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Pettis Mean Convergence of Vector-valued Asymptotic Martingales

J.J. Uhl, Jr.*

Department of Mathematics, University of Illinois, Urbana, Ill. 61801, USA

Austin, Edgar and Tulcea [1] have shown that an L_1-bounded asymptotic martingale of scalar-valued functions converges almost everywhere. Consequently an L_1-bounded uniformly integrable martingale of scalar-valued functions converges in L_1 norm as well. This is no longer the case for asymptotic martingales of Bochner integrable functions. Indeed, Chacon and Sucheston [2] have shown by example that there is an asymptotic martingale of Bochner integrable functions on $[0,1]$ with values in a Hilbert space H that is $L_1([0,1],H)$-bounded and uniformly integrable but fails to converge in $L_1([0,1],H)$ norm. Chacon and Sucheston did show that such an asymptotic martingale necessarily converges weakly almost everywhere. We shall show that such an asymptotic martingale also converges in the norm of Pettis integrable functions. We shall also see that, roughly, the difference between the mean convergence properties of vector-valued martingales and vector-valued asymptotic martingales is the difference between Bochner integrals and Pettis integrals.

Let us briefly collect some terminology and notation. Throughout this note (Ω, Σ, μ) is a finite measure space and X is a Banach space. The space of all μ-Bochner integrable functions with values in X will be denoted by $L_1(\mu, X)$. The *Pettis norm* of a function f in $L_1(\mu, X)$ is defined to be $\sup\{\int_\Omega |x^* f|\, d\mu \colon x^* \in X^*, \|x^*\| \leqq 1\}$ where X^* is the dual of X. If (B_n) is an increasing sequence of sub-σ-fields of Σ and (f_n) is a sequence in $L_1(\mu, X)$ such that f_n is B_n-measurable for all n, then (f_n, B_n) is an asymptotic martingale if $\lim \int_\Omega f_\tau\, d\mu$ exists when the limit is taken over the directed set of all bounded stopping times τ.

The following lemma is basically a restatement of Chacon and Sucheston [2; Lemma 2].

Lemma 1. *Let (f_n, B_n) be an asymptotic martingale in $L_1(\mu, X)$. For each $\varepsilon > 0$ there is a bounded stopping time τ_1 such that if σ and τ are bounded stopping times*

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with $\tau_1 \leqq \tau \leqq \sigma$, *then*

$$\left\| \int_E f_\sigma \, d\mu - \int_E f_\tau \, d\mu \right\| < \varepsilon$$

for all $E \in B_\tau$ *where* B_τ *is the* σ*-field consisting of all finite unions of sets of the form* $E \cap \{s: \tau(s) = n\}$ *where* $E \in B_n$.

Proof. Let $\varepsilon > 0$ and pick a bounded stopping time τ_1 such that if $\sigma, \tau \geqq \tau_1$ then $\left\| \int_\Omega f_\sigma \, d\mu - \int_\Omega f_\tau \, d\mu \right\| < \varepsilon$. Fix $\sigma \geqq \tau \geqq \tau_1$ and suppose $E \in B_\tau$. Define new stopping times $\bar{\sigma}$ and $\bar{\tau}$ by $\bar{\sigma} = \sigma$ on E, $\bar{\tau} = \tau$ on E and $\bar{\sigma} = \bar{\tau} = n_1$ on $\Omega \setminus E$ where n_1 is a positive integer larger than $\max(\sigma, \tau)$. Then $\bar{\sigma} \geqq \bar{\tau} \geqq \tau \geqq \tau_1$ and

$$\left\| \int_E f_\sigma \, d\mu - \int_E f_\tau \, d\mu \right\| = \left\| \int_\Omega f_{\bar{\sigma}} \, d\mu - \int_\Omega f_{\bar{\tau}} \, d\mu \right\| < \varepsilon.$$

This completes the proof.

It follows immediately that if (f_n, B_n) is an asymptotic martingale in $L_1(\mu, X)$, then $\lim_n \int_E f_n \, d\mu = F(E)$ exists for every $E \in \bigcup_n B_n$. The vector measure F will be called the limit measure of (f_n, B_n). The next theorem shows that the range of the limit measure of an asymptotic martingale partially governs the convergence properties of the asymptotic martingale.

Theorem 2. *An asymptotic martingale in* $L_1(\mu, X)$ *is Cauchy in the Pettis norm if and only if its limit measure is* μ*-continuous and has a relatively norm compact range.*

Proof. The proof is based on the elementary observation that a sequence (g_n) in $L_1(\mu, X)$ is Cauchy in the Pettis norm if and only if the sequences of integrals $\int_E g_n \, d\mu$ are uniformly Cauchy in $E \in \Sigma$. This fact is, in turn, an easy consequence of the inequalities

$$\sup_{E \in \Sigma} \left| \int_E h \, d\mu \right| \leqq \int_\Omega |h| \, d\mu \leqq 4 \sup_{E \in \Sigma} \left| \int_E h \, d\mu \right|$$

for *scalar-valued* functions $h \in L_1(\mu)$.

Since the range of an indefinite Bochner integral is norm relatively compact [9], the only if part is a trivial consequence of the above observation.

The if part is a consequence of a theorem of Hoffman-Jørgensen's. Suppose (f_n, B_n) is an asymptotic martingale in $L_1(\mu, X)$ with limit measure F. If F is μ-continuous and has a norm relatively compact range, an appeal to [4, Theorem 9] produces for each $\varepsilon > 0$ a function f_ε of the form $f_\varepsilon = \sum_{i=1}^m x_i \chi_{E_i}$ where $x_i \in X$ and $E_i \in \bigcup_n B_n$ such that $\left\| F(E) - \int_E f_\varepsilon \, d\mu \right\| < \varepsilon$ for all $E \in \bigcup_n B_n$. We have to show that

$$\lim_{i,j} \left\| \int_E f_i \, d\mu - \int_E f_j \, d\mu \right\| = 0$$

uniformly in $E \in \Sigma$. For this it suffices to show that this limit is uniform in $E \in \bigcup_n B_n$ (since all the f_n's are measurable relative to sub-σ-fields of the field $\bigcup_n B_n$). By the triangle inequality and the definition of the limit measure, it suffices to show that

$$\lim_i \left\| \int_E f_i \, d\mu - F(E) \right\| = 0$$

uniformly in $E \in \bigcup_n B_n$. To this end, let $\varepsilon > 0$ be fixed and let $g = f_{(\varepsilon/4)}$. Then, for each fixed positive integer i,

$$\sup_{E \in \bigcup_n B_n} \left\| \int_E f_i \, d\mu - F(E) \right\| \leqq \sup_{E \in \bigcup_n B_n} \left\| \int_E f_i \, d\mu - \int_E g \, d\mu \right\| + \varepsilon/4,$$

by the definition of g and the triangle inequality. Next pick i_0 such that g is measurable relative to B_{i_0}. Then for $i \geqq i_0$ the above quantity is equal to

$$\sup_{E \in B_i} \left\| \int_E f_i \, d\mu - \int_E g \, d\mu \right\| + \varepsilon/4$$

since both f_i and g are B_i-measurable. Again, by the triangle inequality, this quantity is no greater than

$$\sup_{E \in B_i} \left\| \int_E f_i \, d\mu - F(E) \right\| + \sup_{E \in B_i} \left\| F(E) - \int_E g \, d\mu \right\| + \varepsilon/4$$
$$\leqq \sup_{E \in B_i} \left\| \int_E f_i \, d\mu - F(E) \right\| + \varepsilon/2 \qquad \text{for } i \geqq i_0,$$

by the definition of g. But now an appeal to Lemma 1 produces a positive integer $i_1 \geqq i_0$ such that the first term in the above sum is smaller than $\varepsilon/2$ for all $i \geqq i_1$. This completes the proof.

Unfortunately, if μ has a non-atomic set, the space of μ-Pettis integrable functions that are locally Bochner integrable is incomplete by a theorem found in Thomas [7]. On the other hand, if μ is purely atomic, then it is easily seen that the space of Pettis integrable functions is complete and that every μ-continuous vector measure has a relatively norm compact range. Thus, if μ is purely atomic, then a uniformly integrable[1] (not necessarily $L_1(\mu, X)$-bounded) asymptotic martingale converges in the Pettis norm to a locally Bochner integrable Pettis integrable function.

Special properties of the Banach space X can also force uniformly integrable $L_1(\mu, X)$ (not necessarily $L_1(\mu, X)$-bounded) asymptotic martingales to be Cauchy in the Pettis norm. For, if an asymptotic martingale is uniformly integrable, it is easily checked that its limit measure is continuous relative to the underlying measure. Hence, by [8], the range of the limit measure is relatively weakly compact. Thus, if X has the Schur property (every weakly convergent sequence is norm convergent (e.g. l_1)), then a uniformly integrable asymptotic martingale in $L_1(\mu, X)$ is Cauchy in the Pettis norm for every finite measure μ.

In general, convergence of asymptotic martingales in the Pettis norm is evidently related to Radon-Nikodym derivatives.

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1 An asymptotic martingale (f_n, B_n) is uniformly integrable if $\lim_{\mu(E) \to 0} \int_E f_n \, d\mu = 0$ uniformly in n

Corollary 3. *An asymptotic martingale in $L_1(\mu, X)$ converges in the Pettis norm to a member of $L_1(\mu, X)$ if and only if its limit measure has a Radon-Nikodym derivative in $L_1(\mu, X)$.*

Proof. The proof is an essentially trivial consequence of Theorem 2. If the limit measure has a Radon-Nikodym derivative in $L_1(\mu, X)$, then the limit measure is μ-continuous and has a norm relatively compact range [8]. By Theorem 2, it is Cauchy in the Pettis norm and it is a simple matter to prove that it converges to the Radon-Nikodym derivative of its limit measure.

Conversely, if an asymptotic martingale in $L_1(\mu, X)$ converges in the Pettis norm to a member of $L_1(\mu, X)$, the very definition of the limit measure implies that the limit measure has the Pettis norm limit as a Radon-Nikodym derivative. This completes the proof.

In some cases it is possible to use the results of [6] to see when the limit measure has a derivative (see [10]). For most purposes the following corollary should be enough.

Corollary 4. *Suppose X has the Radon-Nikodym property. Then an $L_1(\mu, X)$-bounded uniformly integrable asymptotic martingale in $L_1(\mu, X)$ converges in the Pettis norm to a member of $L_1(\mu, X)$.*

Proof. The uniform integrability assumption implies that the limit measure is μ-continuous. The boundedness assumption implies that the limit measure is of bounded variation (see Chacon and Sucheston [2]). The assumption that X has the Radon-Nikodym property then guarantees that the limit measure has a Radon-Nikodym derivative in $L_1(\mu, X)$. An appeal to Corollary 3 completes the proof.

All the above results remain true in the context of locally Bochner integrable Pettis integrable functions and can be deduced from the above results by showing that an asymptotic martingale of locally Bochner integrable Pettis integrable functions can be approximated in the Pettis norm by an asymptotic martingale of Bochner integrable functions. The details are omitted.

The following question is motivated in part by the asymptotic martingale constructed by Huff in his important paper [5]. If X lacks the Radon-Nikodym property, is there an $L_1([0,1], X)$-bounded uniformly integrable asymptotic martingale in $L_1([0,1], X)$ that is Cauchy in the Pettis norm but is not convergent? If X has the Schur property, the answer is yes. It can also be shown that the answer is yes for $X = L_1[0,1]$ or $X = c_0$.

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