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Several Stability Properties of the Class of Asymptotic Martingales

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Introduction

The notion of asymptotic martingale emerges as an important and useful concept in the last few years. It permits simple, elegant proofs for some of the fundamental a.e. convergence theorems and it provides a unified treatment for martingales, submartingales, supermartingales, quasimartingales (see [1, 7, 13, 4]). The present paper is divided as follows:

§1. A Variant of the Double Limit Lemma
§2. Stability Properties
§3. An Example

The main results of this article were announced in [3].

§1. A Variant of the Double Limit Lemma

We assume in this section that D is a set, T a directed set "filtering to the right" for $\leqq$, and $(D_t)_{t\in T}$ an increasing family of subsets of D, that is

$$s \leqq t \Rightarrow D_s \subset D_t.$$

The following lemma may be regarded as a variant of E.H. Moore's double limit lemma (see [12], p. 28):

Lemma 1. *For each $t\in T$, let $f_t\colon D_t \to R$. We assume that*

i) *The family $(f_t(a))_t$ converges in R to a limit, $f_\infty(a)$, for each $a \in D_\infty = \bigcup_{t\in T} D_t$, and the convergence is "uniform" on D_∞ in the sense that for each $\varepsilon > 0$ there is $t_0 \in T$ such that*

$$s\in T, \quad s \geqq t_0 \Rightarrow |f_s(a) - f_\infty(a)| \leqq \varepsilon \quad \text{for all } a \in D_s.$$

Then $\lim_{t\in T} (\sup_{D_t} f_t)$ and $\lim_{t\in T} (\inf_{D_t} f_t)$ exist in the extended real line $\bar{R}$ and equal $\sup_{D_\infty} f_\infty$ and $\inf_{D_\infty} f_\infty$, respectively.

The proof of Lemma 1 is elementary. However, instead of giving its proof, we shall derive it from a more general onesided version of this lemma, suggested by Aryeh Dvoretzky.

Consider the extended real line $\bar{R}=[-\infty,+\infty]$, the interval $I=[-1,1]$ and the bijection $\varphi: \bar{R}\to I$ given by

$$\varphi(x)=\frac{x}{1+|x|}, \qquad x\in\bar{R}$$

$(\varphi(+\infty)=1, \varphi(-\infty)=-1)$. Let $d(u,v)=|u-v|$ and let d' be the distance on $\bar{R}$ obtained by transporting d to $\bar{R}$ under φ, i.e.

$$d'(x,y)=|\varphi(x)-\varphi(y)|=\left|\frac{x}{1+|x|}-\frac{y}{1+|y|}\right| \quad \text{for } x\in\bar{R},\ y\in\bar{R}.$$

Clearly φ is an isometry of $(\bar{R},d')$ onto (I,d) and an order isomorphism, that is $x<y$ if and only if $\varphi(x)<\varphi(y)$. Below we shall *consider $\bar{R}$ equipped with the distance d'.*

Lemma 1 follows from Lemma 2 below by applying it first to the family (f_t) and then to the family $(-f_t)$. (Note that $\varphi\,|\,R$ is uniformly continuous as a mapping of (R,d) into (R,d) and therefore assumption i) in Lemma 1 implies assumption j) in Lemma 2 below.) In connection with Lemma 2 see also Proposition VII in [11].

Lemma 2 (one-sided version). *For each $t\in T$, let $f_t: D_t\to\bar{R}$. We assume that*

j) *For each $\varepsilon>0$ there exists $t_\varepsilon\in T$ such that*

$$t\geqq s\geqq t_\varepsilon \Rightarrow \frac{f_t(a)}{1+|f_t(a)|}\geqq\frac{f_s(a)}{1+|f_s(a)|}-\varepsilon \quad \textit{for all } a\in D_s.$$

Then:

(a) *$f_\infty(a)=\lim_t f_t(a)$ exists in $\bar{R}$ for every*

$$a\in D_\infty=\bigcup_{t\in T} D_t.$$

(b) *Setting*

$$M_t=\sup_{a\in D_t} f_t(a) \quad \textit{for } t\in T, \quad \textit{and} \quad M_\infty=\sup_{a\in D_\infty} f_\infty(a)$$

we have $\lim_{t\in T} M_t=M_\infty$.

Proof. Note first that for uniformly bounded f_t condition j) is equivalent with the following.

For each $\varepsilon>0$ there is $t'_\varepsilon\in T$ such that

$$t\geqq s\geqq t'_\varepsilon \Rightarrow f_t(a)\geqq f_s(a)-\varepsilon \quad \text{for all } a\in D_s. \tag{1}$$

It suffices then to prove the Lemma under the additional assumption that $|f_t|\leqq 1$ for all $t\in T$ (the general case follows from this one by composing with φ).

To prove (a), let $f_\infty(a)=\limsup_t f_t(a)$, for each $a\in D_\infty$. Given $a\in D_\infty$ and $\varepsilon>0$, there exists $s\in T$, $s\geqq t'_\varepsilon$ for which $a\in D_s$ and $f_s(a)\geqq f_\infty(a)-\varepsilon$, and then by (1)

$$f_t(a)\geqq f_\infty(a)-2\varepsilon \qquad \text{for all } t\geqq s,$$

whence

$$\liminf_t f_t(a)\geqq f_\infty(a)-2\varepsilon.$$

Since $\varepsilon>0$ was arbitrary, (a) is proved.

To prove (b) note first that by (1), we have

$$t\geqq s\geqq t'_\varepsilon \Rightarrow M_t\geqq M_s-\varepsilon.$$

The argument proving (a) shows then that $M=\lim_t M_t$ exists.

Now given $\varepsilon>0$, there exists $a\in D_\infty$ with $f_\infty(a)\geqq M_\infty-\varepsilon$. By (a) and (1) there exists $s\in T$, $s\geqq t'_\varepsilon$ with $a\in D_s$ such that

$$t\geqq s\Rightarrow f_t(a)\geqq f_s(a)-\varepsilon\geqq f_\infty(a)-2\varepsilon\geqq M_\infty-3\varepsilon.$$

This implies

$$M_t\geqq M_\infty-3\varepsilon, \qquad \text{for all } t\geqq s.$$

It follows that

$$M=\lim_t M_t\geqq M_\infty. \tag{2}$$

Again by (1) we have

$$s\geqq t'_\varepsilon\Rightarrow f_\infty(a)\geqq f_s(a)-\varepsilon \qquad \text{for all } a\in D_s,$$

whence

$$s\geqq t'_\varepsilon\Rightarrow M_\infty\geqq M_s-\varepsilon.$$

We deduce

$$M_\infty\geqq M. \tag{3}$$

Conclusion (b) follows from (2) and (3). This completes the proof of Lemma 2.

Application. Let Ω be a set. For any real-valued (finitely) additive set function v defined on a Boolean algebra Σ of subsets of Ω, we set:

$$v^+(\Omega)=\sup_{A\in\Sigma} v(A), \tag{4}$$

$$v^-(\Omega)=-\inf_{A\in\Sigma} v(A)=\sup_{A\in\Sigma}(-v(A)) \tag{5}$$

(in the case when v is bounded, $v=v^+-v^-$ is just the Jordan decomposition of v; see [12], p. 98).

We assume below that: $\mathscr{A}$ is a Boolean algebra of subsets of Ω, T a directed set "filtering to the right" for $\leqq$, and $(\mathscr{A}_t)_{t\in D}$ an increasing family of subalgebras of $\mathscr{A}$, that is:

$$s \leqq t \Rightarrow \mathscr{A}_s \subset \mathscr{A}_t.$$

Corollary 1. *For each $t \in T$, let μ_t: $\mathscr{A}_t \to R$ be a bounded additive set function. We assume that:*

j) *The family $(\mu_t(A))_t$ converges in R to a limit, $\mu_\infty(A)$, for each $A \in \mathscr{A}_\infty = \bigcup_{t \in T} \mathscr{A}_t$, and the convergence is "uniform" on $\mathscr{A}_\infty$ in the sense that for each $\varepsilon > 0$ there is $t_0 \in T$ such that:*

$$s \in T, \quad s \geqq t_0 \Rightarrow |\mu_s(A) - \mu_\infty(A)| \leqq \varepsilon \quad \textit{for all } A \in \mathscr{A}_s.$$

Then $\lim_{t \in T} \mu_t^+(\Omega)$ and $\lim_{t \in T} \mu_t^-(\Omega)$ exists in the extended real line and equals $\mu_\infty^+(\Omega)$ and $\mu_\infty^-(\Omega)$, respectively.

§ 2. Stability Properties

We assume from now on that $(\Omega, \mathscr{F}, P)$ is a *probability space*. Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ and let $(\mathscr{F}_n)_{n \in \mathbb{N}}$ be *an increasing sequence of sub-σ-algebras of $\mathscr{F}$*, i.e., if $n \leqq m$ then $\mathscr{F}_n \subset \mathscr{F}_m$. A *bounded stopping time* (with respect to the sequence $(\mathscr{F}_n)_{n \in \mathbb{N}}$) is a mapping τ: $\Omega \to \mathbb{N}$ such that $\{\omega \in \Omega \mid \tau(\omega) = n\} \in \mathscr{F}_n$ for all $n \in \mathbb{N}$ and τ assumes only finitely many values. Let T be the *set of all bounded stopping times*. With the definition $\tau \leqq \sigma$ if $\tau(\omega) \leqq \sigma(\omega)$ for all $\omega \in \Omega$, T is a directed set "filtering to the right" (note that if $\tau \in T$, $\sigma \in T$ then $\tau \vee \sigma \in T$, and $\tau \wedge \sigma \in T$). Recall that

$$\mathscr{F}_\tau = \{A \in \mathscr{F} \mid A \cap \{\tau = n\} \in \mathscr{F}_n \quad \text{for all } n \in \mathbb{N}\},$$

and that $\tau \leqq \sigma$ implies $\mathscr{F}_\tau \subset \mathscr{F}_\sigma$.

Let E be a *Banach space*. Let X_n: $\Omega \to E$ for each $n \in \mathbb{N}$. The sequence $(X_n)_{n \in \mathbb{N}}$ is called *adapted* if X_n: $\Omega \to E$ is Bochner $\mathscr{F}_n$-measurable for each $n \in \mathbb{N}$. For each $\tau \in T$ we denote by X_τ the random variable defined by

$$(X_\tau)(\omega) = X_{\tau(\omega)}(\omega), \qquad \text{for } \omega \in \Omega.$$

We now recall the definition of *asymptotic martingale*, which is basic for the remainder of this paper:

Definition 1. An adapted sequence $(X_n)_{n \in \mathbb{N}}$ of E-valued random variables is called an E-valued asymptotic martingale if X_n is Bochner integrable, i.e.

$$\int \|X_n(\omega)\| \, dP(\omega) < \infty$$

for each $n \in \mathbb{N}$, and

$(\int X_\tau)_{\tau \in T}$ converges in the norm topology of E.

Note that for each $\tau \in T$, X_τ is measurable with respect to $\mathscr{F}_\tau$.

The notion of asymptotic martingale is much stronger than it would appear at first glance from its definition. This is illustrated by Theorem 1 below. This theorem is known in one form or another (it is almost part of the folklore of asymptotic martingales by now). Because of its great simplicity and importance, we state it and prove it in complete detail:

Theorem 1. *Let E be a Banach space. Let $(X_n)_{n\in\mathbb{N}}$ be an E-valued asymptotic martingale. We have:*

1) $\sup_{\tau\in T} \|\int X_\tau\| < \infty$.

2) *For each $\tau\in T$ define*

$$\mu_\tau(A) = \int_A X_\tau, \quad \text{for } A\in\mathscr{F}_\tau.$$

Then

2a) *The family $(\mu_\tau(A))_\tau$ converges to a limit, $\mu_\infty(A)$, for each*

$$A\in\mathscr{F}_\infty = \bigcup_{\tau\in T}\mathscr{F}_\tau = \bigcup_{n\in\mathbb{N}}\mathscr{F}_n,$$

and the convergence is "uniform" on $\mathscr{F}_\infty$ in the sense that for each $\varepsilon>0$ there is $\tau(\varepsilon)\in T$ such that:

$$\sigma\in T, \quad \sigma\geqq\tau(\varepsilon)\Rightarrow \|\mu_\sigma(A)-\mu_\infty(A)\|\leqq\varepsilon \quad \text{for all } A\in\mathscr{F}_\sigma.$$

2b) *If in addition $\sup_{n\in\mathbb{N}}\int\|X_n\|<\infty$, then there is a constant $M\geqq 0$ such that*

$$\|\mu_\tau(A)\|\leqq M \quad \text{for each } \tau\in T \text{ and } A\in\mathscr{F}_\tau.$$

Proof. 1) By assumption there is $n'\in\mathbb{N}$ such that

$$\sigma\geqq n'\Rightarrow\|\int X_\sigma-\int X_{n'}\|\leqq 1. \tag{6}$$

For arbitrary $\tau\in T$ note that

$$X_\tau+X_{n'}=X_{\tau\vee n'}+X_{\tau\wedge n'}$$

whence

$$\begin{aligned}\|\int X_\tau\| &= \|\int X_{\tau\vee n'}+\int X_{\tau\wedge n'}-\int X_{n'}\|\\ &\leqq 1+\int(\sup_{1\leqq k\leqq n'}\|X_k\|).\end{aligned}$$

and hence 1) is proved.

2a) Since the net $(\int X_\tau)_{\tau\in T}$ is convergent in E, given $\varepsilon>0$, there is $\tau(\varepsilon)\in T$ such that:

$$\tau'\geqq\tau(\varepsilon), \quad \tau''\geqq\tau(\varepsilon)\Rightarrow\|\int X_{\tau'}-\int X_{\tau''}\|\leqq\varepsilon. \tag{7}$$

Let now $\tau\geqq\sigma\geqq\tau(\varepsilon)$ and $A\in\mathscr{F}_\sigma$. Choose $n\in\mathbb{N}$ such that $n\geqq\tau\geqq\sigma$ and define

$$\tau_1=\begin{cases}\tau & \text{on } A\\ n & \text{on } \Omega-A\end{cases}, \qquad \sigma_1=\begin{cases}\sigma & \text{on } A\\ n & \text{on } \Omega-A\end{cases};$$

then $\tau_1\in T$, $\sigma_1\in T$, $\tau_1\geqq\sigma_1\geqq\tau(\varepsilon)$ and

$$\mu_\tau(A)-\mu_\sigma(A)=\int X_{\tau_1}-\int X_{\sigma_1}$$

and hence, by (7) above

$$\|\mu_\tau(A)-\mu_\sigma(A)\| \leqq \varepsilon.$$

Thus we showed that for each $\varepsilon>0$, there is $\tau(\varepsilon)\in T$ such that

$$\tau \geqq \sigma \geqq \tau(\varepsilon) \Rightarrow \|\mu_\tau(A)-\mu_\sigma(A)\| \leqq \varepsilon, \quad \text{for each } A\in\mathscr{F}_\sigma. \tag{8}$$

Let now $A\in\mathscr{F}_\infty$. Then $A\in\mathscr{F}_p$ for some $p\in\mathbb{N}$. Now $\mu_\tau(A)$ is defined for all $\tau\geqq p$ and $(\mu_\tau(A))_{\tau\geqq p}$ is Cauchy in E, since for each $\varepsilon>0$, we have by (8), for $\sigma\geqq\tau(\varepsilon)\vee p$ and $\tau\geqq\tau(\varepsilon)\vee p$,

$$\|\mu_\tau(A)-\mu_\sigma(A)\| \leqq \|\mu_\tau(A)-\mu_{\tau(\varepsilon)\vee p}(A)\| + \|\mu_\sigma(A)-\mu_{\tau(\varepsilon)\vee p}(A)\| \leqq 2\varepsilon.$$

Hence the limit $\mu_\infty(A)$ exists in E, for every $A\in\mathscr{F}_\infty$. We may now pass to the limit with τ in (8) and thus statement 2a) is proved.

2b) Let $n'\in\mathbb{N}$ be as in relation (6).

For $\tau\geqq n'$ and $A\in\mathscr{F}_\tau$, define τ' by setting $\tau'=\tau$ on A and $\tau'=n$ on $\Omega-A$, for some integer $n\geqq\tau$. Then $\tau'\in T$, $\tau'\geqq n'$ and

$$\mu_\tau(A)=\int_\Omega X_{\tau'} - \int_{\Omega-A} X_n$$

whence using (6),

$$\begin{aligned}\|\mu_\tau(A)\| &\leqq \left\|\int X_{\tau'}\right\| + \int \|X_n\| \\ &\leqq \left(1+\left\|\int X_{n'}\right\|\right)+\int\|X_n\| \leqq 1+2\left\{\sup_{k\in\mathbb{N}}\int\|X_n\|\right\}.\end{aligned} \tag{9}$$

Now for an arbitrary $\sigma\in T$ and $B\in\mathscr{F}_\sigma$, note that $B\cap\{\sigma>n'\}\in\mathscr{F}_\sigma\subset\mathscr{F}_{\sigma\vee n'}$; we have

$$\begin{aligned}\int_B X_\sigma &= \int_{B\cap\{\sigma\leqq n'\}} X_\sigma + \int_{B\cap\{\sigma>n'\}} X_\sigma \\ &= \int_{B\cap\{\sigma\leqq n'\}} X_{\sigma\wedge n'} + \int_{B\cap\{\sigma>n'\}} X_{\sigma\vee n'}.\end{aligned}$$

Using the previous computation (9), we get

$$\left\|\int_B X_\sigma\right\| \leqq \int\left(\sup_{1\leqq j\leqq n'}\|X_j\|\right)+1+2\left\{\sup_{k\in\mathbb{N}}\int\|X_k\|\right\}.$$

With

$$M=\int\left(\sup_{1\leqq j\leqq n'}\|X_j\|\right)+1+2\left\{\sup_{k\in\mathbb{N}}\int\|X_k\|\right\},$$

statement 2b) of the theorem is proved. This completes the proof of Theorem 1.

Remarks. 1) The proof of the fact that if $(X_n)_{n\in\mathbb{N}}$ is an E-valued asymptotic martingale, then $(\int X_\tau)_{\tau\in T}$ is bounded in E is the same as in the real-valued case (see [13], Lemma 1.2).

2) Statement 2a) of Theorem 1 may be regarded as a refinement of Lemma 2 of [7]; the convergence of the net $(\mu_\tau(A))_\tau$ to a limit for each $A\in\bigcup_{n\in\mathbb{N}}\mathscr{F}_n$ was already

established there, as well as the fact that for each *fixed* $n \in \mathbb{N}$ the convergence is uniform on $\mathscr{F}_n$.

3) In connection with statement 2b) of Theorem 1 note that even though the ranges of the measures $\mu_\tau (\tau \in T)$ are uniformly bounded in E, their total variations need not be, that is $\sup_{\tau \in T} \int \|X_\tau\|$ need not be finite; see the Example at the end of Section 2 in [7].

4) The technique used in the proof of Theorem 1, namely changing a stopping time $\tau \in T$ on a set $A \in \mathscr{F}_\tau$, is a familiar one. It has been consistently used in [2, 1, 7], and [13]; whence the similarity in all these proofs.

The following result is now an immediate consequence of Theorem 1 and Corollary 1:

Corollary 2. *Let $(X_n)_{n \in \mathbb{N}}$ be a real-valued asymptotic martingale. Then*

a) $\lim_{\tau \in T} (\int X_\tau^+)$ *and* $\lim_{\tau \in T} (\int X_\tau^-)$ *exist in the extended real line.*

b) *Suppose in addition that* $\sup_{n \in \mathbb{N}} \int |X_n| < \infty$. *Then* $(X_n^+)_{n \in \mathbb{N}}$, $(X_n^-)_{n \in \mathbb{N}}$ *and hence also* $(|X_n|)_{n \in \mathbb{N}}$ *are asymptotic martingales.*

Proof. With the notation of Theorem 1, it is enough to note that for each $\tau \in T$, μ_τ is a bounded measure on $\mathscr{F}_\tau$, that X_τ is $\mathscr{F}_\tau$-measurable, and that

$$\mu_\tau^+(\Omega) = \int_\Omega X_\tau^+ \quad \text{and} \quad \mu_\tau^-(\Omega) = \int_\Omega X_\tau^- .$$

Remark. Statement b) of Corollary 2 is due to [1] (see Lemma 2). The generalization given in statement a) is due to A. Dvoretzky (see [11], Proposition VII).

Definition 2. We say that a family $(X_j)_{j \in J}$ of E-valued integrable random variables is L^1-bounded if $\sup_{j \in J} \int \|X_j\| < \infty$.

It is clear that the class of all L^1-bounded asymptotic martingales is a *linear space.* From Corollary 2 follows easily the "*lattice property*" for the class of real-valued L^1-bounded asymptotic martingales:

Corollary 3. *Let $(X_n)_{n \in \mathbb{N}}$ and $(Y_n)_{n \in \mathbb{N}}$ be real-valued L^1-bounded asymptotic martingales. Then $(X_n \vee Y_n)_{n \in \mathbb{N}}$ and $(X_n \wedge Y_n)_{n \in \mathbb{N}}$ are L^1-bounded asymptotic martingales. In particular, for each $a > 0$, $(-a \vee X_n \wedge a)_{n \in \mathbb{N}}$ is an L^1-bounded asymptotic martingale.*

Proof. It is enough to note that for each $n \in \mathbb{N}$

$$X_n \vee Y_n = \tfrac{1}{2}(X_n + Y_n + |X_n - Y_n|),$$
$$X_n \wedge Y_n = \tfrac{1}{2}(X_n + Y_n - |X_n - Y_n|).$$

The second statement of Corollary 3 follows from the first since a constant sequence is obviously an L^1-bounded asymptotic martingale. (See also [13].)

We now recall the fundamental a.e. convergence results for asymptotic martingales. We begin with the following elementary but **Basic Lemma** first proven in [1].

I) *Let $(X_n)_{n \in \mathbb{N}}$ be an adapted sequence of E-valued random variables. Let Y be a random variable measurable with respect to $\sigma(\bigcup_{n \in \mathbb{N}} \mathscr{F}_n)$, the σ-algebra spanned by*

$\bigcup_{n\in\mathbb{N}} \mathscr{F}_n$, *and such that for each* $\omega\in\Omega$, $Y(\omega)$ *is a cluster value of the sequence* $(X_n(\omega))_{n\in\mathbb{N}}$.
Then there exist $\tau_k\in T$, *with* $\tau_{k+1}\geqq\tau_k\geqq k$ *for all* k, *such that*

$$\lim_{k\in\mathbb{N}} X_{\tau_k}(\omega)=Y(\omega), \qquad \text{a.e.}$$

The proof is the same as in the real-valued case: see Lemma 1 of [1].

II) *Let* $(X_n)_{n\in\mathbb{N}}$ *be an adapted sequence of real-valued random variables, such that* $\sup_{n\in\mathbb{N}} |X_n|\in L^1$. *Then the following assertions are equivalent:*

(i) $(X_n)_{n\in\mathbb{N}}$ *converges to a limit* a.e.

(ii) $(X_n)_{n\in\mathbb{N}}$ *is an asymptotic martingale.*

Statement II) follows from I) above, as an easy application of the Lebesgue Dominated Convergence: note that $\limsup_{n\in\mathbb{N}} X_n$ and $\liminf_{n\in\mathbb{N}} X_n$ are random variables Y satisfying the assumptions in I) above (for details see [1], Corollary 1). The result of statement II) has by now become classical. In the case of uniformly bounded random variables (even for continuous parameter processes) it goes back to P.A. Meyer (see [20], p. 232), who apparently was the first to consider "stopping time directed convergence." See also W. Sudderth [26], J.F. Mertens [18], and J.R. Baxter [2].

III) *(The real-valued asymptotic martingale). Let* $(X_n)_{n\in\mathbb{N}}$ *be a real valued* L^1-*bounded asymptotic martingale. Then* $(X_n)_{n\in\mathbb{N}}$ *converges to a limit* a.e.

This basic a.e. convergence theorem for real-valued asymptotic martingales is due to D.G. Austin – G.A. Edgar – A. Ionescu Tulcea: Theorem 2 in [1]. See also [6], for a related result. It was J.R. Baxter's pretty paper [2] and his clever use of the stopping time technique that inspired us in [1]. The term "asymptotic martingale" was actually introduced later by R.V. Chacon and L. Sucheston, in their paper [7].[1]

Simpler, elegant proofs of Theorem III) are now available (see for instance [13]). We sketch below a very simple proof due to A. Dvoretzky (based on his idea of an $\bar{S}$-martingale; see Proposition VI in [11]):

Proof of III) (following Dvoretzky). For each $a>0$ denote by c^a the "truncation at a":

$$c^a(x)=-a\vee x\wedge a, \qquad \text{for } x\in\bar{R}=[-\infty,+\infty].$$

Note that a sequence $(x_n)_{n\in\mathbb{N}}$ of elements of the extended real line $\bar{R}$ converges in $\bar{R}$ if and only if $(c^p(x_n))_{n\in\mathbb{N}}$ converges in R, for each $p\in\mathbb{N}$. By Corollary 3, for each $p\in\mathbb{N}$, $(c^p(X_n))_{n\in\mathbb{N}}$ is an asymptotic martingale, and since $|c^p(X_n)|\leqq p$ for all $n\in\mathbb{N}$, the sequence $(c^p(X_n))_{n\in\mathbb{N}}$ converges to a limit a.e. by Theorem II) above. It follows that for a.e. $\omega\in\Omega$, the sequence $(X_n(\omega))_{n\in\mathbb{N}}$ converges in $\bar{R}$. The fact that the limit is finite a.e. follows from Fatou's Lemma.

See also [13] which contains the first systematic study of "amarts", both in the ascending and the descending case.

[1] The term "amart" is now also in use (see [13])

Remark. The fact that, in the real-valued case, the martingale, the submartingale, the supermartingale are examples of asymptotic martingales was shown in [1]. The fact that the quasi martingale is also an asymptotic martingale was shown in [13] (for the theory of quasimartingales see [14, 23], and [25]).

To simplify the terminology we make use of the following definition (see [7]):

Definition 3. We say that an E-valued asymptotic martingale $(X_n)_{n\in\mathbb{N}}$ is of class (B) if the family $(X_\tau)_{\tau\in T}$ is L^1-bounded (that is, $\sup_{\tau\in T}\int\|X_\tau\|<\infty$).

IV) *(The vector-valued asymptotic martingale). Assume that the Banach space E has the Radon-Nikodym property and that the dual E' is separable. Let $(X_n)_{n\in\mathbb{N}}$ be an E-valued asymptotic martingale of class* (B). *Then there exists an E-valued random variable X_∞ such that the sequence $(X_n(\omega))_{n\in\mathbb{N}}$ converges to $X_\infty(\omega)$ weakly in E, for* a.e. $\omega\in\Omega$.

This basic a.e. convergence theorem for vector-valued asymptotic martingales is due to R.V.Chacon-L.Sucheston: see their paper [7]. Examples were given in [7] for the case of the Banach space $E=\ell_p$ $(1<p<\infty)$, showing that strong convergence of the sequence $(X_n(\omega))_{n\in\mathbb{N}}$ to $X_\infty(\omega)$ for a.e. $\omega\in\Omega$ need not hold. This is all the more striking since in the case of *martingales*, strong convergence a.e. always obtains, as long as the Banach space has the Radon-Nikodym property.

The above considerations led to the following theorem recently proved in [4]:

V) *For a Banach space E the following assertions are equivalent:*

(i) *E is of finite dimension.*

(ii) *Every E-valued asymptotic martingale of class* (B) *converges to a limit strongly* a.e.

(iii) *Every E-valued asymptotic martingale $(X_n)_{n\in\mathbb{N}}$ such that $\|X_n(\omega)\|\leqq 1$ for all $n\in\mathbb{N}$ and $\omega\in\Omega$ converges to a limit strongly* a.e.

Theorem V) had been conjectured by Gilles Pisier and the question explicitly raised by Louis Sucheston at the San Antonio Meeting (January 1976).

The key tool in the proof is the Dvoretzky-Rogers Lemma – the beautiful lemma on which Dvoretzky and Rogers based the proof of their celebrated theorem on absolute and unconditional convergence in Banach spaces ([10]).

A similar method permits to show that *if E is a Banach space of infinite dimension, there exists an E-valued asymptotic martingale which is L^1-bounded, but is not of class* (B).

Theorems IV) and V) strikingly illustrate the difference between the vector-valued martingale and the vector-valued asymptotic martingale (for the vector-valued martingale see [16, 8, 22], and [7]).

Before giving the next stability property of the class of real-valued asymptotic martingales, we need some preliminary observations, which we shall state in the form of lemmas:

Lemma 3. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of real-valued random variables. Suppose that for each $0<\varepsilon<1$ we have $X_n=Y_n^{(\varepsilon)}+Z_n^{(\varepsilon)}$, where $(Y_n^{(\varepsilon)})_{n\in\mathbb{N}}$ is an L^1-bounded* *asymptotic martingale and* $(Z_n^{(\varepsilon)})_{n\in\mathbb{N}}$ *is an adapted sequence such that*

$$\int |Z_\tau^{(\varepsilon)}| \leqq \varepsilon \quad \text{for all } \tau \in T.$$

Then $(X_n)_{n\in\mathbb{N}}$ *is an* L^1*-bounded asymptotic martingale.*

Proof. The L^1-boundedness of $(X_n)_{n\in\mathbb{N}}$ is obvious, and one can easily check that $(\int X_\tau)_{\tau\in T}$ is a Cauchy net.

Below, when the sequence of σ-algebras is not explicitly mentioned, it is assumed that $(\mathscr{F}_n)_{n\in\mathbb{N}}$ is the "minimal" sequence, that is $\mathscr{F}_n=\sigma(X_1, \ldots, X_n)$ for each $n\in\mathbb{N}$.

Lemma 4. *Let* $\Omega=[0,1]$, $\mathscr{F}=$ *the* σ*-algebra of Borel sets and* $P=$ *Lebesgue measure. Let* $(a_n)_{n\in\mathbb{N}}$ *and* $(b_n)_{n\in\mathbb{N}}$ *be sequences of real numbers with* $b_n>1$, $b_n \nearrow +\infty$. *For each* $n\in\mathbb{N}$ *let*

$$X_n(\omega)=a_n 1_{\left[0,\frac{1}{b_n}\right)}(\omega), \quad \text{for } \omega\in\Omega.$$

Then $(X_n)_{n\in\mathbb{N}}$ *is* L^1*-bounded if and only if* $\left(\frac{a_n}{b_n}\right)_{n\in\mathbb{N}}$ *is a bounded sequence, and* $(X_n)_{n\in\mathbb{N}}$ *is an* L^1*-bounded asymptotic martingale if and only if* $\left(\frac{a_n}{b_n}\right)_{n\in\mathbb{N}}$ *is a convergent sequence.*

Proof. It suffices to note that

$$\int X_n=\frac{a_n}{b_n}, \qquad \int |X_n|=\frac{|a_n|}{b_n}$$

and that if $\tau\in T$ and $\tau(0)=k$, then $X_\tau=X_k$.

We may now state and prove the next stability theorem:

Theorem 2. *For a function* $G\colon R\to R$ *the following assertions are equivalent:*

(i) *The function G satisfies the conditions:*

(ia) $G\colon R\to R$ *is continuous:*

(ib) $\lim\limits_{x\to+\infty}\frac{G(x)}{x}$ *and* $\lim\limits_{x\to-\infty}\frac{G(x)}{x}$ *exist and are finite.*

(ii) *If* $(X_n)_{n\in\mathbb{N}}$ *is any real-valued* L^1*-bounded asymptotic martingale*[1], *then* $(G(X_n))_{n\in\mathbb{N}}$ *is an* L^1*-bounded asymptotic martingale.*

Proof. Let $\mathscr{G}$ be the set of all $G\colon R\to R$ satisfying ia) and ib).

(i) $\Rightarrow$ (ii). We claim that it is enough to prove:

(P) For any $G\in\mathscr{G}$ and any asymptotic martingale $(X_n)_{n\in\mathbb{N}}$ with $X_n\geqq 0$ for all $n\in N$, the sequence $(G(X_n))_{n\in\mathbb{N}}$ is an L^1-bounded asymptotic martingale.

In fact, assume (P) proved. Let $G\in\mathscr{G}$ and $(X_n)_{n\in\mathbb{N}}$ an *arbitrary* L^1-bounded asymptotic martingale. Let $G_1(x)=G(x)-G(0)$, for $x\in R$. Then $G_1\in\mathscr{G}$ and $G(X_n)=G_1(X_n)+G(0)$ for all $n\in\mathbb{N}$; thus $(G(X_n))_{n\in\mathbb{N}}$ is an L^1-bounded asymptotic martingale if and only if $(G_1(X_n))_{n\in\mathbb{N}}$ is. Now $(X_n^+)_{n\in\mathbb{N}}$ and $(X_n^-)_{n\in\mathbb{N}}$ are asymptotic martingales

[1] On any probability space

(Corollary 2b)) and

$$G_1(X_n)=G_1(X_n^+)+G_1(-X_n^-)=G_1(X_n^+)+G_1^*(X_n^-),$$

where $G_1^*(x)=G_1(-x)$, for $x\in R$, and clearly $G_1^*\in\mathcal{G}$. Thus $(G_1(X_n))_{n\in\mathbb{N}}$ is an L^1-bounded asymptotic martingale.

We shall now prove (P). Let $G\in\mathcal{G}$ and let $(X_n)_{n\in\mathbb{N}}$ be an asymptotic martingale with $X_n\geqq 0$ for all $n\in\mathbb{N}$. We may assume, without loss of generality that

$$G(0)=0 \quad\text{and}\quad \lim_{x\to+\infty}\frac{G(x)}{x}=0 \tag{10}$$

(otherwise we replace G by H, where

$$H(x)=G(x)-G(0)-\alpha x, \qquad \text{for } x\in R,\ \alpha=\lim_{x\to+\infty}\frac{G(x)}{x};$$

clearly $H\in\mathcal{G}$ and $H(X_n)=G(X_n)-G(0)-\alpha X_n$, so that $(G(X_n))_{n\in\mathbb{N}}$ is an L^1-bounded asymptotic martingale if and only if $(H(X_n))_{n\in\mathbb{N}}$ is).

By Theorem 1, part 1), there is $M>0$ such that

$$\int X_\tau\leqq M \qquad \text{for each } \tau\in T. \tag{11}$$

By Theorem III), there are a random variable X_∞ and a set A_∞, $P(A_\infty)=0$, such that $\lim_{n\in\mathbb{N}} X_n(\omega)=X_\infty(\omega)$ for each $\omega\notin A_\infty$.

Let now $0<\varepsilon<1$. By (10) there is $K>0$ such that

$$x\geqq K \Rightarrow |G(x)|\leqq\left(\frac{\varepsilon}{M}\right)x;$$

we shall also choose K so that if $B_\infty=\{X_\infty=K\}$ then $P(B_\infty)=0$.

We now define for each $n\in\mathbb{N}$

$$U_n=1_{\{X_n<K\}}X_n, \quad\text{and}\quad V_n=1_{\{X_n\geqq K\}}X_n;$$

then $X_n=U_n+V_n$ and since U_n and V_n have disjoint supports

$$G(X_n)=G(U_n)+G(V_n). \tag{12}$$

Let now $\omega\notin A_\infty\cup B_\infty$; if $X_\infty(\omega)<K$, then $U_n(\omega)=X_n(\omega)<K$ for all n large enough, while if $X_\infty(\omega)>K$, then $X_n(\omega)>K$ and so $U_n(\omega)=0$ for all large n. In any case $\lim_{n\in\mathbb{N}} U_n(\omega)$ exists for each $\omega\notin A_\infty\cup B_\infty$ and $P(A_\infty\cup B_\infty)=0$. Since $(U_n)_{n\in\mathbb{N}}$ is adapted, since $0\leqq U_n\leqq K$ for all $n\in\mathbb{N}$ and since G is continuous, an application of Theorem II) yields that

$$\left.\begin{array}{ll}(G(U_n))_{n\in\mathbb{N}} & \text{is uniformly bounded}\\ (G(U_n))_{n\in\mathbb{N}} & \text{is an asymptotic martingale.}\end{array}\right. \tag{13}$$

On the other hand it is clear that $0\leqq V_n\leqq X_n$ and that $|G(V_n)|\leqq\left(\frac{\varepsilon}{M}\right)V_n$, for each $n\in\mathbb{N}$, whence

$$|G(V_\tau)|\leqq\left(\frac{\varepsilon}{M}\right)V_\tau\leqq\left(\frac{\varepsilon}{M}\right)X_\tau, \qquad \text{for each } \tau\in T.$$

We deduce (use also (11))

$$\int |G(V_\tau)| \leqq \varepsilon, \qquad \text{for each } \tau \in T. \tag{14}$$

Clearly the sequence $(G(V_n))_{n\in\mathbb{N}}$ is also adapted. Hence by (12), (13), (14) and Lemma 3, the implication (i) $\Rightarrow$ (ii) is proved.

(ii) $\Rightarrow$ (i). Assume (ii). That G satisfies i a) is clear, since a sequence of real numbers is an asymptotic martingale if and only if it is convergent.

As before we may and shall assume that

$$G(0)=0.$$

To show that G satisfies i b) we reason by contradiction: suppose that i b) were false. We shall only consider the case of the limit at $+\infty$; the case of the limit at $-\infty$ can be treated similarly.

We have to consider separately the following two cases:

Case I.

$$-\infty < \alpha = \liminf_{x\to+\infty} \frac{G(x)}{x} < \limsup_{x\to+\infty} \frac{G(x)}{x} = \beta < +\infty.$$

Case II.

$$\liminf_{x\to+\infty} \frac{G(x)}{x} \quad \text{or} \quad \limsup_{x\to+\infty} \frac{G(x)}{x}$$

(possibly both) are infinite ($\pm\infty$).

We now take $(\Omega, \mathscr{F}, P)$ as in Lemma 4.

Case I. Let $y_n > 1$, $y_n \nearrow +\infty$ and so that

$$\frac{G(y_{2n})}{y_{2n}} \to \beta, \qquad \frac{G(y_{2n-1})}{y_{2n-1}} \to \alpha$$

and set

$$X_n(\omega) = y_n 1_{[0, \frac{1}{y_n})}(\omega), \qquad \text{for } \omega \in \Omega.$$

Then

$$G(X_n)(\omega) = G(y_n) 1_{[0, \frac{1}{y_n})}(\omega), \qquad \text{for } \omega \in \Omega.$$

If we apply Lemma 4 first with $a_n = b_n = y_n$ and then with $a_n = G(y_n)$, $b_n = y_n$, then it is clear that $(X_n)_{n\in\mathbb{N}}$ is an L^1-bounded asymptotic martingale, that $(G(X_n))_{n\in\mathbb{N}}$ is L^1-bounded, but that $(G(X_n))_{n\in\mathbb{N}}$ is not an asymptotic martingale.

Case II. We shall consider the following subcases:

$$\text{II}_1) \quad \limsup_{x\to+\infty} \frac{G(x)}{x} = +\infty.$$

By induction construct a sequence $(x_n)_{n\in\mathbb{N}}$, $x_n>1$, $x_n \nearrow +\infty$ such that for each $n\in\mathbb{N}$

$$\frac{G(x_n)}{x_n}>2^n \quad \text{and} \quad \frac{G(x_n)}{x_{n+1}}<1$$

Since $t\to\frac{G(x_k)}{t}$ is decreasing on the interval $[x_k, x_{k+1}]$, we can also find a sequence $(y_k)_{k\in\mathbb{N}}$, $x_k<y_k<x_{k+1}$ so that

$$\frac{G(x_{2n})}{y_{2n}}=2, \qquad \frac{G(x_{2n-1})}{y_{2n-1}}=1.$$

We then have

$$\frac{x_{2n}}{y_{2n}}=\frac{2x_{2n}}{G(x_{2n})}<\frac{2}{2^n} \quad \text{and} \quad \frac{x_{2n-1}}{y_{2n-1}}=\frac{x_{2n-1}}{G(x_{2n-1})}<\frac{1}{2^n}$$

for all n. We now define for each $n\in\mathbb{N}$

$$X_n(\omega)=x_n 1_{[0,\frac{1}{y_n})}(\omega), \qquad \text{for } \omega\in\Omega.$$

Then

$$G(X_n)(\omega)=G(x_n)1_{[0,\frac{1}{y_n})}(\omega), \qquad \text{for } \omega\in\Omega.$$

If we apply Lemma 4 first with $a_n=x_n$ and $b_n=y_n$ and then with $a_n=G(x_n)$ and $b_n=y_n$, then it is clear that $(X_n)_{n\in\mathbb{N}}$ is an L^1-bounded asymptotic martingale, that $(G(X_n))_{n\in\mathbb{N}}$ is L^1-bounded but that $(G(X_n))_{n\in\mathbb{N}}$ is not an asymptotic martingale.

$\mathrm{II}_2)$ $\liminf_{x\to+\infty}\frac{G(x)}{x}=-\infty.$

By induction again construct a sequence $(x_n)_{n\in\mathbb{N}}$, $x_n>1$, $x_n\nearrow +\infty$ such that for each $n\in\mathbb{N}$

$$\frac{G(x_n)}{x_n}<-2^n \quad \text{and} \quad \frac{G(x_n)}{x_{n+1}}>-1.$$

Then find a sequence $(y_k)_{k\in\mathbb{N}}$, $x_k<y_k<x_{k+1}$ so that

$$\frac{G(x_{2n})}{y_{2n}}=-2, \qquad \frac{G(x_{2n-1})}{y_{2n-1}}=-1,$$

and set for each $n\in\mathbb{N}$

$$X_n(\omega)=x_n 1_{[0,\frac{1}{y_n})}(\omega), \qquad \text{for } \omega\in\Omega.$$

The conclusion is the same as before: $(X_n)_{n\in\mathbb{N}}$ is an L^1-bounded asymptotic martingale, $(G(X_n))_{n\in\mathbb{N}}$ is L^1-bounded but is not an asymptotic martingale.

$\mathrm{II}_3)$ $\limsup_{x\to+\infty}\frac{G(x)}{x}=-\infty.$

Then $\liminf_{x\to+\infty}\frac{G(x)}{x}=-\infty$ also and we are reduced to case II_2).

II_4) $\liminf_{x\to+\infty}\frac{G(x)}{x}=+\infty$.

Then $\limsup_{x\to+\infty}\frac{G(x)}{x}=+\infty$ also and we are reduced to case II_1). This completes the proof of the implication (ii) $\Rightarrow$ (i) and hence of the theorem.

Remarks. 1) In connection with Theorem 2 above note that the class of continuous functions $G: R\to R$ for which

$$\lim_{x\to+\infty}\frac{G(x)}{x} \quad\text{and}\quad \lim_{x\to-\infty}\frac{G(x)}{x}$$

exist (finite or infinite) is quite large; it includes the piecewise linear functions, the concave functions, the convex functions (see [5], Chap. I, p. 52, Excercise 6), the subadditive functions (see [15], p. 244, Theorem 7.6.2).

2) Let us call a function $G: R\to R$ a "stability function" for the class of L^1-bounded asymptotic martingales if whenever $(X_n)_{n\in\mathbb{N}}$ is a real-valued L^1-bounded asymptotic martingale (on any probability space) such that $(G(X_n))_{n\in\mathbb{N}}$ is L^1-bounded, then $(G(X_n))_{n\in\mathbb{N}}$ is an asymptotic martingale. The proof of Theorem 2 (see in particular the proof of the implication (ii) $\Rightarrow$ (i)) shows that conditions i a) and i b) are necessary as well as sufficient for $G: R\to R$ to be a stability function for the class of L^1-bounded asymptotic martingales.

3) The standard examples of functions $G: R\to R$ satisfying $\lim_{x\to+\infty}\frac{G(x)}{x}=+\infty$ are

$$|x|\log^+|x| \quad\text{and}\quad |x|^p \quad (p>1).$$

As Theorem 2 shows, the classical stability theorems from «martingale theory» concerning these functions (see [9], pp. 295–296; alternatively see [17, 19, 21, 22]) do not carry over to asymptotic martingales.

§ 3. An Example

We present here an instance when the asymptotic martingale property fails. We assume below that $\Omega=[0,1)$, $\mathscr{F}=$ the σ-algebra of Borel sets and P a non-atomic probability measure. Note also that since the sequence of σ-algebras is not mentioned explicitly, it is assumed that $(\mathscr{F}_n)_{n\in\mathbb{N}}$ is the "minimal" sequence, that is, $\mathscr{F}_n=\sigma(X_1,\ldots,X_n)$ for each $n\in\mathbb{N}$. The following example was suggested to us by Proposition 2.4 in [13]:

Example. Let $S: \Omega\to\Omega$ be an ergodic measure-preserving transformation. There are then functions $f\in L^1_+$ such that if we set

$$X_n=\frac{f+f\circ S+\cdots+f\circ S^{n-1}}{n}, \quad \text{for each } n\in\mathbb{N}$$

then $(X_n)_{n\in\mathbb{N}}$ is not an asymptotic martingale.

Proof. Let $f \in L^1_+$ be such that $f \log^+ f \notin L^1$ and $\sigma(f) = \mathscr{F}$. (Such functions are easily constructed. In fact, let $C > 0$ be a constant such that

$$C\left(\sum_{n=2}^{\infty} \frac{1}{n^2 (\log n)^2}\right) = 1.$$

Let $(I_n)_{n \geqq 2}$ be a partition of $[0, 1)$ into successive intervals $I_n = [a_n, b_n)$ such that

$$P(I_n) = \frac{C}{n^2 (\log n)^2}, \quad \text{for } n \geqq 2.$$

Define now f: $\Omega \to R$ as follows. On $I_n = [a_n, b_n)$ set

$f(a_n) = n, \quad f(b_n) = n + 1$
f is linear on $[a_n, b_n)$.

Then $f \in L^1_+$ since

$$\int_{I_n} f \leqq (n+1) \frac{C}{n^2 (\log n)^2} = \frac{C}{n (\log n)^2} + \frac{C}{n^2 (\log n)^2}$$

and $\sum_{n=2}^{\infty} \frac{1}{n (\log n)^2} < \infty$. On the other hand, $f \log^+ f \notin L^1$, since

$$\int_{I_n} f \log^+ f \geqq (n \log^+ n) \frac{C}{n^2 (\log n)^2} = \frac{C}{n \log n}$$

and $\sum_{n=2}^{\infty} \frac{1}{n \log n} = +\infty$. Furthermore it is clear that the σ-algebra spanned by f is all of $\mathscr{F}$.)

Now it is obvious that $\mathscr{F}_n = \sigma(X_1, \ldots, X_n) = \mathscr{F}$ for all $n \in \mathbb{N}$. It follows that every $\mathscr{F}$-measurable mapping τ: $\Omega \to \mathbb{N}$ is a stopping time. In particular $X_n^* = \sup(X_1, \ldots, X_n)$ can be written in the form

$$X_n^* = X_{\tau_n}, \quad \text{for some } \tau_n \in T, \ \tau_n \leqq n.$$

But by Ornstein's theorem (see [24]), $X^* = \sup_{j \in \mathbb{N}} X_j \notin L^1$; hence

$$\int X_{\tau_n} = \int X_n^* \nearrow +\infty.$$

It follows that the family $(\int X_\tau)_{\tau \in T}$ is not bounded. Since the boundedness of $(\int X_\tau)_{\tau \in T}$ is a necessary condition for $(X_n)_{n \in \mathbb{N}}$ to be an asymptotic martingale, (see statement 1) of Theorem 1), the desired conclusion is reached.

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