ at the point $x \in \mathbb{Z}^\nu$ the spin value at x, S_Λ for $\Lambda \subset \mathbb{Z}^\nu$ the restriction of the configuration $\mathbf{S}$ to the region Λ, i.e. the spins in Λ. $\mathfrak{X}$ is a topological space with product topology inherited from $\mathbb{R}$, it is a Polish space, i.e. it is metrizable with a metric for which it is separable and complete. A possible metric is

$$d(\mathbf{S},\mathbf{S}')=\sum_{x\in\mathbb{Z}^\nu}\bar{d}(S_x,S'_x)\,2^{-|x|},$$

$$|x|=\sup_{1\leq i\leq\nu}|x^i|,\qquad x=(x_i)_{i=1,\dots,\nu}\in\mathbb{Z}^\nu,$$

$$d(S_x,S'_x)=\min\{|S_x-S'_x|,1\}.$$

We denote by $M(\mathfrak{X})$ the set of Borel probability measures on $\mathfrak{X}$. We introduce on $M(\mathfrak{X})$ a topology determined by the continuous bounded and cylindrical functions on $\mathfrak{X}$ [f is cylindrical with basis Λ (bounded) if $f(\mathbf{S})=f(\mathbf{S}')$ whenever $S_\Lambda=S'_\Lambda$]. A neighbourhood of μ, $W_{\varepsilon,f}(\mu)$ is therefore

$$W_{\varepsilon,f}(\mu)=\{\nu\in M(\mathfrak{X})\,|\,|\nu(f)-\mu(f)|<\varepsilon,\ f\text{ is a continuous bounded cylindrical function on }\mathfrak{X}\}.$$

Hereafter we will call the above topology the local weak topology on $M(\mathfrak{X})$. $M(\mathfrak{X})$ is also a Polish space. To prove this we use the following argument. Let Λ_n be an increasing sequence invading $\mathbb{Z}^\nu$. By the Kolmogorov theorem [10] any measure μ is uniquely determined by the sequence of measures $\mu_n(dS_{\Lambda_n})\cdot(\mu_n(dS_{\Lambda_n})$ is the relativization of μ to $\{S_{\Lambda_n}\}\equiv\mathbb{R}^{|\Lambda_n|}$). It is known [10] that the weak topology of measures on $\mathbb{R}^{|\Lambda_n|}$ is a Polish space; let $\rho_n(\mu_n,\nu_n)$ be a metric which makes it complete and separable. From this a metric ρ can be introduced on $M(\mathfrak{X})$ as

$$\rho(\mu,\nu)=\sum_{n=0}^{\infty}2^{-n}\bar{\rho}_n(\mu_n,\nu_n),\qquad \bar{\rho}_n=\min(\rho_n,1).$$

It is easy to check that ρ makes $M(\mathfrak{X})$ a Polish space and that the induced topology is the same as the local weak topology on $M(\mathfrak{X})$.

Definition 1.2 (Assumptions). In the case we are considering here, for each spin S_x is given a free measure $\mu(dS_x)$, the same goes for each site x; the spins further interact via a pair potential. We assume that

$$\mu(dS_x)=dS_x\exp\{-F(S_x)\}. \tag{1.2}$$

Where $F(S_x)$: $\mathbb{R}\to\mathbb{R}$ is defined as follows: there exists a positive number S_0 such that

$$F(S_x)=\int_0^{S_x}\Phi_r(S'_x)\,dS'_x,\qquad S_x>S_0, \tag{1.3a}$$

$$F(S_x)=\int_{S_x}^{0}\Phi_l(S'_x)\,dS'_x,\qquad S_x<-S_0. \tag{1.3b}$$

Where Φ_r: $[0,+\infty)\to\mathbb{R}^+$ is a C^1 convex positive increasing function and Φ_l: $(-\infty,0]\to\mathbb{R}^+$ is a C^1 positive decreasing convex function. The interaction is so defined: for Λ bounded $\subseteq\mathbb{Z}^\nu$

$$U(S_\Lambda)=-\sum_{\{x,y\}\subset\Lambda}J_{xy}S_xS_y \tag{1.4}$$

where the sum is over all the non ordered pairs $\{x, y\}$ of distinct points in Λ and

$$|J_{xy}| \leqq J(|x-y|), \qquad \sum_{x \in \mathbb{Z}^\nu} J(|x-y|) < +\infty. \tag{1.5}$$

We remark that the one body interaction can be included in the free measure so that in general the free measure is temperature dependent.

Definition 1.3 (Superstability). There are $A>0$ and $c \in \mathbb{R}$ such that, for every S_Λ

$$\sum_{x \in \Lambda} F(S_x) + \beta U(S_\Lambda) \geqq \sum_{x \in \Lambda} [A S_x^2 - c]. \tag{1.6}$$

Consider the function Φ which appears in Equation (1.3). We distinguish two cases:

a) Φ is linear,

b) Φ is strictly convex.

We note that superstability condition is guaranteed in the case b). It holds even in the case a): $\Phi(x) = \alpha(x - x_0)$ (which corresponds to an asymptotically gaussian behaviour of the free spin distribution) if α is sufficiently large:

$$\alpha > \beta \sum_{z \neq 0} J(|z|). \tag{1.7}$$

Definition 1.4 (DLR equation and Gibbs states). For each bounded region $\Lambda \subset \mathbb{Z}^\nu$ we consider the sets

$$\mathfrak{X}(\Lambda^c, \mathrm{T}) = \{\mathrm{S} \in \mathfrak{X} \mid \sup_{x \in \Lambda} \sum_{y \notin \Lambda} J(|x-y|)\, |S_y| \leqq T,\ T>0\} \tag{1.8a}$$

where Λ^c denotes the set $\mathbb{Z}^\nu \backslash \Lambda$.

Let

$$\mathfrak{X}^G(\Lambda) = \bigsqcup_{T>0} \mathfrak{X}(\Lambda^c, T)$$

$\mathfrak{X}^G(\Lambda)$ is a Borel set. For $\mathrm{S} \in \mathfrak{X}^G(\Lambda)$ the (conditional) Gibbs probability at finite volume Λ with boundary conditions (b.c.) S_{Λ^c} is defined as the measure on $\mathfrak{X}(\Lambda) = \{S_\Lambda\}$ given by

$$\begin{aligned}
&q(dS_\Lambda \mid S_{\Lambda^c}) = Z_\Lambda(S_{\Lambda^c})^{-1}\, \mu(dS_\Lambda) \exp[-\beta U(S_\Lambda) - \beta W(S_\Lambda \mid S_{\Lambda^c})],\\
&W(S_\Lambda \mid S_{\Lambda^c}) = \sum_{x \in \Lambda} \sum_{y \notin \Lambda} J_{xy} S_x S_y,\\
&\quad \mu(dS_\Lambda) = \prod_{x \in \Lambda} \mu(dS_x),\\
&\quad Z_\Lambda(S_{\Lambda^c}) = \int \mu(dS_\Lambda) \exp[-\beta U(S_\Lambda) - \beta W(S_\Lambda \mid S_{\Lambda^c})].
\end{aligned} \tag{1.8b}$$

Let λ be a probability measure on $\mathfrak{X}$ such that

$$\lambda[\mathfrak{X}^G(\Lambda)] = 1. \tag{1.9a}$$

We define the Gibbs measure at finite volume Λ with external b.c. given by λ the following probability measure

$$\nu_\Lambda(d\mathbf{S};\lambda)=\lambda(dS_{\Lambda^c})\,q(dS_\Lambda|S_{\Lambda^c}). \tag{1.9b}$$

Notice that the conditional probability of $\nu_\Lambda(d\mathbf{S};\lambda)$ with respect to S_{Λ^c} is λ-modulo zero $q(dS_\Lambda|S_{\Lambda^c})$ while on $\mathfrak{X}(\Lambda^c)$ $\nu_\Lambda(d\mathbf{S};\lambda)$ is equal to λ.

We define a DLR measure as a probability measure which is for every bounded Λ a Gibbs measure for that region, that is, its conditional probabilities are given by $q(dS_\Lambda|S_{\Lambda^c})$.

In this section we study the relationship between the DLR measures and the thermodynamic limits of Gibbs measures at finite volume Λ as Λ invades $\mathbb{Z}^\nu$. More precisely we will select a class $\mathfrak{E}$, see Theorem 1.2, of DLR measures via a regularity condition (any $|S_x|$ is L^1 uniformly in $x\in\mathbb{Z}^\nu$ w.r.t. each measure in $\mathfrak{E}$). We will then show that there is a wide class of b.c. measures which, in the thermodynamic limit, leads to all measures in $\mathfrak{E}$. Results along this direction have already been obtained for a subclass $\mathfrak{E}^0$ of $\mathfrak{E}$ (the "superstable" measures)

Theorem 1.1 [3b]. *Let the superstability condition, see Definition 1.3, hold. Let $\mathfrak{E}^0$ be the set of* DLR *measures carried by the union over N of the sets*

$$R_N=\{\mathbf{S}\in\mathfrak{X}\,|\,\forall j\in\mathbb{Z}^+:\ \sum_{|x|\leqq j} S_x^2\leqq N^2(2j+1)^\nu\} \tag{1.10}$$

then $\mathfrak{E}^0$ is a (locally) weakly compact simplex. Further there are $\gamma>0$ and δ such that for every $\nu\in\mathfrak{E}^0$ and every bounded $\Lambda\subset\mathbb{Z}^\nu$

$$\nu(dS_\Lambda)\leqq dS_\Lambda\exp\{-\sum_{x\in\Lambda}(\gamma S_x^2-\delta)\}. \tag{1.11}$$

Let λ be any external b.c. *in the sense of Definition 1.4 with the additional hypothesis that for every bounded $\Lambda\subset\mathbb{Z}^\nu$*

$$\lambda(dS_\Lambda)\leqq dS_\Lambda\exp\{-\sum_{x\in\Lambda}(\gamma S_x^2-\delta)\}. \tag{1.12}$$

Then for any $\Lambda_n\uparrow\mathbb{Z}^\nu$ the sequence of measures $\nu_{\Lambda_n}(dS_{\Lambda_n};\lambda)$ has limiting points (in the local weak topology) and they are all in $\mathfrak{E}^0$.

If a power behavior is assumed for $J(r)$ at infinity

$$J(r)<\mathrm{const}\,r^{-\lambda'},\qquad \lambda'=\nu+\varepsilon,\ \varepsilon>0$$

then the above results extend to b.c. *given by $\delta_{\hat{\mathbf{S}}}(d\mathbf{S})=\lambda(d\mathbf{S})$,* i.e. *measures carried by a single configuration $\hat{\mathbf{S}}$, when*

$$\sup_{x\in\mathbb{Z}^\nu}\frac{|\hat{S}_x|}{\log_+|x|}<+\infty,\qquad \log_+|x|=\max\{1,\log|x|\}.$$

In this paper we consider the following b.c.

Definition 1.5 (Allowed b.c.). We introduce

$$\mathscr{C}_a(\mathbf{h})=\{\lambda\in M(\mathfrak{X}):\ \int\lambda(d\mathbf{S})|S_x|\leqq ah_x,\ x\in\mathbb{Z}^\nu\}, \tag{1.13a}$$

$$\mathscr{C}_a^\Lambda(\mathbf{h})=\{\lambda\in M(\mathfrak{X}):\ \int\lambda(d\mathbf{S})\,|S_x|\leqq ah_x,\ x\notin\Lambda\} \tag{1.13b}$$

where $\mathbf{h}: \mathbb{Z}^\nu \to [1, \infty)$ is such that

$$\sum_{y \neq x} J(|x-y|)\, h_y \leqq d h_x \quad \text{and} \quad d < +\infty. \tag{1.14}$$

Then we define the Gibbs measure at finite volume Λ with b.c. in $\mathscr{C}_a^\Lambda(\mathbf{h})$ as

$$\mathscr{G}_a^\Lambda(\mathbf{h}) = \{\nu \in M(\mathfrak{X}):\ \nu(d\mathbf{S}) = \lambda(dS_{\Lambda^c})\, q(dS_\Lambda | S_{\Lambda^c}), \lambda \in \mathscr{C}_a^\Lambda(\mathbf{h})\}. \tag{1.15}$$

The above is well defined because of

Lemma 1.1. *Let* $\lambda \in \mathscr{C}_a^\Lambda(\mathbf{h})$, *then for any bounded* $\Lambda' \subset \mathbb{Z}^\nu$, $\Lambda' \supset \Lambda$

$$\lambda[\mathfrak{X}^G(\Lambda)] = 1.$$

Proof. The function $\psi: \mathfrak{X} \to \mathbb{R}^+ \sqcup \infty$

$$\psi(\mathbf{S}) = \psi(S_{\Lambda^c}) = \sum_{x \in \Lambda} \sum_{y \notin \Lambda} J(|x-y|)\, |S_y|$$

Is a non-negative measurable function and by Equation (1.14)

$$\int \lambda(d\mathbf{S})\, \psi(\mathbf{S}) \leqq \sum_{x \in \Lambda} \sum_{y \notin \Lambda} J(|x-y|) a h_y \leqq a \sum_{x \in \Lambda} d h_x < +\infty.$$

Therefore

$$\lambda[\{\mathbf{S} \in \mathfrak{X} \,|\, \psi(\mathbf{S}) = \infty\}] = 0.$$

We also introduce

$$\mathscr{J}^{\bar{d}} = \{\mathbf{h}: \text{Equation (1.14) holds with } d \leqq \bar{d}\} \tag{1.16a}$$

and

$$\mathscr{G}_a^{\Lambda, d} = \bigsqcup_{\mathbf{h} \in \mathscr{J}^d} \mathscr{G}_a^\Lambda(\mathbf{h}). \tag{1.16b}$$

Remark 1.1. We give explicit examples in which Equation (1.14) holds and which will be used later on.

(i) $\mathbf{h} = (h_x, h_x = 1 \text{ for } x \in \mathbb{Z}^\nu)$ and $d = \sum_{y \neq x^y} J(|x-y|)$,

(ii) let $J(|x-y|) = 0$ whenever $|x-y| > R$. Then $\mathbf{h} = (h_x = \exp(\gamma |x|),\ \gamma > 0$ and $x \in \mathbb{Z}^\nu)$ satisfies Equation (1.14) with

$$d \geqq \exp(\gamma R) \sum_{y \notin x^y} J(|y-x|),$$

(iii) let $J(|x-y|)$ be such that

$$\sum_{z \neq 0} J(|z|) \ln |z| < +\infty$$

then it can easily be proven that $\mathbf{h} = (h_x = \ln_+ |\gamma x|,\ x \in \mathbb{Z}^\nu,\ \ln_+ a = \max(\ln a, 1))$ satisfies Equation (1.16) with a bound for d given by

$$\sum_{z \neq 0} J(|z|)(1 + \ln(1 + |z|)).$$

In particular it can be shown that as $\gamma \to 0$ in fact d can be chosen to approach the value $\sum_{z \neq 0} J(|z|)$.

The link between thermodynamic limits of Gibbs measure in $\mathscr{G}_a^\Lambda(\mathbf{h})$ as Λ invades $\mathbb{Z}^\nu$ and the DLR measures in $\mathfrak{E}$ will be obtained as a consequence of the following Lemma 1.2 which will be proven in Appendix A.

Lemma 1.2. *Let the assumptions in Definition 2 and 1.3 hold and let* Φ_r, Φ_l *in Definition 1.2 be strictly convex functions. Then the following is true:*

(i) *for any* $a>0$, Λ *bounded,* $\mathbf{h}$ *which satisfies Equation* (1.14) $\mathscr{G}_a^\Lambda(\mathbf{h})$ *is locally weakly compact,*

(ii) *let* $a, \mathbf{h}, \Lambda$, *be as in* (i), *let* $\Delta \subset \Lambda$ *then there is* $a' > a$ *independent of* Λ *and* Δ *such that* $\mathscr{G}_a^\Lambda(\mathbf{h}) \subset \mathscr{G}_{a'}^\Delta(\mathbf{h})$,

(iii) *let* a *and* $\mathbf{h}$ *be as in* (i) *let* Λ_n *be a sequence of increasing bounded regions invading* $\mathbb{Z}^\nu$, *then*

$$\overline{\bigsqcup_n \mathscr{G}_a^{\Lambda_n}(\mathbf{h})}$$

is locally weakly compact (the closure is taken in the local weak topology).

Notice that (iii) is a direct consequence of (i) and (ii). In the non-strictly convex case the above holds for any $\mathbf{h} \in \mathscr{J}^d$ with $d < \beta^{-1} 2\alpha$ such that

$$\Phi(S_x) = \alpha(S_x - S_x^0)$$

where α is actually the minimum between the linear coefficients of Φ_r and Φ_l.

Theorem 1.2. *Let the assumptions in Definition 1.2 and 1.3 hold. Let* $\mathfrak{E} = \{\nu \in M(\mathfrak{X}) |\ \nu$ is DLR *and there is* $c(\nu) < +\infty$ *such that*

$$\sup_{x \in \mathbb{Z}^\nu} \int \nu(d\mathbf{S})\, |S_x| < c(\nu)\}$$

then $\mathfrak{E}$ *is locally weakly compact, it is a Choquet simplex and there is* $c < +\infty$ *such that*

$$\sup_{\nu \in \mathfrak{E}} c(\nu) \leqq c. \tag{1.17}$$

Let Φ *be a strictly convex function. Let* Λ_n *be an increasing sequence of bounded regions invading* $\mathbb{Z}^\nu$, *correspondingly let* $\nu_n(d\mathbf{S}; \lambda_n)$ *with* $\lambda_n \in \mathscr{C}_a^{\Lambda_n}(\mathbf{h})$ *(see Eq.* (1.13)), *then a subsequence exists which converges to a measure in* $\mathfrak{E}$. *Naturally any measure in* $\mathfrak{E}$ *can be obtained with such a limit.*

For Φ *linear the above holds for those* $\mathbf{h}$ *which correspond to a* d *such that* $d < \beta^{-1} 2\alpha$ *(*α *is defined as in Lemma 1.2).*

Proof. We first prove the existence of the thermodynamic limits. These are

$$\bigsqcap_n \overline{\{\bigsqcup_{m \geqq n} \mathscr{G}_a^{\Lambda_m}(\mathbf{h})\}}. \tag{1.18}$$

The above set is non empty because

$$\overline{\bigsqcup \mathscr{G}^{\Lambda_m}(\mathbf{h})}$$

are compact by Lemma 1.2 (iii) and because they are decreasing as n increases. The set in Equation (1.18) is made up of DLR measures because of Lemma 1.2 (i) and (ii):

$$\bigsqcup_{m \geqq n} \mathscr{G}_a^{\Lambda_m}(\mathbf{h}) \subseteq \mathscr{G}_{a'}^{\Lambda}(\mathbf{h}), \qquad \forall \Delta \subset \Lambda_n$$

therefore

$$\bigsqcap_n \overline{\{ \bigsqcup_{m \geqq n} \mathscr{G}_a^{\Lambda_m}(\mathbf{h})\}} \subseteq \bigcap_m \mathscr{G}_{a'}^{\Lambda_m}(\mathbf{h})$$

and so they are DLR. In fact it can also be proven that they are in $\mathfrak{E}$ and that Equation (1.17) holds (see Appendix A). We now prove that $\mathfrak{E}$ is locally weakly compact. We denote by $\mathbf{1}$ the configuration such that $1_x = 1$ and we consider $\mathscr{G}_c^{\Lambda}(\mathbf{1})$ with c given by Equation (1.18) we have

$$\mathfrak{E} \subseteq \mathscr{G}_c^{\Lambda}(\mathbf{1}) \Rightarrow \bar{\mathfrak{E}} \subseteq \mathscr{G}_c^{\Lambda}(\mathbf{1})$$

for any bounded Λ because of Lemma 1.2 (i). Then

$$\bar{\mathfrak{E}} \subset \bigsqcap_n \mathscr{G}_c^{\Lambda_n}(\mathbf{1}) \subset \mathfrak{E} \subset \mathscr{G}_c^{\Lambda}(\mathbf{1})$$

and so $\mathfrak{E}$ is closed and compact because $\mathscr{G}_c^{\Lambda}(\mathbf{1})$ is compact by Lemma 1.2 (i). The proof that $\mathfrak{E}$ is a simplex follows the arguments used by Ruelle in [1f].

Remark 1.2. Theorem 1.2 gives a sufficient condition for the b.c. probabilities to lead, in the thermodynamic limit, to a measure in $\mathfrak{E}$. In the one dimensional Gaussian case ($F(S_x) = \alpha S_x^2, \alpha > 0$) with nearest neighbour interactions (with coefficients βJ) it is known [11] that if the external spins S_x grow as

$$|S_x| = \gamma_1 \exp(\gamma_2 |x|)$$

then the limiting state is in $\mathfrak{E}$ iff

$$\gamma_2 < \log \frac{\alpha + \sqrt{\alpha^2 - \beta^2 J^2}}{\beta J}.$$

This condition is just the same as $d < \beta^{-1} 2\alpha$ when d is chosen as in Equation (1.14) which now becomes

$$\sum_{|y-x|=1} h_y J \leqq d h_x, \qquad h_x = 1 \exp(\gamma_2 |x|) \;\; x \in \mathbb{Z}^\nu.$$

Section 2. Uniqueness

In this section we give a sufficient condition for the set $\mathfrak{E}$, see Theorem 1.2, to consist of only one measure. The condition is obtained from a theorem by Dobrušin [2] and to introduce it we first give the definition of the Vaserštein distance between two probability measures on $\mathbb{R}$.

Definition 2.1 (Vaserštein distance). Let μ, ν be two Borel probability measures on $\mathbb{R}$. We say that $\hat{P}$ is a joint representation of μ and ν if $\hat{P}$ is a regular probability measure on $\mathbb{R}^2$ and its relativizations to the two real axes give respectively μ and ν, that is

$$\int \hat{P}(dx\,dy)\, f(x) = \int \mu(dx)\, f(x) \qquad f \in L^1(\mathbb{R}, \mu), \tag{2.1a}$$

$$\int \hat{P}(dx\,dy)\, g(y) = \int \nu(dy)\, g(y) \qquad g \in L^1(\mathbb{R}, \nu). \tag{2.1b}$$

We define the V. distance $R(\mu, \nu)$ as

$$R(\mu, \nu) = \inf_{P} \int \hat{P}(dx\,dy)\, |x-y| \tag{2.2}$$

where the inf is taken among all the joint representations of μ and ν. For the properties of the above defined distance we refer to [2]. In this paper we just use the following

Theorem 2.1 [2]. *Let there exist a non-negative function r_{xy} such that*

(i) $r_{xx} = 0$, r_{xy} *depends on* $|x-y|$,

(ii) $\sum_y r_{xy} < 1, \quad \forall x \in \mathbb{Z}^\nu$, (2.3)

(iii) $R[q(dS_x \mid S_{\{x\}^c}),\, q(dS_x \mid S'_{\{x\}^c})] \leqq \sum_y r_{xy} |S_y - S'_y|$, $\forall S_{\{x\}^c}, S'_{\{x\}^c}$ then $\operatorname{Card} \mathfrak{E} = 1$.

Theorem 2.1 reduces the study of uniqueness for $\mathfrak{E}$ to the estimate (iii) that is the V. distance between conditional probabilities at a single site.

In reference [6] the problem of evaluating the V. distance between probability measures on $\mathbb{R}$ is studied and the main result is given by the following:

Theorem 2.2 [6]. *If μ and ν are Borel probability measures on $\mathbb{R}$ with finite first moments, then their* V. *distance is*

$$R(\mu, \nu) = \int_{-\infty}^{+\infty} dx \left| \int_{-\infty}^{x} (\mu(dy) - \nu(dy)) \right|. \tag{2.4}$$

Using the above theorem we obtain an explicit condition of uniqueness

Theorem 2.3. *The best possible choice for r_{xy} in Theorem 2.1* (iii) *is given by*

$$r_{xy} = \beta J(|x-y|) \{\sup_{t \in \mathbb{R}} \int q(dS_x | S_{\{x\}^c})(S_x - \bar{S}_x(t))^2\}, \tag{2.5a}$$

$$t = \beta \sum_{y \neq x} J_{xy} S_y, \qquad \bar{S}_x(t) = \int q(dS_x | S_{\{x\}^c})\, S_x. \tag{2.5b}$$

Therefore condition (ii) *of Theorem 2.1 becomes*

$$\sum_{y \neq x} \beta J(|x-y|) \{\sup_{t \in \mathbb{R}} \int q(dS_x | S_{\{x\}^c})(S_x - \bar{S}_x(t))^2\} < 1 \tag{2.5c}$$

which is then a sufficient condition for uniqueness in $\mathfrak{E}$.

Proof. We introduce the function D: $\mathbb{R}^2 \to \mathbb{R}^+$ given by

$$D(y,t) = \left[\int_{-\infty}^{+\infty} dy \exp[-F(y)+yt]\right]^{-1} \exp[-F(y)+yt] \tag{2.6}$$

where $F(y)$ has been defined in Equation (1.2). For the Gibbs conditional probability at a single site x we have

$$q(dS_x \mid S_{\{x\}^c}) \equiv q(dS_x \mid t) = dS_x D(S_x, t) \tag{2.7}$$

with t as in Equation (2.5b). Equation (2.4) becomes

$$R[q(dS_x|t_0), q(dS_x|t_1)] = \int_{-\infty}^{+\infty} dS_x \left| \int_{t_0}^{t_1} dt \int_{-\infty}^{S_x} \frac{\partial}{\partial t} D(S'_x, t)\, dS'_x \right| \tag{2.8}$$

with $t_0 \leqq t_1$. We have

$$R[q(dS_x|t_0), q(dS_x|t_1)] \leqq \int_{t_0}^{t_1} dt \int_{-\infty}^{+\infty} dx \left| \int_{-\infty}^{x} \frac{\partial}{\partial t} D(y,t)\, dy \right|. \tag{2.9}$$

As it is shown in the following Lemma the error made in the evaluation of $R[q(dS_x|t_0), q(dS_x|t_1)]$ by using the bound of Equation (2.9) can be made arbitrarily small as t_0 and t_1 are close enough.

Lemma 2.1. *For any* $t_0 \in \mathbb{R}$, $\varepsilon > 0, \exists T(\varepsilon) > 0$ such that $\forall t_1$: $0 \leqq t_1 - t_0 < T(\varepsilon)$

$$\int_{-\infty}^{+\infty} dx \left| \int_{t_0}^{t_1} dt \int_{-\infty}^{x} \frac{\partial}{\partial t} D(y,t)\, dy \right| \geqq -\varepsilon(t_1 - t_0) + \int_{t_0}^{t_1} dt \int_{-\infty}^{+\infty} dx \left| \int_{-\infty}^{x} \frac{\partial}{\partial t} D(y,t)\, dy \right|.$$

Proof. see Appendix B.

By calculation we see that

$$\int_{-\infty}^{+\infty} dx \left| \int_{-\infty}^{x} \frac{\partial}{\partial t} D(y,t)\, dy \right| = \int_{-\infty}^{+\infty} q(dS_x|t)[S_x - \bar{S}_x(t)]^2.$$

Since we look for a bound of the kind

$$R[q(dS_x|t_0), q(dS_x|t_1)] \leqq \text{const}\, |t_1 - t_0| \tag{2.10}$$

which is uniform on the choice of t_1 and t_0 we see that we have to choose the

$$\text{const. in Equation (2.10)} = \sup_{t \in \mathbb{R}} \int_{\mathbb{R}} q(dS_x|t)[S_x - \bar{S}_x(t)]^2.$$

Let $\bar{t}$ be the value for which the sup is obtained ($\bar{t}$ is finite as a consequence of Lemma A.1). By taking $t_0 = \bar{t}$, $t_1 = \bar{t} + \delta$ and by considering the limit $\delta \to 0$ from Lemma 2.1 it is evident that the choice in Equation (2.5a) is the best for the constant in Equation (2.10). This proves Theorem 2.3.

Some Applications of Theorem 2.3

Gaussian Case. Let

$$F(S_x)=\alpha(S_x-S^0)^2.$$

Equation (2.5c) gives Card $\mathfrak{E}=1$ if

$$(2\alpha)^{-1}\beta\sum_{y\neq x} J(|x-y|)<1$$

which is just the superstability condition; this proves the absence of phase transitions for Gaussian models [9, 12].

Large Temperatures. Consider Φ to be strictly convex, then it is easy to convience oneself that Equation (2.3) (ii) is satisfied at large temperatures. Notice, as an example, that the integral

$$I(\beta)=[\int_{\mathbb{R}} dS_x \exp(-\beta A S_x^{2+\varepsilon})]^{-1}\int_{\mathbb{R}} dS_x S_x^2 \exp(-\beta A S_x^{2+\varepsilon}) \qquad A,\varepsilon>0$$

behaves like

$$I(\beta)=\text{const}\,\beta^{-\frac{2}{2+\varepsilon}}$$

therefore

$$r_{xy}=\text{const}\,J(|x-y|)\,\beta^{\frac{\varepsilon}{2+\varepsilon}}.$$

To clarify the physical meaning of the result in Theorem 2.3 we conclude this section with the following remarks

Remark 2.1. We think it worth mentioning that the bound given by Equation (2.5c) is in the ferromagnetic case the well-known mean field result that guarantees the absence of phase transitions and spontaneous magnetization: Theorem 2.3 therefore proves that the mean field critical temperature is a rigorous upper bound for the true critical temperature.

Remark 2.2. Also for non ferromagnetic systems Condition (2.5c) has a natural physical interpretation. Let us consider a site $x\in\mathbb{Z}^\nu$ and let us fix the spins S_y outside of x. As a consequence we get a mean value for S_x conditioned on the external spins to be S_y:

$$\bar{S}_x(S_{\{x\}^c})=\int q(dS_x|S_{\{x\}^c})\,S_x. \tag{2.12}$$

Let us now change the outside configuration by changing by the same amount (except for the sign) each external spin: that is we fix a function $\boldsymbol{\eta}:\mathbb{Z}^\nu\backslash x\to\{-1,+1\}$ and a number δ so that the new configuration is

$$S_y(\boldsymbol{\eta},\delta)=S_y+\delta\eta_y. \tag{2.13}$$

Therefore

$$\bar{S}_x(S_{\{x\}^c}(\boldsymbol{\eta}, \delta)) = \int q(dS_x | S_{\{x\}^c}(\boldsymbol{\eta}, \delta))\, S_x. \tag{2.14}$$

We define the mean field theory estimate for the critical temperature as the following:

$$\sup_{\{S_{\{x\}^c}\}} \sup_{\boldsymbol{\eta}} \left| \frac{\partial}{\partial \delta} \bar{S}_x[S_{\{x\}^c}(\boldsymbol{\eta}, \delta)] \right| < 1. \tag{2.15}$$

Equation (2.15) means that starting from any configuration $S_{\{x\}^c}$ if we want to change the mean value of S_x of the amount δ we need to change each of the spins outside of a strictly larger amount than δ. The validity of Equation (2.15) ensures therefore some kind of stability for the distribution given by a measure $\nu \in \mathfrak{E}$. This is sufficient to give uniqueness because it is quite easy to see that Equation (2.15) is just the same condition as the one in Theorem 2.3.

It would seem likely by the above considerations that Equation (2.15) is stronger than what is really needed to prove uniqueness. One would think that rather than the uniformity required by taking the sup in the l.h.s. of Equation (2.15) it ought to be sufficient to have that

$$\frac{\partial}{\partial \delta} \bar{S}_x[S_{\{x\}^c}(\boldsymbol{\eta}, \delta)] < 1$$

for sufficiently many $S_{\{x\}^c}$. This would amount to prove Dobrušin's theorem by assuming a weaker hypothesis of type

$$R[q(dS_x | S_{\{x\}^c}), q(dS_x | S'_{\{x\}^c})] \leqq \sum_y r_{xy} |S_y - S'_y|, \tag{2.16}$$

$$\sum_y r_{xy} \leqq d < 1 \qquad \forall x \in \mathbb{Z}^\nu \tag{2.17}$$

and where Equation (2.16) is required to hold only for some configurations $S_{\{x\}^c}$, $S'_{\{x\}^c}$ which are a-priori known to have large probability to occur.

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Appendix A

In this appendix we prove Lemma 1.2, Equation (1.17) and that

$$\bigcap_n \{\overline{\bigcup_{m \geqq n} \mathscr{G}_a^{\Lambda_m}(\mathbf{h})}\} \subset \mathfrak{E}.$$

All this will be proven according to the ideas contained in Dobrušin's paper [2].

The main tool will be Prohorov's theorem which will be used in the following form

Theorem A.1. *Let* $\mathcal{N} \subset M(\mathfrak{X})$ *be such that*

$$\sup_{\nu\in\mathcal{N}} \int \nu(d\mathbf{S})|S_x| \leqq c_x \quad \forall x\in\mathbb{Z}^\nu, \qquad \mathbf{c}=\{c_x\in\mathbb{R}^+, x\in\mathbb{Z}^\nu\}. \tag{A.1}$$

Then $\bar{\mathcal{N}}$, *the local weak closure of* $\mathcal{N}$, *is compact and Equation* (A.1) *holds in all of* $\bar{\mathcal{N}}$.

Proof. Let Λ_n be an increasing sequence of bounded regions which invades $\mathbb{Z}^\nu$. For any $\nu\in M(\mathfrak{X})$ let $\nu_n(dS_{\Lambda_n})$ be its relativization to $\{S_{\Lambda_n}\}$. To prove that $\bar{\mathcal{N}}$ is compact it is enough to prove that given any sequence $\nu^{(m)}\in\mathcal{N}$ there is a cluster point ν (since $M(\mathfrak{X})$ is a Polish space, see Definition 1.1, this amounts to proving that there is a subsequence of $\nu^{(m)}$ which is locally weakly convergent to ν). We fix Λ, and we consider the sequence $\nu^{(m)}_{\Lambda_1}(dS_{\Lambda_1})$. In the space $\{S_{\Lambda_1}\}$ the set $\mathcal{N}_{\Lambda_1}=\{\nu(dS_{\Lambda_1})$, with $\nu\in\mathcal{N}\}$ are relatively compact by

$$\sup_{\nu\in\mathcal{N}} \int \nu(dS_\Lambda) \sum_{x\in\Lambda} |S_x| \leqq \sum_{x\in\Lambda} c_x < +\infty$$

and the Prohorov theorem [2] (because $\sum_{x\in\Lambda} |S_x|$ is a compact function in $\mathbb{R}^{|\Lambda|}$). Therefore there is a subsequence $\nu^{(m^1)}_{\Lambda_1}$ converging weakly to $\nu_{\Lambda_1}(dS_{\Lambda_1})$. We can then iterate the argument for Λ_2 by choosing a subsequence of $m^{(1)}$ s.t. $\nu^{m^2}_{\Lambda_2} \to \nu_{\Lambda_2}(dS_{\Lambda_2})$. Note that the relativization of $\nu_{\Lambda_2}(dS_{\Lambda_2})$ to $\{S_{\Lambda_1}\}$ is $\nu_{\Lambda_1}(dS_{\Lambda_1})$. In this way we define a sequence $\nu_{\Lambda_n}(dS_{\Lambda_n})$ of compatible measures which by Kolmogorov theorem define uniquely a measure $\nu\in M(\mathfrak{X})$. This measure is by construction a locally weak cluster point of $\nu^{(n)}$.

We will apply Prohorov's theorem by means of the following technical lemma:

Lemma A.1. *Consider the function* Φ *defined in Section 1. If* $\Phi(x)=\alpha(x-x_0)$ *there is* $c_2<+\infty$ *such that*

$$\int q(dS_x|S_{\{x\}^c})|S_x| \leqq \sum_{y\neq x} r_{xy}|S_y| + c_2$$

where

$$r_{xy}=\beta J(|x-y|)(2\alpha)^{-1}.$$

If Φ *is strictly convex the same estimate holds with an arbitrarily small number* c_1 *in place of* $(2\alpha)^{-1}$ *i.e.* $\forall c_1>0$ $\exists c_2$ *such that Equation* (A.1) *holds with* $r_{xy}=\beta J(|x-y|)\,c_1$.

Proof. We have to study the following quantity

$$\langle |S_x|\rangle_t = \left[\int dS_x \exp[-F(S_x)+S_x t]\right]^{-1} \int dS_x |S_x| \exp[-F(S_x)+S_x t], \tag{A.2}$$

$$t=\beta \sum_{y\neq x} J_{x,y} S_y.$$

Since the behavior for positive and negative t is symmetric we will consider only $t>0$. First we note that, for $t>0$

$$\frac{\int_{-\infty}^{0} dx\,|x| \exp[-F(x)+xt]}{\int_{-\infty}^{+\infty} dx \exp[-F(x)+xt]} \leqq \frac{\int_{-\infty}^{0} dx\,|x| \exp[-F(x)]}{\int_{0}^{\infty} dx \exp[-F(x)]} = K. \tag{A.3}$$

Then we have

$$\langle|x|\rangle_t \leqq K+S_0+\frac{\int_{S_0}^{\infty} dx\,|x|\exp[-F(x)+xt]}{\int_{S_0}^{\infty} dx\exp[-F(x)+xt]}. \tag{A.4}$$

Let us consider now $\bar{x}(t)$ as the value of x for which $-F(x)+xt$ achieves its maximum value:

$$\Phi'(\bar{x}(t))=t.$$

Recalling that Φ is a convex continuous increasing function of x it is evident that $\bar{x}(t)$ exists when t is sufficiently large and $\bar{x}(t)\to\infty$ when $t\to\infty$.

We can write:

$$\langle|x|\rangle_t \leqq K+S_0+|\bar{x}(t)|+\frac{\int_{S_0}^{\infty} dx\,|x-\bar{x}(t)|\exp[-F(x)+xt]}{\int_{S_0}^{\infty} dx\exp[-F(x)+xt]}. \tag{A.5}$$

In case a) by explicit calculations we get

$$\bar{x}=(2\alpha)^{-1}t+x_0; \qquad \lim_{t\to\infty}\frac{\int_{S_0}^{\infty} dx\exp[-F(x)+xt]\,|x-\bar{x}(t)|}{\int_{S_0}^{\infty} dx\exp[-F(x)+xt]}=0 \tag{A.6}$$

which completes the proof of the first part of the lemma.

For case b) it is sufficient to prove that

$$\lim_{t\to\infty}|t|^{-1}\langle|x|\rangle_t=0. \tag{A.7}$$

In fact Equation (A.7) implies that $\forall c_1>0$, there is $|t_1|$ such that for

$$|t|>|t_1|, \qquad |t|^{-1}\langle|x|\rangle_t\leqq c_1.$$

Therefore if $c_2=\sup_{|t|\leqq|t_1|}\langle|x|\rangle_t$ we have

$$\langle|x|\rangle_t\leqq c_1|t|+c_2.$$

Now from the strict convexity of the function $\Phi(x)$ it follows that $F''(x)=\Phi'(x)$ is an increasing function of x.

Equation (A.5) then becomes

$$\langle|x|\rangle_t\leqq K+S_0+\bar{x}(t)+\frac{\int_{\bar{x}(t)}^{\infty} dx\,|x-\bar{x}|\exp[-F(\bar{x})+t\bar{x}-\frac{1}{2}F''(\bar{x})(x-\bar{x})^2]}{\int_{S_0}^{\bar{x}(t)} dx\exp[-F(\bar{x})+t\bar{x}-\frac{1}{2}F''(\bar{x})(x-\bar{x})^2]}.$$

Now noting that the last term in the r.h.s. of the above equation tends to zero as $t \to \infty$ and that

$$\lim_{t \to \infty} |t|^{-1} \bar{x}(t) = 0$$

we obtain Equation (A.7) so that the proof of the lemma is concluded.

Lemma A.1 allows us to apply Theorem A.1 to our case. In fact we can prove the following theorem and corollary which will be stated after the following:

Definition A.1. We denote by

$$\mathbf{B}(\lambda, \Lambda) \in \mathfrak{X} \tag{A.8a}$$

the configuration

$$B_x(\lambda, \Lambda) = \int v_\Lambda(d\mathbf{S}; \lambda) |S_x| \qquad v_\Lambda \in \mathscr{G}_a^\Lambda(\mathbf{h}). \tag{A.8b}$$

We consider, for $\mathbf{h} \in \mathscr{J}^d$ the subset of $\mathfrak{X}$

$$\mathscr{B} = \{\mathbf{S} \in \mathfrak{X} \,|\, \sup_x h_x^{-1} |S_x| \equiv |||\mathbf{S}||| \geqq +\infty\} \tag{A.9}$$

and $\mathscr{B}$ is a Banach space with the norm $|||\cdot|||$. We will also consider the projection of the configurations to finite regions Δ so we will have

$$\mathscr{B}_\Delta, \qquad B_\Delta(\lambda, \Lambda) \in \mathscr{B}_\Delta$$

simply as restrictions (with $x \in \Delta$) of the definitions Equation (A.8), (A.9).

Theorem A.2. *Let $\mathscr{C}_a(\mathbf{h})$ be fixed and let d be its corresponding value according to Equation* (1.14). *Fix c_1 in Lemma A.1 so that*

$$c_1 \beta d < 1.$$

This is always possible if $\Phi_{r,l}$ are strictly convex functions, see Definition 1.2, if Φ is linear then d must be such that $d < \beta^{-1} 2\alpha$. Define

$$\eta = 1 - c_1 \beta d > 0$$

then

$$|||B(\lambda, \Lambda)||| \leqq \eta^{-1}(c_2 + a(1-\eta))$$

for every bounded Λ [c_2 is the value corresponding to the above fixed c_1 in Lemma A.1].

Proof. By use of Lemma A.1 we have for $z \in \Lambda$

$$\begin{aligned} B_x(\lambda, \Lambda) &\leqq c_2 + \sum_{y \in \Lambda, y \neq x} r_{xy} B_y(\lambda, \Lambda) + \sum_{y \notin \Lambda} r_{xy} a h_y \\ &\leqq (c_2 + a \sum_{y \neq x} r_{xy} h_y) + \sum_{y \in \Lambda, y \neq x} r_{xy} B_y(\lambda, \Lambda) \\ &\leqq h_x [c_2 + (1-\eta) a] + \sum_{y \in \Lambda, y \neq x} r_{xy} B_y(\lambda, \Lambda). \end{aligned} \tag{A.10a}$$

We rewrite Equation (A.10a)

$$\begin{aligned} &B_\Lambda(\lambda, \Lambda) \leqq A_\Lambda + R_\Lambda B_\Lambda(\lambda, \Lambda), \\ &|||A_\Lambda||| = c_2 + (1-\eta)\,a. \end{aligned} \tag{A.10b}$$

It is easy to see that the norm of the operator R_Λ is less than $1-\eta$ in fact

$$\begin{aligned} \sum_{y\in\Lambda, y\neq x} r_{xy} D_y &\leqq \sum_{y\in\Lambda, y\neq x} r_{xy}\, |||D|||_\Lambda h_y \leqq c_1 \beta d h_x |||D|||_\Lambda \\ &= h_x(1-\eta)\,|||D|||_\Lambda. \end{aligned}$$

Therefore if we prove that $|||B_\Lambda|||_\Lambda$ is bounded we have also proven that:

$$|||B_\Lambda|||_\Lambda \leqq |||(1-R_\Lambda)^{-1} A_\Lambda|||_\Lambda \leqq \eta^{-1}(c_2 + (1-\eta)\,a). \tag{A.11}$$

Since $B_x(\lambda, \Lambda) \leqq a h_x$ for $x \notin \Lambda$ if Equation (A.11) holds then

$$|||B||| \leqq \eta^{-1}(c_2 + (1-\eta)\,a)$$

and the thesis would be proven. We are therefore only left with the proof that $|||B_\Lambda|||_\Lambda$ is finite.

Let $T > 0$ and $\mathfrak{X}(\Lambda^c, T)$ as in Equation (1.8); define

$$B_x^T(\lambda, \Lambda) = \int_{\mathfrak{X}(\Lambda^c, T)} \nu_\Lambda(d\mathbf{S}; \lambda)\,|S_x| \qquad x\in\Lambda.$$

It is easy to see that $B_x^T(\lambda, \Lambda)$ is finite for any $x\in\Lambda$ (by using the superstability conditions on the interaction), then we can apply the same inequalities as above and we get the analogous of Equation (A.11) which now reads:

$$|||B_\Lambda^T|||_\Lambda \leqq \eta^{-1}(c_2 + (1-\eta)\,a).$$

Since $\nu_\Lambda[\bigsqcup_{T>0} \mathfrak{X}(\Lambda^c, T)] = 1$ the assertion is proven.

Lemma A.3. *Let $S_{\Lambda^c} \in \mathfrak{X}^G(\Lambda)$, let $\eta = 1 - \beta c_1 d$ be as in Theorem A.2 then*

$$\int q(dS_\Lambda | S_{\Lambda^c})\,|S_x| \leqq \eta^{-1} \sup_{x\in\Lambda} (c_2 + \sum_{y\notin\Lambda} r_{xy}\,|S_y|).$$

Proof. It is the same proof as in Theorem A.2, the only difference being that we do not integrate over the external spins $S_y \notin \Lambda$.

Lemma A.4. *In the hypothesis of Lemma 1.2, we have*

$$\mathscr{G}_a^\Lambda(\mathbf{h}) \subseteq \mathscr{G}_{a'}^\Delta(\mathbf{h}) \qquad \textit{for any } \Lambda;\ \Delta;\ \Lambda \supset \Delta$$

with

$$a' = \eta^{-1}(c_2 + (1-\eta)\,a).$$

Proof. Let $\nu_\Lambda(d\mathbf{s}) = \lambda(dS_{\Lambda^c})\, q(dS_\Lambda \,|\, S_{\Lambda^c})$ $\lambda \in \mathscr{C}_a^\Lambda(\mathbf{h})$, then we can write for

$$\Delta \subset \Lambda \qquad \nu_\Lambda(d\mathbf{s}) = \nu_\Lambda(dS_{\Delta^c})\, q(dS_\Delta \,|\, S_{\Delta^c})$$

therefore we have to prove that $\nu_\Lambda(dS_{\Lambda^c}) \in \mathscr{C}^\Lambda_{a'}(\mathbf{h})$. We have

$$\int \nu_\Lambda(d\mathbf{s})|S_x| = B_x(\lambda, \Lambda) \leqq |||B(\lambda, \Lambda)||| \, h_x \leqq \eta^{-1}(c_2 + (1-\eta)\, a)\, h_x.$$

Lemma A.5. *Let the assumptions in Lemma 1.2 hold. Let $\{\Lambda_n\}$ increase to $\mathbb{Z}^\nu$ then*

$$\mathscr{A} \equiv \bigcap_n \overline{\{ \bigsqcup_{m \geqq n} \mathscr{G}_a^{1_m}(\mathbf{h})\}} \subset \mathfrak{E}.$$

Proof. By Theorem A.2 and Theorem A.1 any $\nu \in \mathscr{A}$ is such that

$$B_x(\nu) \equiv \int \nu(d\mathbf{S})\, |S_x| \leqq \eta^{-1}(c_2 + a(1-\eta))\, h_x. \tag{A.12}$$

As in Theorem A.2 we obtain

$$B_x(\nu) \leqq c_2 + \sum_{\substack{y \\ y \neq x}} r_{xy} B_y(\nu). \tag{A.13}$$

By use of Equation (A.12) in Equation (A.13) we have

$$\begin{aligned} B_x(\nu) &\leqq c_2 + r c_2 + \sum_{y \neq x} r_{xy} h_y \eta^{-1}(c_2 + (1-\eta)\, a), \\ r &= \sum_{y \neq x} r_{xy} \leqq 1 - \eta < 1. \end{aligned} \tag{A.14}$$

By iterating the same procedure we have

$$B_x(\nu) \leqq \eta^{-1} c_2 + \lim_{m \to \infty} (R^m \mathbf{h})_x \eta^{-1}(c_2 + (1-\eta)\, a) \leqq \eta^{-1} c_2,$$

$$(R\mathbf{h})_x = \sum_{y \neq x} r_{xy} h_y.$$

Lemma A.6. *In the assumptions of Definition* 1.2 *we have that* (*see Eq.* (1.19))

$$\sup_{\nu \in \mathfrak{E}} c(\nu) \leqq c < +\infty.$$

Proof. We use a slight modification of the previous argument. $\forall \nu \in \mathfrak{E}$ define the configuration $\mathbf{B}(\nu)$:

$$B_x(\nu) = \int \nu(d\mathbf{S})\, |S_x|$$

consider the set $\mathscr{B}_0 \subset \mathfrak{X}$:

$$\mathscr{B}_0 = \{\mathbf{S} \in \mathfrak{X} \mid \sup_x |S_x| = \|\mathbf{S}\| < +\infty.$$

$\mathscr{B}_0$ is a Banach space with $\|\cdot\|$ norm. By hypothesis we know that $\|\mathbf{B}(\nu)\| \leqq \mathbf{c}(\nu)$. From Theorem A.1, all we need in order to obtain the assertion above is to show that

$$\sup_{\nu \in \mathfrak{E}} \|\mathbf{B}(\nu)\| < \infty.$$

By Lemma A.1 we have

$$B_x(v) \leqq c_2 + \sum_{y \neq x} r_{xy} B_y(v).$$

We rewrite the last equation in the form

$$\mathbf{B}(v) \leqq \mathbf{c} + RB(v),$$
$$\|\mathbf{c}\| = c_2.$$

It is easy to see that the norm of the operator R is less than η^{-1}. Since the norm $\|\mathbf{B}(v)\|$ is finite we can use the estimate

$$\|\mathbf{B}(v)\| \leqq \|(1-R)^{-1}\mathbf{c}\|$$

to conclude that

$$\|\mathbf{B}(v)\| \leqq \eta^{-1} c_2.$$

Now we must prove the following

Lemma A.8. *In the assumptions of Definition* 1.2 *we have that* $\mathscr{G}_a^\Lambda(\mathbf{h})$ *is closed in the local weak topology.*

To prove this lemma some intermediate steps are needed.

Lemma A.9. *Let* Λ *be a bounded region, for any* $\lambda \in \mathscr{C}_a^\Lambda(\mathbf{h})$ *we have the following:* $\forall \varepsilon > 0\ \exists \Delta \supset \Lambda$, Δ *bounded, such that*

$$\lambda(M) \leqq \varepsilon$$

where

$$M = \{\mathbf{S} \in \mathfrak{X} \,|\, \sup_{x \in \Lambda} \sum_{y \notin \Delta} \beta J(|x-y|)\,|S_y| > \varepsilon\}.$$

Proof. We have

$$\int \lambda(d\mathbf{S})\{\sum_{y \notin \Delta} \beta J(|x-y|)\,|S_y|\} \leqq \beta a \sum_{y \notin \Delta} J(|x-y|)\,h_y.$$

By Equation (1.14) we can choose $\Delta \supset \Lambda$ such that

$$\sup_{x \in \Lambda} \sum_{y \notin \Delta} \beta J(|x-y|)\,a\,h_y < \varepsilon^2$$

then

$$\varepsilon^2 > \int \lambda(d\mathbf{S})(\sum_{y \notin \Delta} \beta J(|x-y|)\,|S_y|) \geqq \int_M \lambda(d\mathbf{S})(\sum_{y \notin \Delta} \beta J(|x-y|)\,|S_y|) \geqq \varepsilon \lambda(M).$$

Lemma A.10. *Let* φ *be a cylindrical (in* $\Gamma \supset \Lambda$*) continuous function with* $|\varphi(\mathbf{S})| \leqq 1$*. Let* $\psi\colon \mathfrak{X}^G(\Lambda) \to \mathbb{R}$ *be*

$$\psi(\mathbf{S}) = \psi(S_{\Lambda^c}) = \int q(dS_\Lambda \,|\, S_{\Lambda^c})\,\varphi(S_\Gamma).$$

Now we fix $\Delta \supset \Gamma$ and define for $0 \leqq u \leqq 1$

$$S_y^u = \begin{cases} S_y & \text{if } y \in \Delta \\ uS_y & \text{if } y \notin \Delta \end{cases}$$

we call $\psi^u(\mathbf{S}) = \psi(\mathbf{S}^u)$ *and* $\psi^\Delta(\mathbf{S}) = \psi^0(\mathbf{S})$ *[so that* $\psi^\Delta(\mathbf{S})$ *is cylindrical in* Δ*]. We have*

$$|\psi(\mathbf{S}) - \psi^\Delta(\mathbf{S})| \leqq \sum_{y \notin \Delta} \sum_{x \in \Lambda} \beta J(|x-y|)\, |S_y| \cdot \sup_{0 \leqq u \leqq 1} 2 \int q(dS_\Lambda \mid S_{\Lambda^c}^u)\, |S_x|.$$

Proof. For $\mathbf{S} \in \mathfrak{X}^G(\Lambda)$, $\psi^u(\mathbf{S})$ is a C^1-function of u, so that

$$\psi(\mathbf{S}) - \psi^0(\mathbf{S}) = \int_0^1 \frac{d}{du} \psi^u(\mathbf{S}).$$

We have

$$\begin{aligned} \frac{d}{du} \psi^u(\mathbf{S}) &= \frac{d}{du} \int q(dS_\Lambda \mid S_{\Lambda^c}^u)\, \varphi(S_\Gamma) \\ &= \int q(dS_\Lambda \mid S_{\Lambda^c}^u)\, \varphi(S_\Gamma) \sum_{x \in \Lambda} \sum_{y \notin \Delta} \beta J_{xy} S_x S_y \\ &\quad - \left[\int q(dS_\Lambda \mid S_{\Lambda^c}^u)\, \varphi(S_\Gamma)\right] \left[\int q(dS_\Lambda \mid S_{\Lambda^c}^u) \sum_{x \in \Lambda} \sum_{y \notin \Delta} \beta J_{xy} S_x S_y\right] \end{aligned}$$

then

$$\left| \frac{d}{du} \psi^u(\mathbf{S}) \right| \leqq 2 \sum_{x \in \Lambda} \sum_{y \notin \Delta} \beta J(|x-y|)\, |S_y| \int q(dS_\Lambda \mid S_{\Lambda^c}^u)\, |S_x|.$$

Now we can prove Lemma A.8.

Proof. We must prove that if $\{\lambda_n \in \mathscr{C}_a(\mathbf{h})\}$ converges in the local weak topology to $\lambda \in \mathscr{C}_a(\mathbf{h})$, then $\lambda_n(dS_{\Lambda^c})\, q(dS_\Lambda \mid S_{\Lambda^c})$ converges to $\lambda(dS_{\Lambda^c})\, q(dS_\Lambda \mid S_{\Lambda^c})$ in the local weak topology. Now we fix φ as in Lemma A.10 and ε and Δ as in Lemma A.9. Then we have:

$$\begin{aligned} &\left|\int [\lambda(dS_{\Lambda^c}) - \lambda_n(dS_{\Lambda^c})]\, \psi(\mathbf{S})\right| \\ &\leqq \left|\int_M [\lambda(dS_{\Lambda^c}) - \lambda_n(dS_{\Lambda^c})]\, \psi(\mathbf{S})\right| + \left|\int_{M^c} [\lambda(dS_{\Lambda^c}) - \lambda_n(dS_{\Lambda^c})]\, \psi(\mathbf{S})\right| \\ &\leqq \left|\int_{M^c} [\lambda(dS_{\Lambda^c}) - \lambda_n(dS_{\Lambda^c})]\, \psi(\mathbf{S})\right| + 2\varepsilon \\ &\leqq 4\varepsilon + \left|\int [\lambda(dS_{\Lambda^c}) - \lambda_n(dS_{\Lambda^c})]\, \psi^\Delta(\mathbf{S})\right| + \int_{M^c} \lambda(dS_{\Lambda^c})\, |\psi(\mathbf{S}) - \psi^\Delta(\mathbf{S})| \\ &\quad + \int_{M^c} \lambda_n(dS_{\Lambda^c})\, |\psi(\mathbf{S}) - \psi^\Delta(\mathbf{S})|. \end{aligned}$$

Now we have the following estimate which is true for any $\lambda \in \mathscr{C}_a(\mathbf{h})$ by using Lemma A.10 and Lemma A.3.

$$\int_{M^c} \lambda(dS_{\Lambda^c})\,|\psi(\mathbf{S})-\psi^{\Lambda}(\mathbf{S})|$$

$$\leqq 2 \int_{M^c} \lambda(dS_{\Lambda^c}) \sum_{y\notin\Lambda} \sum_{x\in\Lambda} \beta J(|x-y|)\,|S_y| \sup_{0\leqq u\leqq 1} \int q(dS_\Lambda\,|\,S^u_{\Lambda^c})\,|S_x|$$

$$\leqq 2\varepsilon \int_{M^c} \lambda(dS_{\Lambda^c}) \sup_u \sum_{x\in\Lambda} \int q(dS_\Lambda\,|\,S^u_{\Lambda^c})\,|S_x|$$

$$\leqq 2\varepsilon \int_{M^c} \lambda(dS_{\Lambda^c}) \sup_{x\in\Lambda} |\Lambda|\,\eta^{-1}(c_2+\sum_{y\notin\Lambda} r_{xy}\,|S_y|)$$

$$\leqq 2\varepsilon\eta^{-1}\,|\Lambda|(c_2+\sup_{x\in\Lambda}\sum_{y\notin\Lambda} r_{xy}ah_y)$$

$$\leqq 2\varepsilon\,|\Lambda|\,\eta^{-1}(c_2+ad\,\sup_{x\in\Lambda} h_x).$$

Then

$$|\int[\lambda(dS_{\Lambda^c})-\lambda_n(dS_{\Lambda^c})]\,\psi(\mathbf{S})|$$

$$\leqq 2\varepsilon+2\varepsilon\,|\Lambda|\,\eta^{-1}(c_2+ad\sup_{x\in\Lambda} h_x)+|\int[\lambda(dS_{\Lambda^c})-\lambda_n(dS_{\Lambda^c})]\,\psi^{\Lambda}(\mathbf{S})|$$

where $\psi^{\Lambda}(\mathbf{S})$ is a cylindrical function.

Appendix B

Proof of Lemma 2.1. Call

$$g(x,t)=\frac{\partial}{\partial t}\int_{-\infty}^{x} dy\, D(y,t)$$

where $D(y,t)$ is defined in Equation (2.6).

We start proving that for any fixed t_0 and N there exists a number $T_N(\varepsilon)$ s.t. when $t_1-t_0<T_N(\varepsilon)$ the following inequality holds

$$\left|\int_{-N}^{+N} dx\left|\int_{t_0}^{t_1} dt\, g(x,t)\right| - \int_{-N}^{+N} dx\int_{t_0}^{t_1} dt\,|g(x,t)|\right| < \varepsilon/2(t_1-t_0).$$

In fact

$$\left|\int_{-N}^{+N} dx\left\{\left|\int_{t_0}^{t_1} dt\, g(x,t)\right| - \int_{t_0}^{t_1} dt\,|g(x,t_0)|\right\}\right| \leqq 2NK(N)(t_1-t_0)^2/2,$$

$$\left|\int_{-N}^{+N} dx\left\{\int_{t_0}^{t_1} dt\,|g(x,t)| - \int_{t_0}^{t_1} dt\,|g(x,t_0)|\right\}\right| \leqq 2NK(N)(t_1-t_0)^2/2$$

where

$$K(N)=\sup_{\substack{t_0\leqq t\leqq t_0+\tau\\ -N\leqq x\leqq N}}\left|\frac{\partial}{\partial t}g(x,t)\right|$$

and τ is some positive fixed number. Therefore we can put

$$T_N(\varepsilon)=\min\{(4NK(N))^{-1}\varepsilon, \tau\}.$$

To complete the proof it is sufficient to show that for any $\varepsilon>0$ there exists a $N(\varepsilon)$ s.t. all contributions of the form

$$\int_N^\infty dx\,|g(x,t)|<\varepsilon/8$$

uniformly in t for $t_0\leqq t\leqq t_0+\tau$.

If we notice (Section 2) that

$$\int_{-\infty}^{+\infty} dx\,|g(x,t)|=\int_{-\infty}^{+\infty} dx(x^2-x\langle x\rangle_t)D(x,t)$$

where

$$\langle x\rangle_t=\int_{-\infty}^{+\infty} dx\,xD(x,t)$$

it is easy to convience ourselves that $t_0\leqq t\leqq t_0+\tau$ there exists a $G(x)$ (independent on t!) s.t.

$$\int_{-\infty}^{+\infty} dx|g(x,t)|\leqq\int_{-\infty}^{+\infty} dx\,G(x)<+\infty.$$

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