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Representation of an Isotropic Diffusion as a Skew Product*

By

ALBERTO RAÚL GALMARINO

1. Introduction

The processes considered in this paper have state space R^3 and are characterized by the following conditions:

(i) possess the simple Markov property

(ii) are homogeneous in time

(iii) have continuous paths

(iv) are isotropic

(v) do not pass through the origin at positive times except possibly on a set of paths of zero probability.

These conditions will be made precise in Sec. 2 after the basic notation has been presented.

The sample space of all continuous paths on R^3 will be expressed as the Cartesian product $\Omega \times \Omega'$ of two sample spaces: Ω consists of all continuous paths ω on the radial coordinate space $[0, \infty)$, and Ω' of all continuous paths ω' on the spherical coordinate space S^2.

A diffusion of the type described is expressed by using spherical coordinates, as

$$\text{(1.1)} \qquad \boldsymbol{x}(t, \omega \times \omega') = [\boldsymbol{r}(t, \omega), \boldsymbol{\varphi}(t, \omega')]$$

where $\boldsymbol{r}(t, \omega)$, $\omega \in \Omega$, is the radial motion and $\boldsymbol{\varphi}(t, \omega')$, $\omega' \in \Omega'$ is the spherical motion both associated to $\boldsymbol{x}(t, \omega \times \omega')$.

It is aimed to prove these results:

a) The radial process $\boldsymbol{r}(t, \omega)$ is simple Markov and homogeneous in time.

b) $\boldsymbol{x}(t, \omega \times \omega')$ can be represented as the so-called skew product of the radial process and an independent spherical Brownian motion $\boldsymbol{\vartheta}$ run with a clock $\sigma(t, \omega)$ depending on the radial path ω. That is, it will be shown that, with probability one for all t simultaneously:

$$\text{(1.2)} \qquad \boldsymbol{x}(t, \omega \times \omega') = (\boldsymbol{r}(t, \omega); \boldsymbol{\vartheta}[\sigma(t, \omega), \omega']) .$$

c) $\sigma(t, \omega)$ is a non-negative, continuous non-decreasing function of t for each fixed ω. For each fixed t it is measurable with respect to the sub-σ-field determined

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by the radial motion up to time t. Moreover it accomplishes the following additive property:

(1.3) $$\sigma(t, \omega) = \sigma(t - s, \omega_s^+) + \sigma(s, \omega) \text{ for } s < t$$

with probability one for all pairs (s, t) simultaneously.

ω_s^+ in (1.3) is the path defined by the equation:

(1.4) $$\boldsymbol{r}(t, \omega_s^+) = \boldsymbol{r}(t + s, \omega).$$

In Sec. 3, by considering Green operators, the problem is reduced to the computation of the characteristic functional of $\boldsymbol{\varphi}(t, \omega')$ as indicated in (3.21). Sec. 4 introduces a special Markov property for $\boldsymbol{\varphi}(t, \omega')$. In Sec. 5 the characteristic functional of $\boldsymbol{\varphi}(t, \omega')$ is actually computed and the desired expression (3.21) is obtained except for a term that still must be proved to be zero. In this proof, some ideas from [*8*] and [*3*] are used, although special arguments have to be applied due to the fact that S^2 is not a group and also in order to show that the clock $\sigma(t, \omega)$ is finite. Sec. 6 leads to the construction of a spherical process with the characteristic functional of $\boldsymbol{\varphi}(t, \omega')$ and makes the above-mentioned term correspond to interlarded Poisson jumps in a spherical Brownian motion run with a suitable clock. Sec. 7 proves the equivalence of $\boldsymbol{\varphi}(t, \omega')$ with the process constructed in Sec. 6 by applying the special Markov property of Sec. 4, and produces the final result.

2. Notation and basic definitions

The sample spaces Ω and Ω' already have been introduced in Sec. 1. Ω consists of all continuous functions ω from $[0, \infty)$ into $[0, \infty)$, and Ω' of all continuous functions ω' from $[0, \infty)$ into S^2.

Throughout this paper $\mathscr{B}$ is the Borel σ-field of subsets of $\Omega \times \Omega'$ generated by sets of the type:

$$(\omega \times \omega' : x(t, \omega \times \omega') \in A),\ A \text{ Borel set} \subset R^3.$$

$\mathscr{B}_s$ is the sub-σ-field of $\mathscr{B}$ whose generators are those in the definition of $\mathscr{B}$ in which $t \leqq s$. Analogously $\mathscr{B}_{(s_1, s_2)}$ is that sub-σ-field whose generators have $s_1 < t \leqq s_2$.

$\mathscr{B}\boldsymbol{r}$ is the sub-σ-field generated by sets of the type:

$$(\omega \times \omega' : \boldsymbol{r}(t, \omega) \in A_1),\ A_1 \text{ Borel set} \subset [0, \infty).$$

$\mathscr{B}\boldsymbol{\varphi}$ is the sub-σ-field generated by sets of the type:

$$(\omega \times \omega' : \boldsymbol{\varphi}(t, \omega') \in A_2),\ A_2 \text{ Borel set} \subset S^2,$$

where S^2 is the unit sphere in R^3.

For each point $a \in R^3$ the process defines a probability measure $P_a(B)$ for all $B \in \mathscr{B}$. This means the probability that a continuous path starting at a belongs to the Borel set B. A dot will often be used for a generic point in R^3, for instance, $P_.(B)$. It is understood that, whenever "." is used several times in an argument, it always refers to the same point.

The name "simple Markov" is reserved for the property that, for all $t \geqq 0$, when the process is considered at time t, the probability of any future event is independent of the past, i.e. of the sub-σ-field $\mathscr{B}_t$.

If the process is also homogeneous in time, the simple Markov property is equivalent to the equation:

$$(2.1) \qquad P.(P[\boldsymbol{x}(t, (\omega\times\omega')_s^+) \in A \,|\, \mathscr{B}_s] = P_{\boldsymbol{x}(s)}[\boldsymbol{x}(t-s, \omega\times\omega') \in A]) = 1$$

for all Borel sets $A \subset R^3$, where $(\omega x \omega')_s^+$ is defined as in (1.4).

By isotropy it is meant that the transition probabilities are invariant under all orthogonal transformations that leave the origin fixed.

That is, if $A \in \mathscr{B}$ and g is such a transformation:

$$(2.2) \qquad P_{(r;\, g\varphi)}[gA] = P_{(r;\, \varphi)}[A].$$

As nearly all statements hold with $P.$-probability one, the repetition of the words "with $P.$-probability one" will be avoided whenever it is obvious from the preceding arguments that it is the case.

3. Reduction of the problem by considering the Green operators

Lemma 3.1. *Let $\boldsymbol{r}(t, \omega)$ be a radial motion with the simple Markov property and homogeneous in time. Let $\boldsymbol{\vartheta}(\sigma(t, \omega), \omega')$ be an independent spherical Brownian motion run with a clock $\sigma(t, \omega)$ which depends on the radial path. Moreover let this clock be a non-negative, continuous and non-decreasing function of t for each fixed ω, measurable with respect to $\mathscr{B}_r \cap \mathscr{B}_t$ for each fixed t, satisfying* (1.3).

Then the skew product

$$(3.1) \qquad (\boldsymbol{r}(t, \omega);\, \boldsymbol{\vartheta}(\sigma(t, \omega), \omega'))$$

is a simple Markov process.

Proof: To prove that (3.1) is simple Markov it is enough to show that:

$$\begin{aligned}(3.2) \quad & P.[(\boldsymbol{r}(t_1); \boldsymbol{\vartheta}(\sigma(t_1))) \in (dr_1; d\varphi_1), \ldots, (\boldsymbol{r}(t_n); \boldsymbol{\vartheta}(\sigma(t_n))) \in (dr_n; d\varphi_n)] = \\ & = P.[(\boldsymbol{r}(t_1); \boldsymbol{\vartheta}(\sigma(t_1))) \in (dr_1; d\varphi_1)] \ldots P_{r_{n-1}\varphi_{n-1}}[(\boldsymbol{r}(t_n - t_{n-1}); \\ & \boldsymbol{\vartheta}(\sigma(t_n - t_{n-1}))) \in (dr_n; d\varphi_n)]\end{aligned}$$

for any choice of $0 < t_1 < \cdots < t_n$.

By introducing a conditional probability with respect to the sub-σ-field $\mathscr{B}_r$, the left hand side of (3.2) yields

$$\begin{aligned}(3.3) \quad & P.[(\boldsymbol{r}(t_1); \boldsymbol{\vartheta}(\sigma(t_1))) \in (dr_1; d\varphi_1), \ldots (\boldsymbol{r}(t_n); \boldsymbol{\vartheta}(\sigma(t_n))) \in (dr_n; d\varphi_n)] = \\ & = E.[\boldsymbol{r}(t_1) \in dr_1, \ldots, \boldsymbol{r}(t_n) \in dr_n,\ P.(\boldsymbol{\vartheta}(\sigma(t_1)) \in d\varphi_1 \ldots, \boldsymbol{\vartheta}(\sigma(t_n)) \in d\varphi_n \,|\, \mathscr{B}_r)].\end{aligned}$$

For simplicity, let

$$(3.4) \qquad X_1(\omega) = P.[\boldsymbol{\vartheta}(\sigma(t_1)) \in d\varphi_1 \,|\, \mathscr{B}_r]$$

$$(3.5) \qquad X_h(\omega) = P_{r_{h-1}\varphi_{h-1}}[\boldsymbol{\vartheta}(\sigma(t_h - t_{h-1}) \in d\varphi_h \,|\, \mathscr{B}_r]$$

for $h = 2, \ldots, n$.

For fixed ω, $\vartheta(\sigma(t, \omega), \omega')$ is simple Markov as a process on ω', for the standard spherical Brownian motion $\vartheta(t, \omega')$ is simple Markov and $\sigma(t, \omega)$ is a suitable time scaling.

Hence, for fixed ω,

$$(3.6) \qquad P.[\vartheta(\sigma(t_1) \in d\varphi_1, \dots, \vartheta(\sigma(t_n) \in d\varphi_n \,|\, \mathscr{B}_r] = X_1, (\omega) \dots X_n(\omega)$$

$X_1(\omega)$ is measurable with respect to $\mathscr{B}_r \cap \mathscr{B}_{t_1}$, and also, for $h = 2, \dots, n$, $X_h(\omega^+_{t_{h-1}})$ is measurable with respect to $\mathscr{B}_r \cap \mathscr{B}_{(t_{h-1}; t_h]}$.

By the Markov property of $\boldsymbol{r}(t, \omega)$:

$$(3.7) \qquad \begin{aligned} &E.(\boldsymbol{r}(t_1) \in dr_1, \dots, \boldsymbol{r}(t_n) \in dr_n, X_1(\omega), \dots, \\ &X_n(\omega^+_{t_{n-1}})] = E.[\boldsymbol{r}(t_1) \in dr_1, X_1(\omega)] \dots, \\ &E_{r_{n-1}\varphi_{n-1}}[\boldsymbol{r}(t_n - t_{n-1}) \in dr_n, X_n(\omega)]. \end{aligned}$$

Now (3.7) together with (3.3), (3.4), (3.5) and (3.6) shows that (3.2) holds, and thus lemma 3.1 is proved.

Lemma 3.2. *The radial motion of the process* (1.1) *possesses the simple Markov property and is homogeneous in time.*

Proof: The homogeneity in time is obvious from the similar property of the process (1.1). To prove the simple Markov property first notice that

$$(3.8) \qquad P.[\boldsymbol{r}(t) \in dr \,|\, \mathscr{B}_r \cap \mathscr{B}_s] = E.(P_{(r(s);\, \varphi(s))}(r(t-s) \in dr) \,|\, \mathscr{B}_r \cap \mathscr{B}_s).$$

As the process (1.1) is isotropic and as the event $[\boldsymbol{r}(t-s) \in dr]$ is independent of $\mathscr{B}_s$ and measurable on $\mathscr{B}_r$

$$(3.9) \qquad E.\,(P_{(r(s);\, \varphi(s))}(r(t-s) \in dr) \,|\, \mathscr{B}_r \cap \mathscr{B}_s) = P_{r(s)}(\boldsymbol{r}(t-s) \in dr).$$

From (3.8) and (3.9) the simple Markov property of the radial process is obtained.

Lemma 3.3. *Let $x(t)$ and $y(t)$ be two Markovian processes with continuous paths with the same state space E.*

If $x(t)$ and $y(t)$ have the same Green operators, that is, if for each function $f \in C(E)$

$$(3.10) \qquad E.(\int_0^\infty \exp(-\alpha t) f(x(t))\, dt) = E.(\int_0^\infty \exp(-\alpha t) f(y(t)\, dt),$$

then the two processes are identical in law.

Proof: From (3.10) it follows, by the uniqueness of the inverse of the Laplace transform, that:

$$(3.11) \qquad E.(f(x(t))) = E.(f(y(t)))$$

for all $f \in C(E)$, $t \geqq 0$.

As indicator functions of measurable sets can be approximated by continuous functions (3.11) yields

$$(3.12) \qquad P.[x(t) \in A] = P.[y(t) \in A]$$

for each Borel set $A \subset E$, $t > 0$.

By (3.12) both processes have the same transition probabilities. As their paths are continuous, they are identical in law as stated.

Let S_n^l be the usual spherical functions.

Theorem 3.1. *To prove* (1.2) *it is enough to show that with probability one for each* t:

$$(3.13)\qquad E.(S_n^l[\boldsymbol{\vartheta}(\sigma(t,\omega),\omega')]\,|\,\mathscr{B}_r) = E.(S_n^l(\boldsymbol{\vartheta}(t,\omega'))\,|\,\mathscr{B}_r),$$

where $\sigma(t,\omega)$ *is a suitable clock as described in Sec.* 1 *with property* (1.3).

Proof: Both processes in (1.2) are Markovian; the one on the left hand side by condition (i) at the beginning of Sec. 1; the one on the right hand side by lemmas 3.1 and 3.2. Also both have continuous paths. Hence to prove they are identical in law it is enough to show, by lemma 3.3, that the Green operators are equal, i.e.:

$$(3.14)\qquad E.\left(\int_0^\infty \exp(-\alpha t) f(\boldsymbol{r}(t);\boldsymbol{\vartheta}(\sigma(t)))\,dt\right) = E.\left(\int_0^\infty \exp(-\alpha t) f(\boldsymbol{r}(t);\boldsymbol{\varphi}(t))\,dt\right)$$

for all $f \in C(R^3)$.

To prove (3.14) it is enough to consider only functions of the type $f(r,\varphi) = g(r)h(\varphi)$ with $g(0)=0$.

In this case:

$$(3.15)\qquad E.\left(\int_0^\infty \exp(-\alpha t) f(\boldsymbol{r}(t);\boldsymbol{\vartheta}(\sigma(t)))\,dt\right) = E.\left(\int_0^\infty \exp(-\alpha t)\, g(\boldsymbol{r}(t))\,(E.[h(\boldsymbol{\vartheta}(\sigma(t)))\,|\,\mathscr{B}_r]\,dt\right).$$

Also:

$$(3.16)\qquad E.\left(\int_0^\infty \exp(-\alpha t) f(\boldsymbol{r}(t);\boldsymbol{\varphi}(t))\,dt\right) = E.\left(\int_0^\infty \exp(-\alpha t)\, g(\boldsymbol{r}(t))\,[E.[h(\boldsymbol{\varphi}(t))\,|\,\mathscr{B}_r]]\,dt\right).$$

From (3.15), (3.16) to prove (3.14) it is enough to show that, for each t, with probability one:

$$(3.17)\qquad E.(h[\boldsymbol{\vartheta}(\sigma(t,\omega),\omega')]\,|\,\mathscr{B}_r) = E.(h[\boldsymbol{\varphi}(t,\omega')]\,|\,\mathscr{B}_r)$$

for all $h \in C(S^2)$.

As linear combinations of the spherical functions S_n^l are dense in $C(S^2)$, to prove (3.17) (and hence (1.2)) it is enough to prove (3.13), as theorem 3.1 states.

Theorem 3.2. *For any clock* $\sigma(t,\omega)$ *as in theorem* 3.1, *with probability one:*

$$(3.18)\qquad E.(S_n^l[\boldsymbol{\vartheta}(\sigma(t,\omega),\omega')]\,|\,\mathscr{B}_r) = S_n^l(\cdot)\exp(-1/2\,n(n+1)\,\sigma(t,\omega)).$$

Proof: First it will be shown that

$$(3.19)\qquad E.(S_n^l[\boldsymbol{\vartheta}(t,\omega')]) = S_n^l(\cdot)\exp(-1/2\,n(n+1)\,t)$$

for the spherical standard Brownian motion $\boldsymbol{\vartheta}(t,\omega')$.

In fact, the right hand side of (3.19) is the unique solution of the heat equation:

$$(3.20)\qquad \frac{\partial u(\cdot,t)}{\partial t} = 1/2\,\Delta' u(\cdot,t)$$

with initial condition $u(\cdot, 0^+) = S_n^l(\cdot)$. ($\Delta'$ in (3.20) is the spherical Laplacian.)

From the theory of the spherical Brownian motion, the semigroup operator of this process applied to $S_n^l(\cdot)$, that is

$$H_t S_n^l(\cdot) = E.(S_n^l[\boldsymbol{\vartheta}(t, \omega')])$$

is equal to the unique solution mentioned above, which proves (3.19).

Now, to compute $E.(S_n^l[\boldsymbol{\vartheta}(\sigma(t, \omega), \omega')] \mid \mathscr{B}_r)$, for a fixed ω, amounts to perform the same computation as for (3.19). In fact, by the boundedness and continuity of $\boldsymbol{\vartheta}(\sigma(t, \omega))$ with respect to t, it holds with probability one:

$$E.(S_n^l[\boldsymbol{\vartheta}(\sigma(t, \omega), \omega')] \mid \mathscr{B}_r) = E.(S_n^l[\boldsymbol{\vartheta}(s, \omega')])\big|_{s = \sigma(t, \omega)}.$$

This gives the result (3.18).

Remark: By theorems 3.1 and 3.2 to prove (1.2) it is enough to show that, for each t, with probability one

$$E.[S_n^l(\boldsymbol{\varphi}(t, \omega')) \mid \mathscr{B}_r] = S_n^l(\cdot) \exp(-1/2\, n(n+1)\, \sigma(t, \omega)) \tag{3.21}$$

where $\sigma(t, \omega)$ has all the conditions described in Sec. 1, c).

4. Markov property of the spherical process

The result obtained by DAVID BLACKWELL in theorem 5 of his paper [2] will now be applied to $(\Omega \times \Omega', \mathscr{B}, P.)$.

According to that theorem, for each ω there is a probability measure $P_.^\omega$ on $\mathscr{B}$ such that

$$\text{for fixed } B \in \mathscr{B},\ P_.^\omega(B) \text{ is a } \mathscr{B}_r\text{-measurable function of } \omega \times \omega', \tag{4.1}$$

$$P.[P_.^\omega(B) = P.(B \mid \mathscr{B}_r)] = 1. \tag{4.2}$$

Consider the angular motion $\boldsymbol{\varphi}(t, \omega')$. By Blackwell's theorem it is possible to work with the transition probabilities:

$$p^\omega(t', \vartheta', t, d\vartheta) = P^{\omega_{t'}^+}_{(r(t', \omega), \vartheta')}[\varphi(t - t', \omega') \in d\vartheta] \tag{4.3}$$

for each fixed ω, each pair $t' < t$ and $\vartheta', \vartheta \in S^2$.

The process thus defined for each ω has a Markov property in the sense of theorem 4.1 that follows. This is not quite the usual simple Markov property, for equation (4.4) does not hold with probability one simultaneously for all n-vectors $(t_1, \ldots, t_n)$, but rather on a variable set of probability one depending on $(t_1, \ldots, t_n)$.

Theorem 4.1. *Given $t_1 < \cdots < t_n$, the following equality holds on a set of probability one which depends on $(t_1, \ldots, t_n)$.*

$$\begin{aligned} &P^\omega_{\varphi_0}[\boldsymbol{\varphi}(t_1) \in d\varphi_1, \ldots, \boldsymbol{\varphi}(t_n) \in d\varphi_n] = \\ &P^\omega_{\varphi_0}[\boldsymbol{\varphi}(t_1) \in d\varphi_1] \cdots P^{\omega_{t_{n-1}}^+}_{\varphi_{n-1}}[\boldsymbol{\varphi}(t_n - t_{n-1}) \in d\varphi_n]. \end{aligned} \tag{4.4}$$

The proof is very similar to that of lemma 3.1 and will be, therefore, omitted.

5. Characteristic functional of the spherical process

The purpose of this section is to prove the following theorem:

Theorem 5.1. *On a set of probability one which depends on t it holds:*

$$E.(S_n^l[\boldsymbol{\varphi}(t,\omega')]\,|\,\mathscr{B}_r) = S_n^l(\cdot)\exp[-1/2\,n(n+1)\,\sigma(t,\omega) - \int_{0^+}^{\pi}[1-P_n(\cos\vartheta)]\,\nu([0,t]\times d\vartheta,\omega)] \tag{5.1}$$

where the functions P_n are the usual Legendre Polynomials and

(i) *$\sigma(t,\omega)$ is non-negative, continuous and non-decreasing in t for fixed ω, measurable with respect to $\mathscr{B}_r \cap \mathscr{B}_t$ for fixed t, and has the addition property* (1.3).

(ii) *$\nu(dt\times d\vartheta,\omega)$ is a non-negative measure on $[0,\infty)\times(0,r]$ satisfying:*

$$\int_{0^+}^{\pi}(1-\cos\vartheta)\,\nu([0,t]\times d\vartheta,\omega) < \infty\,. \tag{5.2}$$

Besides $\dot{\nu}([0,t]\times d\vartheta,\omega) = (1-\cos\vartheta)\,\nu([0,t]\times d\vartheta,\omega)$ is weak-star continuous.

(iii) *$\sigma(t,\omega)$ and $\nu(dt\times d\vartheta,\omega)$ in* (5.1) *are uniquely determined.*

Since the proof is rather lengthy it will be preceded by several lemmas.

Lemma 5.1. *Let G be the group of rotations of the sphere S^2, and K be the subgroup of those rotations which leave the North pole N fixed. Let dk be the Haar measure in K. Then for $g_1, g_2 \in G$ it holds:*

$$\int_{k\in K} S_n^l(g_1 k g_2)\,dk = S_n^l(g_1)\,P_n(g_2)\,. \tag{5.3}$$

Proof: For a fixed g_1 the left hand side of (5.3) is a function of g_2, say:

$$f(g_2) = \int_{k\in K} S_n^l(g_1 k g_2)\,dk\,. \tag{5.4}$$

If Δ' is the spherical Laplacian, clearly:

$$\Delta'(f(g_2) = \int_{k\in K}\Delta' S_n^l(g_1 k g_2)\,dk = -n(n+1)\int_{k\in K} S_n^l(g_1 k g_2)\,dk = -n(n+1)\,f(g_2)\,. \tag{5.5}$$

On the other hand, from (5.4) and the fact that S_n^l is a function of left cosets gK it follows that:

$$f(g_2) = f(g_2 K) = f(K g_2)\,. \tag{5.6}$$

By (5.5) and (5.6), $f(g_2)$ must be of the form:

$$f(g_2) = c\,P_n(g_2) \tag{5.7}$$

where c is a constant to be determined.

Let in (5.4) $g_2 = k^* \in K$. Then by (5.7)

$$f(k^*) = c = \int S_n^l(g_1 k k^*)\,dk = S_n^l(g_1)\,. \tag{5.8}$$

(5.4), (5.7), (5.8) prove (5.3) and hence the lemma.

Lemma 5.2. *On a set of probability one which depends on t it holds:*

$$E.(S_n^l[\boldsymbol{\varphi}(t),\omega')]\,|\,\mathscr{B}_r) = S_n^l(\cdot)\,E_N^{\omega}(P_n[\varphi(t,\omega')])\,. \tag{5.9}$$

The random variables

$$c_n(t, \omega) = E_N^\omega(P_n(\boldsymbol{\varphi}(t, \omega))) \tag{5.10}$$

are continuous and non-increasing in t for fixed ω and satisfy:

$$0 \leqq c_n(t, \omega) \leqq 1 \tag{5.11}$$

$$c_n(t, \omega) = c_n(s, \omega)\, c_n(t - s, \omega_s^+), \tag{5.12}$$

with probability one depending on the pair $(s < t)$.

Proof: By (4.2)

$$E_\cdot(S_n^l(\boldsymbol{\varphi}(t, \omega')] \,|\, \mathscr{B}_r) = E_\cdot^\omega[S_n^l(\boldsymbol{\varphi}(t, \omega'))] \tag{5.13}$$

on a set of probability one depending on t.

Let g be a rotation that sends the N north pole N to 0; let k be a rotation around N, and dk the Haar measure on K as in lemma 5.1.

By lemma 5.1 and the isotropy assumption:

$$\begin{aligned} E_\cdot^\omega(S_n^l[\boldsymbol{\varphi}(t, \omega')] &= E_N^\omega(S_n^l[g\,\boldsymbol{\varphi}(t, \omega')]) = E_N^\omega(\int_K S_n^l(g\,k\,\boldsymbol{\varphi}(t, \omega')\,dk) \\ &= S_n^l(\cdot)\, E_N^\omega(P_n(\boldsymbol{\varphi}(t, \omega'))) \end{aligned} \tag{5.14}$$

which is (5.9).

By the Markov property of the spherical process, for $s < t$:

$$\begin{aligned} E_N^\omega(P_n(\boldsymbol{\varphi}(t, \omega'))) &= E_N^\omega(E_N^\omega[P_n(\boldsymbol{\varphi}(t, \omega'))|\mathscr{B}_s]) \\ &= E_N^\omega(E_{\varphi(s)}^{\omega_s^+}[P_n(\boldsymbol{\varphi}(t - s, \omega'))]). \end{aligned} \tag{5.15}$$

By applying to the right hand side of (5.15) the computation in (5.14)

$$E_N^\omega(P_n(\boldsymbol{\varphi}(t))) = E_N^\omega(P_n(\boldsymbol{\varphi}(s)))\, E_N^{\omega_s^+}(P_n(\boldsymbol{\varphi}(t - s))), \tag{5.16}$$

on a set of probability one depending on the pair s, t, as asserted in (5.12).

The assertion that the integral

$$c_n(t, \omega) = \int_{\Omega'} P_n[\boldsymbol{\varphi}(t, \omega')]\, P_N^\omega(d\omega') \tag{5.17}$$

is continuous in t for fixed ω is easily justified from the continuity of P_n and $\boldsymbol{\varphi}$.

To prove inequalities (5.11) first notice that, as $P_n(\boldsymbol{\varphi}) \leqq 1$, the integral (5.17) is also $\leqq 1$. This shows that the inequality on the right holds.

As $c_n(0, \omega) = P_n(N) = 1$, from the continuity already proved it follows that, for a fixed ω, if $c_n(t, \omega)$ is not positive for all $t > 0$, there must exist a least value, say $\boldsymbol{m}_n(\omega) > 0$, such that:

$$c_n(\boldsymbol{m}_n(\omega), \omega) = 0; \quad c_n(t, \omega) > 0 \quad \text{for} \quad t < \boldsymbol{m}_n(\omega). \tag{5.18}$$

By (5.12) it is then clear that:

$$c_n(t, \omega) = 0 \quad \text{for} \quad t > \boldsymbol{m}_n(\omega).$$

So the inequality on the left in (5.10) is also proved.

Finally $c_n(t, \omega)$ is non-increasing for, by (5.12), it is obtained from $c_n(s, \omega)$ by multiplying by a factor not greater than one.

Next the usual spherical coordinates α, β for the unit vector $\boldsymbol{\varphi}$ are introduced. α is the spherical longitude and β the collatitude (i.e. the angular distance from the North pole N).

Lemma 5.3. $c_n(t, \omega) = 0$ *if and only if*

$$P_N^\omega(\max_{s \leqq t} \beta(s, \omega') < a_n) = 0$$

where a_n *is the smallest positive root of the equation* $P_n(\cos\beta) = 0$.

Proof: Clearly:

$$(5.19) \qquad c_n(t, \omega) = E_N^\omega[P_n(\boldsymbol{\varphi}(t)), \ \operatorname*{Max}_{s \geqq t} \beta(s, \omega') < a_n] + \\ + E_N^\omega[P_n(\boldsymbol{\varphi}(t)), \ \operatorname*{Max}_{s \leqq t} \beta(s, \omega') \geqq a_n].$$

Let $\boldsymbol{h}_n(\omega') = \inf(s : \beta(s, \omega') = a_n)$. Then:

$$(5.20) \qquad |E_N^\omega[P_n(\boldsymbol{\varphi}(t)), \ \operatorname*{Max}_{s \leqq t} \beta(s, \omega') \geqq a_n]| = |E_N^\omega[P_n(\boldsymbol{\varphi}(t)), \ \boldsymbol{h}_n \leqq t]| \leqq$$

$$\leqq \sum_{k=1}^{2m} |E_N^\omega((k-1)2^{-m}t < \boldsymbol{h}_n \leqq k2^{-m}t, P_n(\boldsymbol{\varphi}(t)))|$$

$$= \sum_{k=1}^{2m} |E_N^\omega((k-1)2^{-m}t < \boldsymbol{h}_n \leqq k2^{-m}t, E_{\varphi(k2^{-mt})}^{\omega \overset{+}{k} 2^{-mt}}(P_n(\boldsymbol{\varphi}(t - k2^{-m}t))))|$$

(by the Markov property of the spherical process)

$$= \sum_{k=1}^{2m} |E_N^\omega((k-1)2^{-m}t < \boldsymbol{h}_n \leqq k2^{-m}t, \\ P_n(\boldsymbol{\varphi}(k2^{-m}t))E_N^{\omega \overset{+}{k} 2^{-mt})}P_n(\boldsymbol{\varphi}(t - k2^{-m}t))))|$$

(similarly as in (5.14))

$$\leqq \sum_{k=1}^{2m} E_N^\omega[(k-1)2^{-m}t < \boldsymbol{h}_n \leqq k2^{-m}t, |P_n(\boldsymbol{\varphi}(k2^{-m}t))|]$$

$$= (\text{as } 0 \leqq E_N^{\omega \overset{+}{k} 2^{-mt}}[P_n(\boldsymbol{\varphi}(t - k2^{-m}t)] \leqq 1)$$

$$= E_N^\omega[|P_n(\boldsymbol{\varphi}(([[\boldsymbol{h}_n 2^m]] + 1)2^{-m}))|]$$

where $[[\boldsymbol{h}_n 2^m]]$ is the greatest integer smaller than $\boldsymbol{h}_n 2^m$.

Obviously as $m \to \infty$ the right hand side of (5.20) tends to $E_N^\omega[P(\boldsymbol{\varphi}(\boldsymbol{h}_n))] = 0$. According to (5.18) the "if" condition immediately follows. The "only if" condition is also clear from the fact that:

$$E_N^\omega[P_n(\boldsymbol{\varphi}(t), \ \operatorname*{Max}_{s \leqq t} \beta(s, \omega') < a_n] > 0$$

unless

$$P_N^\omega(\operatorname*{Max}_{s \leqq t} \beta(s, \omega') < a_n) = 0.$$

Lemma 5.4. *Let* $\boldsymbol{m}_n(\omega)$ *be defined as in* (5.18). *Then for fixed* ω *the sequence* $\boldsymbol{m}_n(\omega)$ *is strictly decreasing as n increases.*

Proof: By lemma 5.3 and (5.18),

$$\boldsymbol{m}_n(\omega) = \min(t : P_N^\omega(\operatorname*{Max}_{s \leqq t} \beta(s) < a_n) = 0).$$

As obviously it holds, for $n_1 < n_2$, the strict inequality

$$P_N^\omega(\underset{s \leqq t}{\operatorname{Max}} \beta(s) < a_{n_1}) > P_N^\omega(\underset{s \leqq t}{\operatorname{Max}} \beta(s) < a_{n_2}),$$

lemma 5.4 follows.

Lemma 5.5. *Consider for fixed $\bar{t}$, $\bar{n}$, the set $A_{\bar{t}\bar{n}} = (\omega : \boldsymbol{m}_{\bar{n}}(\omega) > \bar{t})$. Then for almost all $\omega \in A_{\bar{t}\bar{n}}$, $t \leqq \bar{t}$ and $n \leqq \bar{n}$ it holds:*

$$(5.21)\quad c_n(t,\omega) = \exp\left[-1/2\, n(n+1)\,\sigma(t,\omega) - \int_{0^+}^{\pi} (1 - P_n(\cos\vartheta))\,\nu([0,t)\times d\vartheta,\omega)\right]$$

where:

(i) *$\sigma(t,\omega)$ is non-negative, finite, non-decreasing in t for fixed ω, measurable with respect to $\mathscr{B}_r \cap \mathscr{B}_t$ for fixed t, and satisfies the additive property* (1.3).

(ii) *$\nu(dt \times d\vartheta, \omega)$ is a measure on $[0,\bar{t}] \times (0,\pi]$ which is non-negative and satisfies*

$$(5.22)\qquad \int_{0^+}^{\pi} (1 - \cos\vartheta)\,\nu([0,t]\times d\vartheta,\omega) < \infty.$$

Proof: Since $c_n(t,\omega) > 0$ holds for $\omega \in A_{\bar{t}\bar{n}}$ and $n \leqq \bar{n}$, $t \leqq \bar{t}$,

$$(5.23)\qquad c_n(t,\omega) = \exp(-f_n(t,\omega)).$$

By lemma 5.2, $f_n(t,\omega)$ is clearly continuous in t for fixed ω, finite, non-negative, non-decreasing in t, and satisfies the additive property:

$$(5.24)\qquad f_n(t,\omega) = f_n(s,\omega) + f_n(t-s,\omega_s^+)$$

for almost all ω in $A_{\bar{t}\bar{n}}$, on a set depending on each pair (s,t). Therefore (5.23) also holds for almost all ω in $A_{\bar{t}\bar{n}}$, for all pairs belonging to a countable dense subset $D \subset [0,\bar{t})$. As $f_n(t,\omega)$ is a continuous function in t for fixed ω, it is determined by its values on D for all such ω. Property (5.23) obviously applies for s or t not in D by a simple continuity argument. Therefore (5.23) holds for almost all $\omega \in A_{\bar{t}\bar{n}}$ for all pairs (s,t) simultaneously.

By (5.23), (5.24),

$$(5.25)\qquad E_N^{\omega_s^+}(P_n[\boldsymbol{\varphi}(t-s)\,\omega']) = \exp[-(f_n(t,\omega) - f_n(s,\omega))].$$

Hence if $0 = t, < t_1 < t_2 < \cdots < t_n = t$, (5.25) yields

$$(5.26)\quad f_n(t_k) - f_n(t_{k-1}) = 1 - E_N^{\omega_{t_{k-1}}^+}(P_n(\boldsymbol{\varphi}(t_k - t_{k-1}))) + \sigma[f_n(t_k) - f_n(t_{k-1})]$$

for $k = 1, 2, \ldots, m$.

Let $\Delta_m = \underset{k}{\operatorname{Max}}\,(t_k - t_{k-1})$.

Then clearly:

$$(5.27)\qquad f_n(t,\omega) = \lim_{\Delta_m \to 0} \sum_{k=1}^{m} (1 - E_N^{\omega_{t_{k-1}}^+}[P_n(\boldsymbol{\varphi}(t_k - t_{k-1}))]).$$

In what follows $P_n(\varphi)$ is replaced by $P_n(\cos\beta)$ since $P_n(\varphi)$ depends only on the collatitude β.

From (5.27),

$$f_n(t,\omega)=\lim_{\Delta_m\to 0}\sum_{k=1}^{m}\int_0^{\pi}(1-P_n(\cos\vartheta))\,P_N^{\omega\overset{+}{t}_{k-1}}[\beta(t_k-t_{k-1})\in d\vartheta] \tag{5.28}$$

$$=\lim_{\Delta_m\to 0}\int_0^{\pi}(1-P_n(\cos\vartheta)\left[\sum_{k=1}^{m}P_N^{\omega\overset{+}{t}_{k-1}}(\beta(t_k-t_{k-1})\in d\vartheta)\right].$$

The following measures are introduced:

$$\nu_m([0,t]\times d\vartheta,\omega)=\sum_{k=1}^{m}P_N^{\omega\overset{+}{t}_{k-1}}[\beta(t_k-t_{k-1})\in d\vartheta], \tag{5.29}$$

$$\dot{\nu}_m([0,t]\times d\vartheta,\omega)=(1-\cos\vartheta)\,\nu_m([0,t]\times d\vartheta,\omega). \tag{5.30}$$

By (5.28), (5.29), (5.30):

$$\lim_{m\to\infty}\int_0^{\pi}\frac{1'-P_n(\cos)\,\vartheta}{1-\cos\vartheta}\,\dot{\nu}_m([0,t]\times d\vartheta,\omega)=f_n(t,\omega). \tag{5.31}$$

For $n=1$, (5.31) yields:

$$\lim_{m\to\infty}\dot{\nu}_m([0,t]\times[0,\pi],\omega)=f_1(t,\omega)<\infty. \tag{5.32}$$

Hence for fixed t and ω, the sequence of measures $(\dot{\nu}_m)$ is weakly compact. There is a subsequence $(\dot{\nu}_{m_i})$ which converges weakly to a certain measure $\dot{\nu}$.

P_n has the well known properties:

$$\frac{1-P_n(\cos\vartheta)}{1-\cos\vartheta}\leqq\lim_{\theta\downarrow 0}\frac{1-P_n(\cos\vartheta)}{1-\cos\vartheta}=1/2\,n(n+1) \tag{5.33}$$

$$|P_n(\cos\vartheta)|\leqq 1 \qquad P_n(\cos 0)=1. \tag{5.34}$$

By (5.31), (5.33) and the convergence of $\dot{\nu}_{m_i}$ to $\dot{\nu}$

$$f_n(t,\omega)=1/2\,n(n+1)\,\sigma(t,\omega)+\int_{0+}^{\pi}+\frac{1-P_n(\cos\vartheta)}{1-\cos\vartheta}\,\dot{\nu}([0,t]\times d\vartheta,\omega) \tag{5.35}$$

for each fixed $t\leqq\bar{t}$, $\omega\in A_{\bar{t}\bar{n}}$, $n\leqq\bar{n}$, where

$$\dot{\nu}([0,t]\times(0),\omega)=\sigma(t,\omega). \tag{5.36}$$

Thus, (5.31) is obtained by simply defining:

$$\nu([0,t]\times d\vartheta,\omega)=\dot{\nu}([0,t]\times d\vartheta,\omega)/(1-\cos\vartheta). \tag{5.37}$$

Property (5.22) of $\nu([0,t]\times d\vartheta,\omega)$ immediately follows from (5.32).

To prove that $\nu(dt\times d\vartheta,\omega)$ is non-negative it is enough to show that $\nu([0,t]\times d\vartheta,\omega)$ is non-negative and increasing in t for fixed ω.

That it is non-negative follows from the way the approximating sequence (ν_m) is defined in (5.29).

It is also plain, from (5.29), that $\nu([0, t]\times d\vartheta, \omega)$ accomplishes the additive property:

$$\nu([0, t\times]d\vartheta, \omega) = \nu([0, s]\times d\vartheta, \omega) + \nu([0, t-s]\times d\vartheta, \omega_s^+) \tag{5.38}$$

for $s < t$.

As $\nu([0, t-s]\times d\vartheta, \omega_s^+) \geqq 0$, (5,38) shows the monotony of $\nu([0, t]\times d\vartheta, \omega)$ with respect to t.

According to the definition of $\sigma(t, \omega)$ in (5.36) a similar argument applies to show that it is non-negative, finite, non-decreasing in t for fixed ω, and satisfies the additive property (1.3). It is also clear that $\sigma(t, \omega)$ is measurable with respect to $\mathscr{B}_r \cap \mathscr{B}_t$ for fixed t. The proof of lemma 5.5 is thus complete.

Lemma 5.6. *For all t and all n*

$$P.[\boldsymbol{m}_n(\omega) < t] = 0\,. \tag{5.39}$$

Proof: Suppose that for, say, $n = \bar{n}$ and $t = \bar{t}$, $P.[\boldsymbol{m}_{\bar{n}}(\omega) < \bar{t}] > 0$.

It is sufficient to consider $\bar{n} > 1$ for, if (5.37) is not true for $n = 1$, it is not true for $n > 1$ either, by lemma 5.4.

For almost all ω in such a set, by lemma 5.5:

$$f_{\bar{n}}(t, \omega) = 1/2\, n(n+1)\, \sigma(t, \omega) + \int_{0^+}^{\pi} (1 - P_{\bar{n}}(\cos\vartheta))\, \nu([0, t]\times d\vartheta, \omega) \tag{5.40}$$

for $t < \boldsymbol{m}_{\bar{n}}(\omega)$.

But by lemma 5.4 $\boldsymbol{m}_{\bar{n}}(\omega) < \boldsymbol{m}_1(\omega)$ and, for $t < \boldsymbol{m}_1(\omega)$:

$$f_1(t, \omega) = \sigma(t, \omega) + \int_{0^+}^{\pi} (1 - \cos\vartheta)\, \nu([0, t]\times d\vartheta, \omega) < \infty\,. \tag{5.41}$$

Comparing (5.40) and (5.41) and applying the continuity of $f_{\bar{n}}(t, \omega)$

$$\lim_{t \to \boldsymbol{m}_{\bar{n}}(\omega)} f_{\bar{n}}(t, \omega) < \infty$$

in contradiction to the definition of $\boldsymbol{m}_n$.

Therefore lemma 5.6 is proved.

Lemma 5.7. *With probability one:*

$$c_n(t, \omega) > 0 \ \textit{for all } t \textit{ and } n. \tag{5.42}$$

Proof: It is an obvious corollary of lemma 5.6.

Proof of Theorem 5.1. By the preceding lemmas it only remains to prove the assertions of continuity and uniqueness.

The proof of the uniqueness of the representation is similar to a proof given by S. BOCHNER [*3*].

The uniqueness of $\sigma(t, \omega)$ follows from the fact that, by (5.33), (5.34):

$$\int_{0^+}^{\pi} [1 - P_n(\cos\vartheta)]\, \nu([0, t]\times d\vartheta, \omega) = o(n^2)\,. \tag{5.43}$$

The uniqueness of $\dot{\nu}([0, t]\times d\vartheta, \omega)$ (and hence of $\nu([0, t] x d\vartheta, \omega)$) follows from the fact that any polynomial in x is a linear combination with constant coefficients of the polynomials of degree $n - 1\, (1 - P_n(x)) \,|\, (1 - x)$ and from the classical theorem on the moment problem.

To prove the continuity of $\sigma(t,\omega)$ it will be first shown that $\lim_{t\downarrow s}\sigma(t,\omega)=\sigma(s,\omega)$.

By the continuity of $f_n(t,\omega)$

$$\lim_{t\uparrow s} f_n(t,\omega)=f_n(s,\omega). \tag{5.44}$$

By the monotony of $\sigma(t,\omega)$ and $\dot\nu([0,t]\times d\vartheta,\omega)$:

$$\sigma(t,\omega)\downarrow\bar\sigma(s,\omega) \text{ as } t\downarrow s. \tag{5.45}$$

$$\dot\nu([0,t]\times d\vartheta,\omega)\downarrow\bar{\dot\nu}([0,s]\times d\vartheta,\omega) \text{ as } t\downarrow s. \tag{5.46}$$

By (5.35), (5.44), (5.45), (5.46) and the uniqueness of the representation (5.35):

$$\bar\sigma(s,\omega)=\sigma(s,\omega)+\bar{\dot\nu}([0,s]\times(0),\omega). \tag{5.47}$$

But:

$$\bar{\dot\nu}([0,s]\times(0),\omega)\leqq\lim_{\varepsilon\downarrow 0}\overline{\lim_{t\downarrow s}}\int_{0^+}^{\varepsilon}\dot\nu([0,t]\times d\vartheta,\omega). \tag{5.48}$$

From (5.48) and the monotony of $\dot\nu$ it follows, for $s'>s$:

$$\bar{\dot\nu}([0,s]\times(0),\omega)\leqq\lim_{\varepsilon\downarrow 0}\bar{\dot\nu}([0,s']\times d\vartheta,\omega)=0. \tag{5.49}$$

From (5.45), (5.47), (5.49) $\lim_{t\downarrow s}\sigma(t,\omega)=\sigma(s,\omega)$.

In a similar way it can be shown that $\lim_{t\uparrow s}\sigma(t,\omega)=\sigma(s,\omega)$. Thus the continuity of $\sigma(t,\omega)$ follows.

To prove that $\dot\nu([0,t]\times d\vartheta,\omega)$ is weak-star-continuous in t notice that

$$\int_{0^+}^{\pi}\frac{1-P_n(\cos\vartheta)}{1-\cos\vartheta}\,\dot\nu([0,t]\times d\vartheta,\omega)$$

is continuous in t, as it is the difference between $f_n(t,\omega)$ and $1/2n(n+1)\sigma(t,\omega)$ which have been proved to be continuous.

As any continuous function $f(x)$ in $(-1,1]$ can be approximated by polynomials in x, and any polynomial in x is a linear combination with constant coefficients of the polynomials $\frac{1-P_n(x)}{1-x}$ of degree $n-1$, it clearly follows that $\dot\nu([0,t]\times d\vartheta,\omega)$ is weak-star-continuous in t. The proof of theorem 5.1 is complete.

6. Construction of a spherical Brownian motion with interlarded Poisson jumps

Consider a non-negative, continuous, non-decreasing function $\sigma(t)$, and a non-negative measure $\nu(dt\times d\vartheta)$ on $[0,\infty)x(0,\pi]$ satisfying:

$$\int_{0^+}^{\pi}(1-\cos\vartheta)\,\nu([0,t]\times d\vartheta)<\infty \text{ for each } t\geqq 0, \tag{6.1}$$

$$(1-\cos\vartheta)\,\nu([0,t]\times d\vartheta) \text{ weak-star-continuous.}$$

This section is devoted to the construction of a sequence of isotropic processes $(\psi_m(t,\omega'))$ on the unit sphere S^2, with the following properties:

(i) Each $\psi_m(t, \omega')$ can be described, in the sense explained below, as a spherical Brownian motion run with a clock $\sigma(t)$, with interlarded Poisson jumps (a finite number for each path).

(ii) The characteristic functionals satisfy:

$$\lim_{m\to\infty} E_N(P_n(\psi_m(t, \omega'))) = \exp[-1/2\, n(n+1)\,\sigma(t) - \int_{0+}^{\pi}(1 - P_n(\cos\vartheta))\,\nu([0, t]\times d\vartheta)]. \tag{6.2}$$

(iii) With probability one, $(\psi_m(t, \omega'))$ converges uniformly on compact time intervals.

The following steps lead to the construction:

(a) Let $\boldsymbol{\vartheta}(\sigma(t))$ be the spherical Brownian motion run with the clock $\sigma(t)$, the same for all paths. First interlard in $\boldsymbol{\vartheta}(\sigma(t))$ one single (isotropic) jump of „length" ϑ at time s. A process, say $\psi'(t, \omega')$ is thus obtained.

What is meant is this: the process $\boldsymbol{\vartheta}(\sigma(t))$ is unchanged for $t < s$. At time s, for each fixed path ω', the process "jumps" isotropically from $\boldsymbol{\vartheta}(\sigma(s), \omega')$ to any point whose angular distance from $\boldsymbol{\vartheta}(\sigma(s), \omega')$ is ϑ. After time s the spherical Brownian motion goes on from $\psi'(s + 0, \omega')$.

Hence, similarly to theorem 3.2:

$$E_N(P_n(\psi'(t, \omega'))) = \exp(-1/2\, n(n+1)\,\sigma(t)), \text{ for } t < s. \tag{6.3}$$

At time s there is a jump that can be described for fixed ω', as gk where g is a fixed rotation of length ϑ from $\boldsymbol{\vartheta}(\sigma(s), \omega') = \psi'(s - 0, \omega')$, and k is a rotation around $\psi'(s - 0, \omega')$. Let dk be the Haar measure defined on K (the set of all rotations k).

Then by lemma 5.1:

$$E_N(P_n(\psi'(s, \omega'))) = E_N(\int_K P_n(g\,k\,\psi'(s - 0, \omega'))\,dk) = P_n(\cos\vartheta)\,E_n(P_n(\psi'(s-0))) = P_n(\cos\vartheta)\exp(-1/2\, n(n+1)\,\sigma(s)). \tag{6.4}$$

Clearly:

$$E_N(P_n(\psi'(t, \omega'))) = P_n(\cos\vartheta)\exp(-1/2\, n(n+1)\,\sigma(t)). \tag{6.5}$$

for $t > s$.

(b) Analogously m_i jumps of lengths ϑ_i, respectively, for $i = 1, \ldots, k$ can be interlarded. Let the resulting process be $\psi''(t, \omega')$. Then:

$$E_N(P_n(\psi''(t, \omega')) = \exp(-1/2\, n(n+1)\,\sigma(t))\prod_{i=1}^{k}[P_n(\cos\vartheta_i)]^{m_i}. \tag{6.6}$$

(c) Let (q_m) be a decreasing sequence of positive numbers such that

$$q_m \downarrow 0 \text{ as } m \to \infty. \tag{6.7}$$

Define a sequence $(\psi_m(t, \omega'))$ from $\boldsymbol{\vartheta}(\sigma(t))$ by interlarding in each path a finite number of isotropic independent Poisson jumps with measure $\nu_m(dt\times d\vartheta)$ given by:

$$\nu_m(dt \times B) = \nu(dt \times B \cap (q_m, \infty)] \tag{6.8}$$

for each Borel set $B \subset (0, \pi]$.

It will be proved that (ψ_m) satisfies properties (i), (ii), (iii) stated at the outset in Sec. 6.

(i) is already clear.

The proof of (ii) and (iii) deserves separate theorems.

Theorem 6.1. *The sequence* (ψ_m) *satisfies* (6.2).

Proof: For a fixed path ω', let $h([0,t]\times d\vartheta, \omega')$ be the number of jumps occurring before time t, whose length is between ϑ and $\vartheta + d\vartheta$.

The identity

$$E_N(P_n(\psi_m(t))) = E_N[P_n(\psi_m(t))\,|\,h([0,t]\times d\vartheta)]) \tag{6.9}$$

will be used in the following argument.

Suppose that the fixed ω' has before time t jumps of lengths $\vartheta_1, \ldots, \vartheta_k$ at times $t_1 < \cdots < t_k$ respectively.

Then, for that fixed ω', similarly as in (a):

$$E_N(P_n(\psi_m(t))\,|\,h|_{\omega'} =$$
$$E_N(P_n(\psi_m(t-t_k-0))\,|\,h)|_{\omega'}\,P_n(\cos\vartheta_k)\exp(-1/2\,n(n+1)\,\sigma(t-t_k)).$$

After k similar steps

$$E_N(P_n(\psi_m(t))\,|\,h)|_{\omega'} = \exp[-1/2\,n(n+1)\,\sigma(t)]\prod_{i=1}^{k} P_n(\cos\vartheta_i) = \tag{6.10}$$
$$\exp(-1/2\,n(n+1)\,\sigma(t)\prod[P_n(\cos\vartheta)]^{h([0,t]\times(\vartheta),\omega')}.$$

From (6.9) and (6.10):

$$E_N(P_n(\psi_m(t))) = \exp[-1/2\,n(n+1)\,\sigma(t)]\,E_N(\prod(P_n(\cos\vartheta))^h). \tag{6.11}$$

If $(a_m, \pi]$ is divided into intervals of length Δ the following computation can be performed:

$$E_N(\prod(P_n(\cos\theta))^h) = E_N(E_N[\prod P_n(\cos\vartheta))^h\,|\,h([0,t]\times(q_m, q_m+\Delta]) = \tag{6.12}$$
$$\sum_{p=0}^{\infty}(p')^{-1}\exp(-\nu([0,t]\times(q_m, q_m+\Delta]))\,[\nu_m([0,t]\times$$
$$(q_m, q_m+\Delta)]^p\times[P_n(\cos\vartheta')]^p\,E_N((P_n(\cos\vartheta))^{h'})$$

where $q_m < \vartheta' < q_m + \Delta$ and $h'([0,t]\times A) = h([0,t]\times A \cap (q_m+\Delta, \pi])$ for all Borel sets $A \subset (q_m, \pi]$.

By repeating the procedure in (6.12)

$$E_N(\prod P_n(\cos\vartheta))^h) = \prod_i\sum_{p=0}^{\infty}(p')^{-1}\exp[-\nu[0,t]\times(q_m+(i-1)\Delta, \tag{6.13}$$
$$q_m+i\Delta))]\times(P_m(\cos\vartheta^i))^p\,(\nu_m([0,t]\times(q_m+(i-1)\Delta, q_m+i\Delta))^p =$$
$$\prod_i\exp[-(1-P_n(\cos\vartheta^i))\,\nu_m([0,t]\times(q_m+(i-1)\Delta, q_m+i\Delta))]$$

where $q_m + (i-1)\Delta < \vartheta^i < q_m + i\Delta$.

From (6.13) it is clear that, by letting $\Delta \to 0$

$$E_N((P_n(\cos\vartheta))^h) = \exp(-\int_{0^+}^{\pi}(1-P_n(\cos\vartheta))\,\nu_m([0,t]\times d\vartheta)). \tag{6.14}$$

From (6.11) and (6.14) it easily follows that (ψ_m) satisfies (6.2) as stated.

Theorem 6.2. *There is a sequence* $(\psi_m(t, \omega'))$ *such as described which, with probability one, converges uniformly on compact time intervals.*

Proof: Let

$$A_t = \int_{0+}^{\pi} (1 - \cos\theta)\,\nu([0, t] \times d\theta) < \infty. \tag{6.15}$$

Define (q_m) by the conditions:

$$\int_{qm+1}^{qm} (1 - \cos\vartheta)\,\nu([0, t] \times d\vartheta) = 2^{-m} A_t; \qquad q_1 = \pi. \tag{6.16}$$

Clearly (q_m) satisfies (6.13). It will be proven that the corresponding (ψ_m) has the required property.

First it is necessary to have the same sample space for all the terms of the sequence (ψ_m). To achieve this an appropriate ordering to the jumps occurring in each sample path must be given. As for each ψ_m each sample path has only a finite number of jumps it suffices the following criterion: A sample path of ψ_m is described by interlarding, in the corresponding sample path of $\vartheta(\sigma(t))$, successive jumps so that:

(I) if two jumps have different lengths, the one with greater length is interlarded first.

(II) if two jumps have the same length, the one occurring first in time is interlarded first.

Clearly the sample space thus obtained can be considered for all ψ_m.

The following remark is important for the proof:

(6.17) $\psi_m(t, \omega')$ is a Martingale with respect to t.

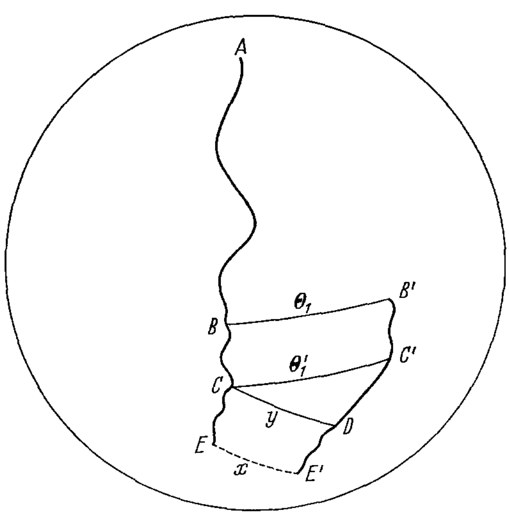

Fig. 1. (corresponding to theorem 6.2.)

Actually, (6.17) is an immediate consequence of the isotropy of the process.

Hence also $\psi_m(t) - \psi_n(t)$ is a Martingale and satisfies the inequality:

$$(6.18) \qquad P_N(\sup_{s \leqq t}(|\psi_m(s) - \psi_n(s)|) > 2^{-n/3}) < (2^{n/3})^2 E_N((|\psi_m(t) - \psi_n(t)|)^2) \text{ for } m > n.$$

In order to estimate $E_N((|\psi_m(t) - \psi_n(t)|)^2)$, consider a fixed path in the process $\psi_n(t)$, say ω_n'. Let ω_m' from the sample space of $\psi_m(t)$ be the path that is made to correspond to ω_n'.

Both paths have the same jumps with length bigger than q_m occurring at the same time. But ω_m' has a finite number of jumps whose length is between q_m and q_n and ω_n' has no such jumps.

Assume first that ω_n' has only two jumps between q_m and q_n (and possibly other jumps with length bigger than q_n) before time t.

In Figure 1 the path $ABCE$ represents ω_n' and $ABB'C'DE'$ represents ω_m'. Both paths coincide up to the first jump of length ϑ_1, between q_m and q_n in ω_m', which occurs at B. From then until the next jump ω_n' describes BC and ω_m' describes $B'C'$. Clearly if the geodesic BB' is considered to be a part of the equator, then C and C' have the same latitude and keep the same difference of longitude as exists between B and B'. So:

$$(6.19) \qquad \vartheta_1' \leqq \vartheta_1.$$

After the second such jump of length ϑ_2 at time t_2, the angular distance between both paths is y. From t_2 to t ω_n' describes CE and $\omega_m' DE'$. At time t the angular distance is x and, by the same argument that yielded (6.19):

$$(6.20) \qquad x \leqq y.$$

Hence

$$(6.21) \qquad (|\psi_m(t, \omega_m') - \psi_n(t, \omega_n)|)^2 = 4\sin^2 x/2 \leqq 4\sin^2 y/2 = 2(1 - \cos y).$$

In the spherical triangle $CC'D$:

$$(6.22) \qquad \cos y = \cos\vartheta_1 \cos\vartheta_2 + \sin\vartheta_1 \sin\vartheta_2 \cos\varphi.$$

The expectation over the paths such as ω_m', with only two jumps of lengths ϑ_1, ϑ_2 between q_n and q_m can be estimated as follows:

$$E_N^{(2)}((|\psi_m(t) - \psi_n(t)|)^2) \leqq E_N^{(2)}[2(1 - \cos\vartheta_1\cos\vartheta_2 - \sin\vartheta_1\sin\vartheta_2\cos\varphi)] =$$
$$= 2 - 2\cos\vartheta_1\cos\vartheta_2 = 4[\sin^2(1/2\,\vartheta_1) + \sin^2(1/2\,\vartheta_2) \leqq 4[\sin^2(1/2\,\vartheta_1) +$$
$$+ \sin^2(1/2\,\vartheta_2)]. \qquad (\text{for } E_N^{(2)}(\cos\varphi) = 0)$$

Considering, by induction, all paths with k jumps before time t with lengths $\vartheta_1, \ldots, \vartheta_k$ between q_n and q_m

$$E_N^{(k)}((|\psi_m(t) - \psi_n(t)|)^2) \leqq 4\sum_{i=1}^{k}\sin^2(1/2\vartheta_i).$$

And in general:

$$E_N((|\psi_m(t) - \psi_n(t)|)^2) \leqq 4 \int_{q_m}^{q_n} \sin^2(1/2\,\vartheta)\,\nu([0,t]\times d\vartheta) = 2\int_{q_m}^{q_n}(1-\cos\vartheta)\,\nu([0,t]\times d\vartheta). \tag{6.23}$$

From the estimate (6.23) and (6.16), (6.18)

$$P_N(\sup_{s\leqq t}(|\psi_m(s) - \psi_n(s)|) > 2^{-n/3}) < 2^{2n/3+1}\sum_{i=0}^{m-n-1} A_t\, 2^{-(n+i)} = A_t\, 2^{-n/3+1}\sum_{i=0}^{m-n-1} 2^{-i}. \tag{6.24}$$

Adding the right hand side of (6.24) over all n,

$$\sum_{n=1}^{\infty} A_t\, 2^{-n/3+1}\sum_{i=0}^{m-n-1} 2^{-i} < 2A_t \sum_{i=1}^{\infty}\sum_{n=1}^{\infty} 2^{-(i+n/3)} < B_t < \infty, \tag{6.25}$$

where B_t is independent of m.

By the Borel-Cantelli lemma it follows that $\psi_m(t)$ converges uniformly on compact time intervals with probability one, as stated.

7. Final result

From the results of Sec. 6 and the properties of the process $\boldsymbol{\varphi}(t, \omega')$ in Sec. 5 the following theorem is obtained:

Theorem 7.1. *For each fixed* ω *there is a sequence* $(\psi_m^\omega(t, \omega'))$ *such as described in Sec.* 6(c) *with the following properties:*

(i) $\lim_{m\to\infty} E_N(P_n(\psi_m^\omega(t, \omega')) = E_N^\omega(P_n\boldsymbol{\varphi}((t, \omega'))$

holds with probability one for each t.

(ii) ψ_m^ω *converges with probability one uniformly on compact time intervals to some* $\psi^\omega(t, \omega')$.

From theorem 7.1 the following fundamental corollary can be inferred:

Corollary 7.1.1. *The measure* $\nu([0, t]\times d\vartheta, \omega)$ *in* (5.1) *vanishes identically with* ω*-probability one.*

Proof: From theorem 7.1 and the results of the preceding sections

$$E_N^\omega(P_n(\boldsymbol{\varphi}(t, \omega'))) = E_N(P_n(\psi^\omega(t, \omega'))) = \exp[-1/2\, n(n+1)\,\sigma(t,\omega) - \int_{0^+}^{\pi}(1 - P_n(\cos\vartheta))\,\nu([0,t]\times d\vartheta, \omega)] \tag{7.1}$$

with probability one for each t.

By (7.1) and (ii) in theorem 7.1 it is clear that:

$$P_N(\nu([0,t]\times d\vartheta, \omega) \equiv 0) = P_N\ (\psi^\omega(t, \omega') \text{ has no jumps}). \tag{7.2}$$

From the uniqueness of characteristic functionals (7.1) yields the equality of the probability distributions:

(7.3) $$P_N^\omega(\boldsymbol{\varphi}(t,\omega') \in d\varphi) = P_N(\psi^\omega(t) \in d\varphi), \qquad d\varphi \subset S^2,$$

for fixed ω, with ω-probability one.

By the Markov property (4.5) of $\boldsymbol{\varphi}(t, \omega')$, the equation (7.3) and the isotropy of both processes which permits the replacement of N by any other point on S^2,

(7.4) $$P_N^\omega[\boldsymbol{\varphi}(t_1) \in d\varphi_1, \ldots, \boldsymbol{\varphi}(t_k) \in d\varphi_k] = P_N[\psi^\omega(t_1) \in d\varphi_1, \ldots, \psi^\omega(t_k) \in d\varphi_k]$$

on a set of probability one depending on $t_1, \ldots, t_k$.

The indicator functions of the events ($\boldsymbol{\varphi}(t)$ has no jumps) and (ψ^ω has no jumps) can be approximated by linear combinations of indicators of events as in (7.4), using only a countable number of time values.

Hence:

(7.5) $$P_N(\psi^\omega(t) \text{ has no jumps} = P_N^\omega(\boldsymbol{\varphi}(t) \text{ has no jumps})$$

with ω-probability one.

But

(7.6) $$P_N(\boldsymbol{\varphi}(t) \text{ has no jumps}) = P_N[P_N^\omega(\boldsymbol{\varphi}(t) \text{ has no jumps}] = 1$$

by the continuity assumptions (iii) and (v) at the outset of Sec. 1.

Then (7.2), (7.5), (7.6) prove the corollary.

Corollary 7.1.2. *The two processes in* (1.2) *are equivalent.*

Proof: Corollary 7.1.1 together with theorem 5.1 completes the proof of (3.21) and thus yields the equivalence (1.2).

Final remarks. The results promised in a) b) and c) of Sec. 1 are achieved by Lemma 3.2, Corollary 7.1.2 and Theorem 5.1 (i), respectively.

The equivalence thus proved can be obviously generalized to any Euclidean space. The proof for more general spaces should need a less special argument than the one used in theorem 6.2 for the uniform convergence of the sequence (ψ_m).

It might be possible to remove assumption (ii) in Sec. 1 about homogeneity in time, by considering the process $(\boldsymbol{x}(t, \omega \times \omega'), t)$ which has state space $R^3 \times [0, \infty)$ and is homogeneous in time.

Assumption (v) is essential for the clock $\sigma(t, \omega)$ to be finite with probability one. The discussion of boundary conditions at the origin constitutes a topic for further research.

Formula (5.1) should have a generalization for homogeneous spaces. It is believed that, if the space is compact, an extension of the method used in theorem 6.2 might be possible.

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Massachusetts Institute of Technology
Northeastern University
and University of Buenos Aires

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