Z. Wahrscheinlichkeitstheorie 1, 340—346 (1963)

Convergence Theorems of Martingales

By

Y. S. Chow

Let $(\Omega, \mathscr{F}, P)$ be a probability space and $(\mathscr{F}_n)$ be a sequence of sub-σ-algebras of $\mathscr{F}$ and $\mathscr{F}_n \subset \mathscr{F}_{n+1}$ for each n. If the random variables x_n are measurable with respect to $\mathscr{F}_n$, then $(x_n, \mathscr{F}_n, n \geqq 1)$ is said to be a stochastic sequence. If moreover $E(x_n^+) < \infty$ and $E(x_{n+1} | \mathscr{F}_n) \geqq x_n$ a.e., then $(x_n, \mathscr{F}_n, \geqq 1)$ is called a submartingale (or semi-martingale). An extended real valued random variable t is called a stopping variable, if $\{t(\omega) = n\} \in \mathscr{F}_n$ for each n. There are two kinds of convergence theorems of submartingales. One requires $\sup E|x_n| < \infty$ and was first proved by Doob [*2*; p. 324] and later improved by Snell [*4*; p. 298]. Another requires $\limsup x_n < \infty$ with some conditions on $\mathscr{F}_n$ and was proved in [*1*; p. 274] and later improved by Doob [*3*]. In this paper, following the line of Doob in [*3*], we prove Theorem 1, which unifies all the results mentioned above together. For the completeness of the paper, we prove Theorem 3, which has been implicitly included in the proof of Theorem 4.2 of [*1*] (but not stated), and it is not included in Theorem 1. Theorem 4 is a generalization of a Theorem proved by Doob [*2*; p. 339] about the sums of independent random variables.

Let $(x_n, \mathscr{F}_n, n \geqq 1)$ be a stochastic sequence and for $b > 0$ put $s(\omega) = \inf\{n | x_n(\omega) \geqq b\}$. For a set A, the indicator function of A will be denoted by I_A.

Definition 1. For $a < 0$, the stochastic sequence $(x_n, \mathscr{F}_n, n \geqq 1)$ is said to satisfy the condition (C, a), if $E(x_n) < \infty$ for each n, if there are random variables $y_n \geqq b$ such that $\int_{(s<\infty)} y_s \, dP < \infty$, and if for each n

$$E(I_{(s=n+1)}(x_s - y_s) | \mathscr{F}_n) \leqq E(\max(a, x_{n+1}) | \mathscr{F}_n) - x_n \tag{1}$$

a.e. in $\{s(\omega) \geqq n+1, x_n(\omega) > a\}$.

Assume that the condition (C, a) is satisfied. Put

$$x'_{n,a} = \max(a, x_n) \quad \text{if} \quad s(\omega) > n \tag{2}$$
$$= y_s \qquad\qquad \text{if} \quad s(\omega) \leqq n .$$

Then $E(x'_{n+1} | x'_1, \ldots, x'_n) \geqq x'_n$ a.e. To prove this, let B be an n-dimensional Borel set and $A = \{(x'_1, \ldots, x'_n) \in B\}$. Then

$$A(s > n) = \{[\max(a, x_1), \ldots, \max(a, x_n)] \in B\}(s > n) \in \mathscr{F}_n .$$

Set $D = A(x_n > a)$ and $E = A(x_n \leqq a)$. Then by (1)

$$\int_D (x'_{n+1} - x'_n)\,dP = \int_{D(s>n)} [\max(a, x_{n+1}) - x_n]\,dP - \int_{D(s=n+1)} (x_{n+1} - y_{n+1})\,dP$$

$$= \int_{D(s>n)} [\max(a, x_{n+1}) - x_n]\,dP - \int_{D(s>n)} I_{(s=n+1)}(x_{n+1} - y_{n+1})\,dP$$

$$= \int_{D(s>n)} [\max(a, x_{n+1}) - x_n]\,dP - \int_{D(s>n)} E(I_{(s=n+1)}(x_{n+1} - y_{n+1})|\mathscr{F}_n)\,dP \geqq 0\,,$$

$$\int_E (x'_{n+1} - x'_n)\,dP = \int_{E(s>n)} (x'_{n+1} - a)\,dP \geqq 0\,,$$

since $x'_{n+1} \geqq a$ a.e. Therefore, $\int_A (x'_{n+1} - x'_n)\,dP \geqq 0$, and then $E(x'_{n+1}|x'_1, \ldots, x'_n) \geqq x'_n$ a.e. Thus we have

Lemma 1. *Under the condition (C, a), the sequence (x'_n) defined by (2) is a submartingale and* $|x'_n| \leqq b - a + I_{(s\leqq n)}\,y_s$.

Theorem 1. *Let $b > 0$ be fixed. If (C, a) is satisfied by $(x_n, \mathscr{F}_n, n \geqq 1)$ for every $a < 0$, then $\lim x_n$ exists* a.e. *in $\{\sup x_n(\omega) < b\}$. If, moreover, $E|x_n| < \infty$ for each n and $y_n = y'_n + y''_n$ such that there exists $K > 0$ satisfying*

$$\int_{(s<\infty)} |y'_s|\,dP \leqq K\,, \quad I_{(s<\infty)} y''_s \leqq -Ka \tag{3}$$

for every $a < -1$, then $\lim x_n$ exists and is finite a.e. *in $\{\sup x_n(\omega) < b\}$.*

Proof. Let $A = \{\sup x_n(\omega) < b\}$. By Lemma 1 and Doob's convergence theorem of submartingales [2; p. 324],

$$\lim x'_{n,a} = x'_a \quad \text{a.e. and} \quad \lim E(x'_{n,a}) = E(x'_a)$$

for every $a < 0$. Hence, $\lim x_n = x_\infty$ a.e. in A. To prove the second part, put $B = A\{x_\infty(\omega) = -\infty\}$ and let $P(B) = \varepsilon > 0$. Then

$$E(x'_a) = \int_{(s=\infty)} \max(a, x_\infty)\,dP + \int_{(s<\infty)} y_s\,dP \leqq a\,\varepsilon + b + \int_{(s<\infty)} y_s\,dP\,,$$

and for every m

$$\lim E(x'_{n,a}) \geqq E(x'_{m,a}) \geqq -E(x^-_m) + \int_{(s\leqq m)} y_s\,dP\,.$$

Hence,

$$a\,\varepsilon \geqq -b - E(x^-_m) - \int_{(m<s<\infty)} (y'_s + y''_s)\,dP \geqq -b - E(x^-_m) + Ka\,P(m < s < \infty) - K\,.$$

Take m_0 so large such that $P(m_0 < s < \infty) < \frac{\varepsilon}{2K}$. Then

$$\frac{a\,\varepsilon}{2} \geqq -b - E(x^-_{m_0}) - K\,.$$

Letting a tend to $-\infty$, we have a contradiction. Therefore $P(B) = 0$.

Definition 2. A stochastic sequence $(x_n, \mathscr{F}_n, n \geqq 1)$ is said to satisfy the condition (D, a) for $a < 0$, if $E(x_n) < \infty$ for each n, if there exist $1 > \delta > 0$ and random variables z_n such that $z_n \geqq b$ and $\int_{(s<\infty)} z_s\,dP < \infty$, and if a.e. in

$$\{s(\omega) \geqq n + 1, x_n(\omega) > a\}$$

at least one of the following two conditions holds:

$$(4') \qquad P(x_{n+1} \geqq b \mid \mathscr{F}_n) \geqq \delta,$$

$$(4'') \qquad E(I_{(s=n+1)}(x_{n+1} - z_{n+1}) \mid \mathscr{F}_n) \leqq E(\max(a, x_{n+1}) \mid \mathscr{F}_n) - x_n.$$

Theorem 2. *Under the condition* (D, a), *by putting* $y_n = \frac{b-a}{\delta} + z_n$, *the condition* (C, a) *is satisfied.*

Proof. Let

$$A = \{P(x_{n+1} \geqq b \mid \mathscr{F}_n)(\omega) \geqq \delta, s(\omega) \geqq n+1, x_n(\omega) > a\}.$$

To prove Theorem 2, we need only to prove that (1) holds a.e. in A. Now in A

$$\begin{aligned} E(I_{(s=n+1)}&(x_{n+1} - y_{n+1}) \mid \mathscr{F}_n) - E(\max(a, x_{n+1}) \mid \mathscr{F}_n) \leqq \\ &\leqq E(I_{(s=n+1)} x_{n+1} \mid \mathscr{F}_n) - b + a - a P(x_{n+1} \leqq a \mid \mathscr{F}_n) - \\ &- E(I_{(x_{n+1} > a)} x_{n+1} \mid \mathscr{F}_n) = - E(I_{(a < x_{n+1} < b)} x_{n+1} \mid \mathscr{F}_n) - \\ &- b + a - a P(x_{n+1} \leqq a \mid \mathscr{F}_n) \leqq - a P(a < x_{n+1} < b \mid \mathscr{F}_n) - \\ &- b + a P(x_{n+1} > a \mid \mathscr{F}_n) = a P(x_{n+1} \geqq b \mid \mathscr{F}_n) - b \leqq - \\ &- b \leqq - x_n. \end{aligned}$$

Therefore (1) holds a.e.in A.

Corollary 1. *If* (D, a) *is satisfied for every* $a < 0$, *then* $\lim x_n$ *exists a.e. in* $\{\sup x_n(\omega) < b\}$. *If moreover* δ *is independent of* a *and* $z = z_n' + z_n''$ *such that there exists* $K > 0$ *satisfying*

$$\int_{(s<\infty)} |z_s'| \, dP \leqq K, \quad I_{(s>\infty)} z_s'' \leqq -Ka$$

for every $a < -1$, *then* $\lim x_n$ *exists and is finite* a.e. *in* $\{\sup x_n(\omega) < b\}$.

Proof. The first part follows from Theorems 1 and 2 immediately. For the second part, $y_n = y_n' + y_n''$, where $y_n' = z_n'$ and $y_n'' = \frac{b-a}{\delta} + z_n''$. Therefore (3) holds for $K' = K + \frac{b+1}{\delta}$. By Theorem 1 again, we finish the proof.

Corollary 2. *Let* $(x_n, \mathscr{F}_n, n \geqq 1)$ *be a submartingale with* $E(x_n) < \infty$ *for each* n. *If for* $b > 0$ *there exists* $c > b$ *such that for every* $a < 0$ *there exists* $1 > \delta > 0$ *so that for almost every point for which* $\max(x_j, j \leqq n) < b$ *and* $x_n \geqq a$ *we have at least one of the following two properties:*

$$(5') \qquad P(x_{n+1} \geqq b \mid \mathscr{F}_n) \geqq \delta,$$

$$(5'') \qquad P(x_{n+1} \geqq c \mid \mathscr{F}_n) = 0,$$

then $\lim x_n$ *exists* a.e. *in* $\{\sup x_n(\omega) < b\}$. *If, moreover, the* δ *is independent of* a *and* $E|x_n| < \infty$ *for each* n, *then* $\lim x_n$ *exists and is finite* a.e. *in* $\{\sup x_n(\omega) < b\}$.

Proof. Define $s(\omega) = \inf\{k \mid x_k(\omega) \geqq b\}$ and put $z_n = c$ for each n. Let

$$A = \{P(x_{n+1} \geqq c \mid \mathscr{F}_n) = 0, s \geqq n+1, x_n \geqq a\}.$$

Then a.e. in A,

$$E(I_{(s=n+1)}(x_{n+1}-z_{n+1})|\mathscr{F}_n) = E(I_{(s=n+1)}(x_{n+1}-c)|\mathscr{F}_n) \leqq$$
$$\leqq E(I_{(x_{n+1}\geqq c)}((x_{n+1}-c)|\mathscr{F}_n) = \lim_{m\to\infty} E(I_{(m\geqq x_{n+1}\geqq c)}(x_{n+1}-c)|\mathscr{F})$$
$$= 0 \leqq E(\max(a, x_{n+1})|\mathscr{F}_n) - x_n.$$

Hence, (4″) holds a.e. in A. Thus, Corollary 2 follows from Corollary 1.

The second half of Corollary 2 has been recently proved by DOOB [3]. To show that if δ is not independent of a, we do not always have the finiteness of the lim x_n, even if $E|x_n| < \infty$ for each n, let $\Omega = (0, 1]$ be the half-open unit interval, $\mathscr{F}$ the class of all Lebesgue measurable sets on Ω, and P the Lebesgue measure.

Let $I_{1,1} = (0, d_{1,1}]$ and $I_{1,2} = (d_{1,1}, 1]$ such that $P(I_{1,1}) = 2P(I_{1,2})$. Define $x_1(\omega) = -1$ for $\omega \in I_{1,2}$. Then $E(x_1) = 0$. Suppose that we have defined $I_{n,1}$, $I_{n,2}, \ldots, I_{n,2}$ n and x_n. For

$$I_{n,i} = (d_{n,i-1}, d_{ni}], \text{ choose } d_{n,i-1} < d_{n+1,2i-1} < d_{n,i},\ I_{n+1,2i-1} =$$
$$= (d_{n,i-1}, d_{n+1,2i-1}) \text{ and } I_{n+1,2i} = (d_{n+1,2i-1}, d_{n,i}]$$

such that

$$2^n P(I_{n+1,2i}) = nP(I_{n+1,2i-1}) + \int_{I_{n,i}} x_n dP.$$

Define $x_{n+1}(\omega) = -n$ for $\omega \in I_{n+1,2i-1}$ and $x_{n+1}(\omega) = 2^n$ for $\omega \in I_{n+1,2i}$, $i = 1, 2, \ldots, 2^n$. Then (x_n) is a martingale and

$$P(x_n = -n) = \frac{2^n}{2^n + n}, \qquad P(x_n = 2^n) = \frac{n}{2^n + n}.$$

For $b > 0$, it is easy to see that for every $a < 0$ for each n either (5′) or (5″) is satisfied, by taking $\delta = \frac{1}{4}$ and $c = b$. But we have $P(\lim x_n = -\infty) = 1$.

From Corollary 2, immediately follow the next two corollaries:

Corollary 3. (CHOW [1]). *Let $(x_n, \mathscr{F}_n, n \geqq 1)$ be a submartingale such that for each $n, E|x_n| < \infty$ and*

(6) $\qquad \mathscr{F}_n$ *is generated by disjoint atoms $I_{n,i}$, $i = 1, 2, \ldots$*

(7) $$\inf\left\{\frac{P(I_{n+1,j})}{P(I_{n,i})}\ \Big|\ i, j, n, I_{n+1,j} \subseteqq I_{n,i}\right\} = \delta > 0.$$

Then lim x_n *exists and if finite* a.e. *in* $\{\limsup x_n(\omega) < \infty\}$.

Corollary 4. *Let $z_1, z_2, \ldots$ be independent random variables with mean zero and $x_n = z_1 + \cdots + z_n$. If for every large b there exists $\delta > 0$ such that for each n either*

(8′) $$P(z_n \geqq b) = 0, \qquad or$$

(8″) $$P(z_n \geqq b) \geqq \delta,$$

then lim x_n *exists* a.e. *in* $\{\limsup x_n(\omega) < \infty\}$.

Corollary 5. (DOOB [2; p. 325]). *Let $(x_n, \mathscr{F}_n, n \geqq 1)$ be a submartingale and $E|x_n| < \infty$ for each n. If $E[\sup\{x_{n+1} - E(x_{n+1}|\mathscr{F}_n)\}] < \infty$, then* lim x_n *exists and is finite* a.e. *in* $\{\limsup x_n(\omega) < \infty\}$.

Proof. For $b > 0$, put $y_{n+1} = \{x_{n+1} - E(x_{n+1}|\mathscr{F}_n)\}^+ + b$ for $n \geqq 1$ and $y_1 = x_1^+ + b$. Define $s(\omega) = \inf\{k|x_k(\omega) \geqq b\}$. Then a.e. in $\{s(\omega) \geqq n+1\}$ we have

$$E\{(I_{(s=n+1)}(x_{n+1} - y_{n+1})|\mathscr{F}_n\} \leqq E\{I_{(s=n+1)} E(x_{n+1} - b|\mathscr{F}_n)|\mathscr{F}_n\} = \\ = E(I_{(s=n+1)}|\mathscr{F}_n) E(x_{n+1} - b|\mathscr{F}_n) \leqq E(x_{n+1} - x_n|\mathscr{F}_n).$$

Hence (1) is satisfied. Since $\int_{(s<\infty)} y_s dP < \infty$, the Corollary follows from Theorem 1.

Corollary 6. *Let* $(x_n, \mathscr{F}_n, n \geqq 1)$ *be a submartingale.* (i) *if* $\int_{(t<\infty)} x_t^+ dP < \infty$ *for every stopping variable t, then* $\lim x_n$ *exists (and is finite) a.e. in* $\{\limsup x_n(\omega) < \infty\}$, *(provided that* $E(x_1) > -\infty$*).* (ii) (Doob [2; p. 324]) *if* $\sup E(x_n^+) < \infty$, *then* $\lim x_n$ *exists a.e. and* $P(\lim x_n < \infty) = 1$. *If moreover,* $E(x_1) > -\infty$, *then* $\lim x_n$ *is finite a.e.* (iii) (Snell [4; p. 293]) $\lim x_n$ *exists and is* $< \infty$ *for almost all* ω *such that* $\inf_k \sup_n E(x_n^+|\mathscr{F}_k) < \infty$.

Proof. For $b > 0$, put $y_n = x_n^+ + b$. Then $\int_{(s<\infty)} y_s dP < \infty$ in case of (i), since s is a stopping variable. Evidently (1) is satisfied and then (i) follows from Theorem 1. For (ii), we need only to prove that $P(\limsup x_n < \infty) = 1$, because the condition $\sup E(x_n^+) < \infty$ implies that $\int_{(t<\infty)} x_t^+ dP < \infty$ for every stopping variable t [1; p. 266]. By the martingale inequality [2; p. 314]

$$c P\{\max x_j^+ \geqq c, j \leqq n\} \leqq E(x_n^+)$$

for every $c > 0$. Hence

$$c P\{\sup x_n^+ \geqq c\} \leqq \sup E(x_n^+) < \infty.$$

Therefore, $P(\sup x_n < \infty) = 1$. To prove (iii), let $A = \{\sup_n E(x_n^+|\mathscr{F}_k) < N\}$, where N is a positive number. Put $y_n = I_A x_n$. Then $(y_n, \mathscr{F}_n, n \geqq k)$ is a submartingale and $\sup E(y_n^+) = \sup E(y_n^+|\mathscr{F}_k)) \leqq N$. Hence by (ii) $P(\lim y_n = y < \infty) = 1$. Since N and k are arbitrary, $\lim x_n = x$ exists and $x < \infty$ for almost all ω such that for some k $\sup_n E(x_n^+|\mathscr{F}_k) < \infty$. (ii) is slightly more general than Doob's original one, but the former can be deduced immediately from the latter by considering $x_n' = \max(c, x_n)$ for $c < 0$. (ii) is a special case of (iii) by taking $F_0 = \{0, W\}$ and $x_0 = E(x_1)$. (i) can be strengthened to the following theorem, which has been implicitly proved in [1; p. 270] and for the completeness of this paper we will repeat the proof here.

Theorem 3. *Let* $(x_n, \mathscr{F}_n, n \geqq 1)$ *be a submartingale such that* $\int_{(t<\infty)} x_t^+ dP < \infty$ *for every stopping variable t. Then* $\lim x_n$ *exists a.e.*

Proof. Put $V = \{\limsup x_n(\omega) > b > a > \liminf x_n(\omega)\}$ and suppose, if possible, that $P(V) \geqq \delta > 0$. For simplicity we may assume that $a = 0$ and $b = 1$. Put $K_n = \{x_n(\omega) > 1\}$. Then for $\varepsilon > 0$ there exists n_1 such that

$$P(V - A_1) < \varepsilon/2,$$

where $A_1 = \bigcup_1^{n_1} K_n$. Put $J_n = \{x_n(\omega) < 0\} A_1$. Then there exists $m_1 \geqq n_1$ such that

$$P(V A_1 - B_1) < \varepsilon/2,$$

where $B_1 = \bigcup_{n_1}^{m_1} J_n$. Define t_1 and t_2 by

$$\begin{aligned} t_1(\omega) &= n, \text{ for } \omega \in K_n = \bigcup_1^{n-1} K_i, \qquad n = 1, 2, \ldots, n_1, \\ &= n_1, \text{ for } \omega \in \Omega - A_1. \\ t_2(\omega) &= n, \quad \text{for } \omega \in A_1 - B_1, \\ &= n_1, \text{ for } \omega \in \Omega - A_1. \end{aligned}$$

Then t_1 and t_2 are bounded stopping variables and $t_1 \leqq t_2$ a.e. Hence [*2*; p. 303]

$$\int_{A_1 - B_1} x_{m_1} dP \geqq \int_{A_1 - B_1} x_{m_1} dP + \sum_{i = n_1}^{m_1} \int_{(t_2 = i) B_1} x_i dP = \int_{A_1} x_{t_2} dP \geqq \int_{A_1} x_{t_1} dP > P(A_1) > \delta - \varepsilon.$$

Put $y_n = I_{B_1} x_n$. Then $(y_n, \mathscr{F}_n, n \geqq m_1)$ is a submartingale and

$$V B_1 \subset \{\limsup y_n > 1, \, 0 > \liminf y_n\}.$$

Since $P(VB_1) \geqq P(V) - P(V - B_1) \geqq P(V) - P(V - VA_1) - P(VA_1 - B_1) > > \delta - \varepsilon$, by repeating the previous argument we can find $m_2 > n_2 \geqq m_1$ and sets $B_2 \subset A_2 \subset B_1$ such that $A_2 \in \mathscr{F}_{n_2}$, $B_2 \in \mathscr{F}_{m_2}$, and

$$\int_{A_2 - B_2} x_{m_2} dP > \delta - \varepsilon - \frac{\varepsilon}{2}, \qquad P(VB_2) > \delta - \varepsilon - \frac{\varepsilon}{2}.$$

By induction, there are $m_i \geqq n_i \geqq m_{i-1}$ and sets $B_{i-1} \supset A_i \supset B_i$ such that $A_i \in \mathscr{F}_{n_i}$, $B_i \in \mathscr{F}_{m_i}$ and

$$\begin{aligned} \int_{A_i - B_i} x_{m_i} dP &> \delta - \varepsilon(1 + 2^{-1} + \cdots + 2^{-i+1}) \\ P(VB_i) &> \delta - \varepsilon(1 + 2^{-1} + \cdots + 2^{-i+1}). \end{aligned}$$

Take $\varepsilon = \delta/2$ and define a stopping variable t by

$$t(\omega) = m_i \text{ if } \omega \in B_{i-1} - B_i \text{ for } i > 1 = m_1 \text{ if } \omega \in B_1.$$

Then

$$\int_{(t < \infty)} x_t^+ dP \geqq \sum^{\infty} \int_{(t = n)} x_t^+ dP \geqq \sum^{\infty} \int_{A_i - B_i} x_{m_i}^+ dP = \infty,$$

which contradicts the assumption.

Theorem 3 is not a simple consequence of Theorem 1 or 2, for example, let $\Omega = [0, 1]$, $F = \{0, \Omega\}$, $P(\Omega) = 1$, $x_n = n$ and $\mathscr{F}_n = \mathscr{F}$ for each n. Then the conditions of Theorem 3 are satisfied, but $P(\limsup x_n = \infty) = 1$. To conclude the paper, we will prove the following theorem about sums of independent random variables, which generalizes a theorem proved by Doob [*2*; p. 339], in which $E(\sup (y_n)^2) < \infty$ ia assumed.

Theorem 4. *Let $y_1, y_2, \ldots$ be independent random variables with $E(y_n) \geqq 0$ such that $E(\sup y_n) < \infty$. Put $x_n = y_1 + \cdots + y_n$.* If $P(\limsup x_n < \infty) = 1$, *then x_n tends to a limit in mean of order* 1.

Proof. From Corollary 5, lim x_n exists and is finite a.e. To prove that x_n converges in mean, let us suppose, if possible, that there exist $\varepsilon > 0$ and subsequences (n_i) and (m_i) such that

$$E\,|x_{m_i} - x_{n_i}| > \varepsilon,\ n_i \leqq m_i \leqq n_{i+1},\ \text{for } i \geqq 1.$$

For $N > 0$, let $t(\omega) = \inf\{k \,|\, x_k(\omega)| > N\}$. Then $P(t = \infty) = P(\sup|x_k| \leqq N) > 0$ for large N. Define

$$x'_n = x_{\min(n,\,t)}.$$

Then (x'_n) is a submartingale [2; p. 302] and $|x'_n| \leqq N + \sup y_j$. Therefore, x'_n converges in mean of order one [2; p. 324]. Choose a subsequence (n_{k_j}) of (n_i) such that

$$E\,|x'_i - x_{n_{k_j}}| < 2^{-j} \text{ for } i \geqq n_{k_j}.$$

Then

$$\sum_{j=1}^{\infty} E|x'_{m_{k_j}} - x'_{n_{k_j}}| < \infty,$$

and

$$\sum_{j=1}^{\infty} E\{|x'_{m_{k_j}} - x'_{n_{k_j}}|\,|\mathscr{F}_{n_{k_j}}\} < \infty \text{ a.e.}$$

Since $E\{|x'_{m_{k_j}} - x'_{n_{k_j}}|\,|\mathscr{F}_{n_{k_j}}\} = E\{|x_{m_{k_j}} - x_{n_{k_j}}|\,|\mathscr{F}_{n_{k_j}}\} = E\,|x_{m_{k_j}} - x_{n_{k_j}}|$

for almost all ω such that $t(\omega) \geqq n_k$,

$$\sum_{1}^{\infty} E\,|x_{m_{k_j}} - x_{n_{k_j}}| < \infty$$

a.e. in $\{t(\omega) = \infty\}$. Therefore, we have a contradiction and the proof is finished. Theorem 4 states that under its conditions (x_n) tends to a finite limit a.e. implies that (x_n) tends to the limit in mean.

References

[1] CHOW, Y. S.: Martingales in σ fl-finite measure space indexed by directed sets. Trans. Amer. math. Soc. **97**, 254—285 (1960).

[2] DOOB, J. L.: Stochastic processes. New York: Wiley 1953.

[3] — Notes on martingale theory. Proc. Fourth Berkeley Sympos. math. Statist. and Probability, **II**, 95—102 (1961).

[4] SNELL, J. L.: Applications of martingale system theorems. Trans. Amer. math. Soc. **73**, 293—312 (1952).

Watson IBM Research Laboratory
612 West 115th Street
New York 25, N. Y.

(Received November 24, 1961)