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Induced Weak Convergence and Random Measures

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§1. Introduction

Let $(\mathscr{X}, \mathscr{T})$ be an arbitrary topological space, with Borel σ-algebra $\mathscr{B}$ (generated by $\mathscr{T}$). $\mathscr{M}^+(\mathscr{X})$ denotes the space of all non-negative finite Borel measures on $\mathscr{X}$. Equip $\mathscr{M}^+(\mathscr{X})$ with its *weak topology*, defined [16] as the weakest topology making the map $\mu \to \mu(\mathscr{X})$ continuous and each of the maps $\mu \to \mu(G)$, where $G \in \mathscr{T}$, lower semicontinuous. Convergence in this topology is denoted by $\Rightarrow$. When $\mathscr{T}$ is metrisable this is just the usual topology of weak convergence used, for example, by Billingsley [2].

Suppose $(\mathscr{X}_i, \mathscr{T}_i)_{i \in I}$ is a family of topological spaces (with Borel σ-algebras $\mathscr{B}_i$), and that Borel measurable maps T_i: $\mathscr{X} \to \mathscr{X}_i$ are given (i.e. $T_i^{-1} \mathscr{B}_i \subseteq \mathscr{B}$). We seek conditions under which the weak convergence $\mu_\alpha \Rightarrow \mu$ in $\mathscr{M}^+(\mathscr{X})$ is implied by the weak convergence of the induced measures $\mu_\alpha^i \Rightarrow \mu^i$ in $\mathscr{M}^+(\mathscr{X}_i)$ for each i, where for any $v \in \mathscr{M}^+(\mathscr{X})$ we write v^i for the measure in $\mathscr{M}^+(\mathscr{X}_i)$ defined by $v^i(A) = v(T_i^{-1} A)$. There are a number of applications for a result of this form. For example, in Section 4 it will be shown that weak convergence of measures on the function space $D[0, \infty)$ follows from convergence of the measures induced on the spaces $D[0, t]$. As with all the applications which we shall cite, this is a well-known result; our aim is merely to point out, and exploit, the simple underlying structure which they have in common.

In Section 2 we begin with the simplest case: if the maps T_i are continuous and if they generate the topology on $\mathscr{X}$, then we give some easily checkable conditions which ensure that $\mu_\alpha^i \Rightarrow \mu^i$ for each i implies $\mu_\alpha \Rightarrow \mu$. This is applied to rederive results for weak convergence on the spaces $\mathbb{R}^\infty$ and $C[0, \infty)$, and in Section 3 we examine the more interesting case of weak convergence of random measures on a locally compact space. An extension to non-continuous T_i's is proved in Section 4.

§2. Weak Convergence Induced by Continuous Maps

To prove our first result on induced weak convergence we shall need to make two assumptions. First we need the target measure μ to be τ-additive: *for every* *downward filtering family* $\{F_\beta\}$ *of closed sets* $\mu(\bigcap_\beta F_\beta)=\inf_\beta \mu(F_\beta)$. See [16] for some of the nice properties of the space $\mathcal{M}^+(\mathcal{X},\tau)$ of all τ-additive members of $\mathcal{M}^+(\mathcal{X})$. [If $\mathcal{X}$ is a separable metric space or, more generally, if there is a countable base to the topology $\mathcal{T}$, then it is easy to show that *every* member of $\mathcal{M}^+(\mathcal{X})$ is τ-additive.] The other assumption constitutes a slight extension of the concept of a family of maps generating a topology. The family $\{T_i: i\in I\}$ is said to $\cup f$*-generate* the topology $\mathcal{T}$ if the class $\{T_i^{-1}G: G\in\mathcal{T}_i, i\in I\}$ forms a basis for $\mathcal{T}$ *which is closed under finite unions.*

If the T_i's only generate the topology in the usual sense, a $\cup f$-generating family can be obtained by forming the larger family of maps T_J, from $\mathcal{X}$ into $\mathcal{X}_J=\prod_{i\in J}\mathcal{X}_i$, defined by taking the i-th coordinate of $T_J x$ to be $T_i x$. Here J runs through the family $\not{f}I$ of all finite subsets of I. The induced measure $\mu\cdot T_J^{-1}$ will be denoted by μ^J.

Theorem 1. *Suppose the family* $\{T_i: i\in I\}$ *of continuous maps* $\cup f$*-generates the topology* $\mathcal{T}$ *on* $\mathcal{X}$ *and that the target measure* μ *is* τ*-additive. Then* $\mu_\alpha\Rightarrow\mu$ *iff* $\mu_\alpha^i\Rightarrow\mu^i$ *for every* $i\in I$.

Proof. The proof of necessity requires but a simple modification of the well-known continuous mapping theorem [2].

If $\mu_\alpha^i\Rightarrow\mu^i$ for every $i\in I$, then it is clear that $\mu_\alpha(\mathcal{X})\to\mu(\mathcal{X})$. Thus it suffices to prove that $\liminf_\alpha \mu_\alpha(H)\geqq\mu(H)$ for every $H\in\mathcal{T}$. By definition of the $\cup f$-generating property, the family of open subsets of H of the form $T_i^{-1}G$, where $G\in\mathcal{T}_i$ for some i, filters up to H. Use τ-additivity of μ to find such a set for which $\mu(H)<\mu(T_i^{-1}G)+\varepsilon$. Then

$$\begin{aligned}\liminf_\alpha \mu_\alpha(H) &\geqq \liminf_\alpha \mu_\alpha(T_i^{-1}G)\\ &=\liminf_\alpha \mu_\alpha^i(G)\\ &\geqq \mu^i(G) \quad \text{since } \mu_\alpha^i\Rightarrow\mu^i\\ &>\mu(H)-\varepsilon. \quad \square\end{aligned}$$

Corollary 1. If the family $\{T_i: i\in I\}$ of continuous maps generates the topology $\mathcal{T}$ (but not necessarily $\cup f$-generates it), and if μ is τ-additive, then $\mu_\alpha\Rightarrow\mu$ iff $\mu_\alpha^J\Rightarrow\mu^J$ for every $J\in\not{f}I$.

Proof. $\{T_J: J\in\not{f}I\}$ satisfies the conditions of the theorem. □

Corollary 2. *If* $\{T_i: i\in I\}$ *is a family of continuous maps* $\cup f$*-generating (generating) the topology on* $\mathcal{X}$ *then the family of maps* $\{\mu\to\mu^i: i\in I\}$ $(\{\mu\to\mu^J: J\in\not{f}I\})$ *generates the weak topology on* $\mathcal{M}^+(\mathcal{X},\tau)$. □

Example 1 (cf. [2, p. 19]). Let $\mathcal{X}=\mathbb{R}^\infty$ (a countable product of real lines), equipped with the product topology. Since $\mathcal{X}$ is separable and metrisable, $\mathcal{M}^+(\mathcal{X})=\mathcal{M}^+(\mathcal{X},\tau)$. Let T_i denote the projection map of $\mathcal{X}$ onto the i-th coordinate space. If, for a net $\{\mu_\alpha\}$ in $\mathcal{M}^+(\mathcal{X})$, $\{\mu_\alpha^J\}$ converges weakly to some $\mu^J\in\mathcal{M}^+(\mathbb{R}^J)$ for each finite $J\subseteq\{1,2,\ldots\}$ then the μ^J's can easily be shown to form a consistent family of distributions. By the Kolmogorov extension theorem, there exists a $\mu\in\mathcal{M}^+(\mathcal{X})$

having these μ^J's as its finite dimensional distributions. It follows immediately from Theorem 1 that $\mu_\alpha \Rightarrow \mu$. A similar analysis can be given for more general product spaces. □

Example 2 (cf. [18]). Let $C[0, \infty)$ be the space of continuous real functions on $[0, \infty)$, equipped with the topology of uniform convergence on compacta. This is a separable metrisable topology. For every $t \in (0, \infty)$ define the continuous map T_t from $C[0, \infty)$ onto $C[0, t]$ by: $T_t(x)$ = the restriction of x to $[0, t]$. It is easy to verify that the family $\{T_t: 0 < t < \infty\}$, or even $\{T_{t_n}: \{t_n\}$ is some sequence of real numbers $\uparrow \infty\}$, satisfies the conditions of Theorem 1. So weak convergence of measures on $C[0, \infty)$ is equivalent to convergence of the measures which they induce on the spaces $C[0, t]$, where $0 < t < \infty$ or even $t \in \{t_1, t_2, \ldots\}$. □

§ 3. Random Measures

In this section we show that weak convergence of random measures on a locally compact space is equivalent to convergence of the so-called finite dimensional distributions (fidis). When the underlying space is second countable further simplifications can be achieved by invoking an existence theorem for random measures (Theorem 3); the argument is similar to that of Example 1. Some comparison is made with the alternative approaches which are to be found in the literature.

Let $\mathscr{S}$ be a locally compact Hausdorff space. Take $\mathscr{X}$ as the space of non-negative Radon measures on $\mathscr{S}$, i.e. the non-negative linear functionals on $\mathscr{C}$ (= the continuous real functions on $\mathscr{S}$ having compact support). Equip $\mathscr{X}$ with its *vague topology* $\mathscr{T}$, the weak topology induced by the maps $x \to x(g)$, $g \in \mathscr{C}$, thus making $(\mathscr{X}, \mathscr{T})$ a completely regular Hausdorff space. A *random measure* on $\mathscr{S}$ is defined to be a *tight* (= inner regular with respect to compact sets) *Borel probability measure on* $\mathscr{X}$. The space $\mathscr{Q}$ of all such random measures is a subset of $\mathscr{M}^+(\mathscr{X}, \tau)$ from which it inherits the relativised topology of weak convergence. This topology on $\mathscr{Q}$ is the same as that employed in [7] and [9] (see [16, p. 40]).

For any finite subset $\Gamma = \{g_1, \ldots, g_k\}$ of $\mathscr{C}$ write T_Γ for the map from $\mathscr{X}$ into $\mathbb{R}^\Gamma$ such that $T_\Gamma(x)$ has g_i-th coordinate $x(g_i)$. Also, if $\Gamma_2 \subseteq \Gamma_1$, write $T_{\Gamma_1 \Gamma_2}$ for the canonical projection of $\mathbb{R}^{\Gamma_1}$ onto $\mathbb{R}^{\Gamma_2}$. Notice that each T_Γ, $T_{\Gamma_1 \Gamma_2}$ is continuous and that the family of all such T_Γ's $\cup f$-generates the vague topology on $\mathscr{X}$. Thus the following result is an immediate consequence of Theorem 1. We write P^Γ for the induced measure $P \cdot T_\Gamma^{-1}$; the class of all such P^Γ's constitutes the *finite dimensional distributions* of the random measure P.

Theorem 2. *A net $\{P_\alpha\}$ of random measures converges weakly to $P \in \mathscr{Q}$ iff $P_\alpha^\Gamma \Rightarrow P^\Gamma$ for every finite $\Gamma \subseteq \mathscr{C}$.* □

Now let us assume that $\mathscr{S}$ is second countable, i.e. there is a countable base for the topology on $\mathscr{S}$. This places us in the setting adopted by Jagers [7] and Kallenberg [9] amongst others. It can be shown that under this extra assumption each of the spaces $\mathscr{S}$, $\mathscr{X}$ and $\mathscr{Q}$ is Polish (cf. [1, pp. 224, 241]). By using these separability properties we can strengthen Theorem 2; it is no longer necessary for the target random measure P to be given a priori. It can actually be constructed from the limiting fidi measures by means of the following existence theorem. The construction is based on a method of Le Cam [12] and Prohorov [14]. We write $\mathscr{C}^+$ for $\{g\in\mathscr{C}: g\geqq 0\}$.

Theorem 3. *Given Borel probabilities P^Γ on $[0,\infty)^\Gamma$, for Γ a finite subset of $\mathscr{C}^+$, there is a uniquely determined random measure P having these as fidis iff:*

(i) *$\Gamma_2\subseteq\Gamma_1$ implies that $P^{\Gamma_2}=P^{\Gamma_1}\cdot T_{\Gamma_1\Gamma_2}^{-1}$;*

(ii) *if $\Gamma=\{g_1,g_2,g_1+g_2\}$ then P^Γ is concentrated on the closed set $\{\psi\in[0,\infty)^\Gamma: \psi(g_1+g_2)=\psi(g_1)+\psi(g_2)\}$.*

Proof. Clearly the two conditions are necessary.

Second countability of $\mathscr{S}$ implies the existence of a countable subset of $\mathscr{C}$ which is dense with respect to the topology of uniform convergence ([1, p. 224]). Without loss of generality it can be supposed that this subset forms a vector lattice over the field of rational numbers. Let $\mathscr{D}$ be its positive cone, and $\mathscr{Y}$ the set of $[0,\infty)$-valued functions on $\mathscr{D}$ which satisfy the additivity condition: $y(g_1+g_2)=y(g_1)+y(g_2)$ for $g_1, g_2\in\mathscr{D}$. Equip $\mathscr{Y}$ with the weakest topology making each of the maps $y\to y(g)$ continuous, $g\in\mathscr{D}$. It is routine to show that the obvious correspondence between $\mathscr{Y}$ and $\mathscr{X}$ establishes a homeomorphism between the two spaces.

From (i) and the Kolmogorov extension theorem, there exists a tight Borel probability measure P_0 on $[0,\infty)^{\mathscr{D}}$ having the required finite dimensional distributions P^Γ for $\Gamma\subseteq\mathscr{D}$. [Notice that the product σ-algebra coincides with the Borel σ-algebra of $[0,\infty)^{\mathscr{D}}$, since $\mathscr{D}$ is countable.]

$\mathscr{Y}$ is a topological subspace of $[0,\infty)^{\mathscr{D}}$, being a countable intersection of closed cylinder sets of the form $\{\psi\in[0,\infty)^{\mathscr{D}}: \psi(g_1+g_2)=\psi(g_1)+\psi(g_2)\}$ where $g_1, g_2\in\mathscr{D}$. From (ii) each of these closed cylinder sets has P_0 measure one, hence $P_0(\mathscr{Y})=1$ also. The required random measure P is obtained by transferring P_0 from $\mathscr{Y}$ to the homeomorphic space $\mathscr{X}$.

This P has the desired fidis for $\Gamma\subseteq\mathscr{D}$. A simple approximation argument can now be used to prove that not only is P uniquely determined, but also that it has the required fidis for every $\Gamma\subseteq\mathscr{C}^+$. [Use the tightness of P and the fact that the maps $x\to x(g)$, $g\in\mathscr{D}$, generate the topology on $\mathscr{X}$.] □

Random measures can also be constructed from the family of setwise fidis $P^{A_1,\ldots,A_k}$ induced by the measurable maps $x\to[x(A_1),\ldots,x(A_k)]$, where the sets $A_1,\ldots,A_k$ run through a suitably large class of bounded Borel subsets of $\mathscr{S}$. Such a method has been employed by Jagers [7] for the locally compact case, Harris [5, 6] and Kerstan, Matthes & Mecke [11] for Polish spaces, and Jiřina [8] in an abstract setting. With our method, a random non-negative linear functional on $\mathscr{C}$ (a Radon measure) is constructed; this corresponds to a random countable additive measure on $\mathscr{S}$ since Radon measures automatically possess the countable additivity property. When working from the setwise fidis it is also straightforward to construct a random *finitely* additive measure, but the countable additivity is no longer automatic. An extra condition is needed to convert finite additivity to countable additivity. The appropriate property – inner approximation by a compact system of sets – is guaranteed by a continuity condition on the setwise fidis [7, p. 193, (2)]. Kallenberg [9] has given yet another existence proof for random measures on a locally compact space, based on a number of preliminary results on weak convergence; but his method is not as direct as ours.

With our existence theorem we can now improve upon Theorem 2. We write P_α^g instead of $P_\alpha^{\{g\}}$.

Theorem 4. *Let $\{P_\alpha\}$ be a net of random measures on a second countable locally compact Hausdorff space $\mathscr{S}$. Then $\{P_\alpha\}$ is weakly convergent in $\mathscr{Q}$ iff $\{P_\alpha^g\}$ converges to a probability measure on $[0,\infty)$ for each $g\in\mathscr{C}^+$. [Notice that no target random measure is specified.]*

Proof (cf. [11, p. 233]). In view of Theorem 2 it suffices to find a $P\in\mathscr{Q}$ such that $P_\alpha^\Gamma\Rightarrow P^\Gamma$ for every $\Gamma\subseteq\mathscr{C}^+$. For if $\Gamma=\{g_1,\dots,g_k\}$ and if $P_\alpha^{\Gamma^*}\Rightarrow P^{\Gamma^*}$ where $\Gamma^*=\{g_1^+,g_1^-,\dots,g_k^+,g_k^-\}$ then the continuity of $T_{\Gamma^*\Gamma}$ implies that $P_\alpha^\Gamma\Rightarrow P^\Gamma$. So choose $\Gamma=\{g_1,\dots,g_k\}\subseteq\mathscr{C}^+$. Then by considering $\{P_\alpha^g\}$ for $g=s_1g_1+\dots+s_kg_k, s_i\geqq 0$, we see that the Laplace transforms of $\{P_\alpha^\Gamma\}$ converge pointwise to a function which is continuous along each ray $\{s\mathbf{e}: s\geqq 0\}$, where $\mathbf{e}\geqq\mathbf{0}$. The multivariate version of the continuity theorem for Laplace transforms [4, p. 431] is thus applicable (even for nets, unlike the corresponding pointwise convergence result for characteristic functions), and it follows that there exists a probability measure P^Γ such that $P_\alpha^\Gamma\Rightarrow P^\Gamma$ for each such Γ. These P^Γ's satisfy the conditions of Theorem 3: if F represents any of the closed sets in condition (ii), $P_\alpha^\Gamma\Rightarrow P^\Gamma$ implies that $P^\Gamma(F)\geqq\limsup_\alpha P_\alpha^\Gamma(F)=1$; and the continuity of $T_{\Gamma_1\Gamma_2}$ takes care of (i). Thus they are the fidis of some $P\in\mathscr{Q}$. It follows that $P_\alpha\Rightarrow P$. □

Various versions of this result are to be found in the literature: see for example [14] (general compact spaces), [17] (σ-compact, locally compact spaces), [6] and [11] (Polish spaces), and [7] and [9] (second countable locally compact spaces). In [6] the problem was transformed to one involving $\mathbb{R}^\infty$ where the result described in our Example 1 could be used, while an argument similar to that which we employed in Theorem 1 was given in [11, pp. 231–232]. The other authors proceeded by demonstrating that tightness (or relative compactness) of a family of random measures is equivalent to the same property for the families of fidis. In Theorem 5 below we show that this result is itself a simple consequence of our Theorems 2 and 3.

It is also possible to characterise weak convergence of random measures in terms of the setwise fidis. But since the maps $x\to x(A)$, A a bounded Borel subset of $\mathscr{S}$, are not necessarily continuous an extra condition such as $P\{x: x(\partial A)>0\}=0$ is required. Here ∂A denotes the boundary of A. Notice that the target random measure P must therefore be specified in advance. It is interesting to note that this result for random measures on Polish spaces was proved in [11] by showing that convergence of suitably many of the setwise fidis implies convergence of the fidis induced by certain continuous maps; arguments like those in our Theorem 1 can then be used. The alternative methods of [7] and [9] (based on deducing relative compactness of $\{P_n\}$ from convergence of the setwise fidis) can be viewed, in the light of our next theorem, as an example of proof by the standard "relative compactness plus convergence on a separating class" type of argument (cf. [3, p. 165] – in this case the separating class consists of almost surely continuous functions).

Theorem 5. *Let $\mathscr{S}$ be second countable. Then $\mathscr{P} \subseteq \mathscr{Q}$ is relatively compact in $\mathscr{Q}$ iff $\mathscr{P}^{\Gamma}(=\{P^{\Gamma}: P \in \mathscr{P}\})$ is relatively compact in $\mathscr{M}^{+}(\mathbb{R}^{\Gamma})$ for every finite $\Gamma \subseteq \mathscr{C}$.*

Proof. Suppose the closure $\overline{\mathscr{P}}$ of $\mathscr{P}$ in $\mathscr{Q}$ is compact. Then $\mathscr{P}^{\Gamma} \subseteq \overline{\mathscr{P}}^{\Gamma} =$ the continuous image of the compact set $\overline{\mathscr{P}}$ under the continuous map $P \to P^{\Gamma}$. Thus $\overline{\mathscr{P}}^{\Gamma}$ is a compact subset of $\mathscr{M}^{+}(\mathbb{R}^{\Gamma})$ containing $\mathscr{P}^{\Gamma}$, as required.

Conversely, we use a possibly little known result valid for any regular Hausdorff space E: $A \subseteq E$ is relatively compact in E iff every universal net $\{e_{\alpha}\}$ on A is convergent in E (see [10, pp. 81, 136] and [16, prelim 7]). If $\{P_{\alpha}\}$ is a universal net on $\mathscr{P}$, then the image net $\{P_{\alpha}^{\Gamma}\}$ is a universal net on $\mathscr{P}^{\Gamma}$, for each $\Gamma \in \not{f}\mathscr{C}$ [10, p. 81], which is therefore convergent in $\mathscr{M}^{+}(\mathbb{R}^{\Gamma})$ if $\mathscr{P}^{\Gamma}$ is relatively compact in $\mathscr{M}^{+}(\mathbb{R}^{\Gamma})$. Say $P_{\alpha}^{\Gamma} \Rightarrow P^{\Gamma} \in \mathscr{M}(\mathbb{R}^{\Gamma})$. Then the P^{Γ}'s are the fidis of a random measure, as in the proof of Theorem 4; and the result follows from Theorem 2. □

Notice that the relative compactness of $\mathscr{P}$ in $\mathscr{Q}$ is in fact equivalent to the relative compactness of $\mathscr{P}^{g}$ in $\mathscr{M}^{+}([0,\infty))$ for all $g \in \mathscr{C}^{+}$, by virtue of Theorem 4. This also follows directly from Theorem 5, by noting that $\{P: P \in \mathscr{P}\}$ is tight iff $\{P^{g}: P \in \mathscr{P}\}$ is tight, where $g = \sum_{g_i \in \Gamma} |g_i|$.

§4. Weak Convergence Induced by Non-Continuous Maps

In this section we extend Theorem 1 to cover the case of noncontinuous T_i's: it then suffices to have arbitrarily large pieces A_{ε} of the space for which the maps $T_i|_{A_{\varepsilon}}$ (= the restriction of T_i to the domain A_{ε}) are continuous in the relative topology on A_{ε}, and so that they $\cup f$-generate that topology. Notice that continuity of $T_i|_A$ is a weaker requirement than continuity of T_i at each point of A. We write μ_{*}, $\mu_{\alpha *}$ for the inner measures associated with μ and μ_{α}.

Theorem 6. *Given a net $\{\mu_{\alpha}\}$ in $\mathscr{M}^{+}(\mathscr{X})$ and a τ-additive target measure μ, if for every $\varepsilon > 0$ there is an $A_{\varepsilon} \subseteq \mathscr{X}$ such that:*

(i) *$\mu_{*}(\mathscr{X} \setminus A_{\varepsilon}) < \varepsilon$ and $\limsup_{\alpha} \mu_{\alpha *}(\mathscr{X} \setminus A_{\varepsilon}) < \varepsilon$,*

(ii) *for every $i \in I$, $T_i|_{A_{\varepsilon}}$ is continous in the relative topology on A_{ε},*

(iii) *$\{T_i|_{A_{\varepsilon}}: i \in I\}$ $\cup f$-generates the relative topology on A_{ε},*

then $\mu_{\alpha}^{i} \Rightarrow \mu^{i}$ for every $i \in I$ implies that $\mu_{\alpha} \Rightarrow \mu$.

Proof. The argument is conceptually the same as for Theorem 1. Trivially $\mu_{\alpha}(\mathscr{X}) \to \mu(\mathscr{X})$, thus we have only to prove that $\liminf_{\alpha} \mu_{\alpha}(H) \geqq \mu(H)$ for each $H \in \mathscr{T}$. For such an H, $A_{\varepsilon} \cap H$ is a relatively open subset of A_{ε} and hence it is the union of an upward filtering (by (iii)) family $\{H_{\beta}\}$ of relatively open subsets of A_{ε} of the form $T_i|_{A_{\varepsilon}}^{-1} G_{\beta} (= A_{\varepsilon} \cap T_i^{-1} G_{\beta})$, where $G_{\beta} \in \mathscr{T}_i$ for some i depending on β. Let $H_{\beta}^{*} = \operatorname{int}(A_{\varepsilon}^{c} \cup H_{\beta})$, so that $\{H_{\beta}^{*}\}$ is an upward filtering family in $\mathscr{T}$ for which $A_{\varepsilon} \cap H_{\beta}^{*} = H_{\beta}$. Since μ is τ-additive, there exists a β_0 such that $\mu(H_{\beta_0}^{*}) \geqq \mu(H^{*}) - \varepsilon$, where $H^{*} = \bigcup_{\beta} H_{\beta}^{*}$.

As $A_{\varepsilon} \cap H_{\beta_0}^{*} = H_{\beta_0} = A_{\varepsilon} \cap T_i^{-1} G_{\beta_0}$, the symmetric difference $T_i^{-1} G_{\beta_0} \Delta H_{\beta_0}^{*}$ is a Borel measurable subset of $\mathscr{X} \setminus A_{\varepsilon}$. Similarly $A_{\varepsilon} \cap H^{*} = \bigcup_{\beta} (A_{\varepsilon} \cap H_{\beta}^{*}) = A_{\varepsilon} \cap H$

implies that $H \Delta H^*$ is also a Borel measurable subset of $\mathscr{X} \setminus A_\varepsilon$. Thus from (i)

$$\begin{aligned}\liminf_\alpha \mu_\alpha(H) &\geqq \liminf_\alpha \mu_\alpha(H^*) - \varepsilon \\ &\geqq \liminf_\alpha \mu_\alpha(H^*_{\beta_0}) - \varepsilon \\ &\geqq \liminf_\alpha \mu_\alpha(T_i^{-1} G_{\beta_0}) - 2\varepsilon \\ &= \liminf_\alpha \mu^i_\alpha(G_{\beta_0}) - 2\varepsilon \\ &\geqq \mu^i(G_{\beta_0}) - 2\varepsilon\end{aligned}$$

since $\mu^i_\alpha \Rightarrow \mu^i$. Similarly

$$\begin{aligned}\mu^i(G_{\beta_0}) &= \mu(T_i^{-1} G_{\beta_0}) \\ &\geqq \mu(H^*_{\beta_0}) - \varepsilon \\ &\geqq \mu(H^*) - 2\varepsilon \\ &\geqq \mu(H) - 3\varepsilon\end{aligned}$$

and the result follows. □

As with Theorem 1, condition (iii) can be relaxed by requiring only that $\{T_i|_{A_\varepsilon} : i \in I\}$ generates the relative topology on A_ε in the usual sense, but then we would need $\mu^J_\alpha \Rightarrow \mu^J$ for every $J \in \not{I}$ in order to deduce that $\mu_\alpha \Rightarrow \mu$.

We conclude with two illustrations of the use of Theorem 6.

Example 3 (cf. [13], [19]). Let $\mathscr{X} = D[0, \infty)$, the space of all real valued functions $x(\cdot)$ on $[0, \infty)$ which are right continuous and have left limits at every point. We equip $\mathscr{X}$ with Stone's [15] separable metric topology, for which x_n converges to x iff there are continuous strictly increasing maps λ_n of $[0, \infty)$ onto $[0, \infty)$ such that $x_n(\lambda_n(\cdot))$ converges uniformly to $x(\cdot)$, and $\lambda_n(t)$ converges uniformly to t, on the compact subsets of $[0, \infty)$.

For any probability $P \in \mathscr{M}^+(\mathscr{X})$ define C_P as the set of those $t \in [0, \infty)$ for which $P\{x: x \text{ is continuous at } t\} = 1$. It is easy to prove that $[0, \infty) \setminus C_P$ is denumerable and that the restriction map $T_t: x \to x|_{[0,t]}$ of $D[0, \infty)$ onto $D[0, t]$ is continuous P a.e. for every $t \in C_P$ [2, p. 124], [13], [19]. So for any sequence of probabilities $\{P_n\}$ in $\mathscr{M}^+(\mathscr{X})$ such that $P_n \Rightarrow P$, $P_n T_t^{-1} \Rightarrow P T_t^{-1}$ for every $t \in C_P$.

Conversely, if $P_n T_t^{-1} \Rightarrow P T_t^{-1}$ for every $t \in C_P$, then $P_n \Rightarrow P$. This results from an application of Theorem 6. For if C is any countable, unbounded subset of $(\bigcap_n C_{P_n}) \cap C_P$, then $A = \{x \in D[0, \infty): x \text{ is continuous at each } t \in C\}$ is a Borel measurable subset of $D[0, \infty)$ for which $P(A) = P_n(A) = 1$ for $n = 1, 2, \ldots$. Also straight from the definition of Stone's topology (cf. Theorem 2.1 of [19]) it follows that the family $\{T_t|_A : t \in C\}$ generates the topology on A, and in fact $\cup f$-generates it: if $t_1 < t_2$ both belong to C then the restriction map $T_{t_2 t_1}: z \to z|_{[0,t_1]}$ is a continuous map from $D[0, t_2] \cap T_{t_2}(A)$ onto $D[0, t_1] \cap T_{t_1}(A)$, satisfying $T_{t_1}|_A = T_{t_2 t_1} \circ T_{t_2}|_A$. □

Example 4 (cf. [2, p. 54]). Let $\mathscr{X} = C[0, 1]$, equipped with the uniform topology. For every finite $S = \{s_1, \ldots, s_k\}$, where $0 \leqq s < \cdots < s_k \leqq 1$, the fidi projection T_S is defined on $\mathscr{X}$ by $T_S(x)=(x(s_1), \ldots, x(s_k)) \in \mathbb{R}^k$. Suppose $\{P, P_n: n=1,2,\ldots\}$ is uniformly tight [2, p. 37] and that $P_n T_S^{-1} \Rightarrow P T_S^{-1}$ for every T_S. Uniform tightness means that for every $\varepsilon>0$ there exists a compact K_ε such that $P(\mathscr{X} \setminus K_\varepsilon)<\varepsilon$, $P_n(\mathscr{X} \setminus K_\varepsilon)<\varepsilon$, $n=1,2,\ldots$. Each of the maps T_S is continuous on $\mathscr{X}$ and the family of $T_S|_{K_\varepsilon}$'s $\cup f$-generates the uniform topology on K_ε *since pointwise and uniform convergence are equivalent on the compact subsets of* $C[0,1]$ (the pointwise topology on K_ε is a Hausdorff topology weaker than the uniform topology, therefore these two topologies must coincide on K_ε [10, p. 141]). From Theorem 6, $P_n \Rightarrow P$. □

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