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A Condition for Absolute Continuity of Infinitely Divisible Distribution Functions

By

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1. Summary***

Throughout this paper the symbols r.v., d.f., ch.f., and i.d. will stand, respectively, for "random variable", "distribution function", "characteristic function", and "infinitely divisible".

Let $F(x)$ be an i.d.d.f.. Hartman and Wintner [5] and Blum and Rosenblatt [1] have given a condition, necessary and sufficient, for $F(x)$ to be a continuous d.f. In this note a sufficient condition for $F(x)$ to be an absolutely continuous d.f. is given.

2. The condition for absolute continuity

As shown by Khintchine [6], the logarithm of the ch.f. $\varphi(v)$ of an i.d.d.f. is representable in the form

$$\log \varphi(v) = i\gamma v - \frac{1}{2}\sigma^2 v^2 + \left[\int_{-\infty}^{0-} + \int_{0+}^{\infty} A(u, v)\, dH(u)\right], \tag{1}$$

where γ and $\sigma (\geqq 0)$ are constants, $H(u)$ is defined and non decreasing for $u < 0$ and for $u > 0$, $H(-\infty) = H(+\infty) = 0$, and, for any $\varepsilon > 0$,

$$\left[\int_{-\varepsilon}^{0-} + \int_{0+}^{\varepsilon} u^2\, dH(u)\right] < \infty .$$

The function $A(u, v)$ is given by the formula

$$A(u, v) = e^{ivu} - 1 - \frac{ivu}{1 + u^2} . \tag{2}$$

Theorem. *Let $F(x)$ be an infinitely divisible distribution function and let $H(u)$ be the function corresponding to $F(x)$ in formula (1). If, for some $u_0 > 0$, in one at least of the intervals $(-u_0, 0)$ and $(0, u_0)$, $H(u)$ is both unbounded and absolutely continuous, $F(x)$ is absolutely continuous.*

Proof. Without restricting the generality of our reasoning, we may take γ in (1) to be 0. Further, if $\sigma > 0$ in (1), $F(x)$ is obviously absolutely continuous. We may (and do) therefore assume that $\sigma = 0$.

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We now remark that an i.d.r.v. X whose ch.f. satisfies formula (1) with $\gamma = \sigma = 0$ may be assumed to be the r.v. $X(1)$ of some stochastic process $\{X(t), 0 \leqq t \leqq 1\}$ that is separable, centered, has independent and stationary increments, and satisfies the equation $X(0) = 0$, and, moreover, is such that the ch.f. $\varphi(t, v)$ of $X(t)$ satisfies, for any $0 \leqq t \leqq 1$, the relation

$$\log \varphi(t, v) = t \left[\int_{-\infty}^{0-} + \int_{0+}^{\infty} A(u, v)\, dH(u) \right], \tag{3}$$

where $A(u, v)$ and $H(u)$ are given by (2) and (1), respectively.

We assume now that in one at least of the intervals $(-u_0, 0)$ and $(0, u_0)$, the function $H(u)$ is unbounded. In other words, we assume that one at least of the relations

$$H(0-) = +\infty, \tag{4}$$

$$H(0+) = -\infty, \tag{4'}$$

holds. If, for instance, (4') holds, then the sample functions of the process $X(t)$ have in any interval $(0, t)$, t being any point in $(0, 1]$, an infinite number of positive jumps, with probability one. Indeed, let N_n denote the number of positive jumps in $(0, t)$ of magnitude m with $2^{-n-1} < m \leqq 2^{-n}$, and let n_0 be a positive integer such that $2^{-n_0} < u_0$. Then the N_n $(n = n_0, n_0 + 1, \ldots)$ form a sequence of independent Poisson variables with

$$\lambda_n = E(N_n) = H(2^{-n}) - H(2^{-n-1}).$$

By (4'), we have $\sum_{n \geqq n_0} \lambda_n = \infty$. This implies (cf. [2], p. 115, Theorem 2.7 (ii)) that $\sum_{n \geqq n_0} N_n = \infty$, with probability one. It follows in particular that the probability that $X(1) = 0$ is zero.

Let $\{\tau_k\}$ $(k = 1, 2, \ldots)$ be a sequence of positive numbers such that $\tau_1 < u_0$ and $\tau_k \searrow 0$ as $k \to \infty$. Denote by ξ_{jk} $(j = 0, 1, \ldots, \nu_k)$, where ν_k is a non-negative, integervalued r.v., the j^{th} jump* of the sample function $X(1)$ whose magnitude m is such that $\tau_{k+1} < m \leqq \tau_k$. Let

$$\xi_k = \sum_{j=0}^{\nu_k} \xi_{jk}. \tag{5}$$

It follows from (4') and the assumption $\sigma = 0$, that $X(1)$ can be represented, with probability one, as

$$X(1) = \sum_{k=1}^{\infty} \xi_k + \xi, \tag{6}$$

where ξ denotes the sum of magnitudes of negative jumps and the jumps of magnitude $\geqq u_0$, if any. ξ_k $(k = 1, 2, \ldots)$ and ξ are independent, and the number of those indices k, for which $\xi_k > 0$, is infinite. If $F_k(x)$ denotes the d.f. of ξ_k, we have

$$F_k(x) = a_k + (1 - a_k) F_k^+(x), \tag{7}$$

* when the points of jumps are arranged from 0 to 1.

where $a_k = P(\nu_k = 0)$ and $F_k^+(x) = P\ (\xi_k \leqq x \mid \nu_k > 0)$. From (6) and (7) it follows that

$$F(x) = \left\{\prod_{k=1}^{\infty} {}^* [a_k + (1 - a_k) F_k^+(x)]\right\} * F^-(x), \tag{8}$$

where * denotes convolution and $F^-(x)$ is the d.f. of ξ. We also note that

$$a_k = \exp\{-[H(\tau_k) - H(\tau_{k+1})]\}. \tag{9}$$

Since $H(0+) = -\infty$, we can choose the constants τ_k such that $0 < a_k < 1$ for all k and $\prod_1^{\infty}(1 - a_k) > 0$. We shall thus assume that the τ_k's have been so chosen.

Suppose we now assume that, in addition to (4′), $H(u)$ is absolutely continuous in the interval $(0, u_0)$. We proceed to prove that the infinite convolution

$$\prod_1^{\infty} {}^* [a_k + (1 - a_k) F_k^+(x)]$$

is absolutely continuous. Since $H(u)$ is absolutely continuous, for any Borel set S of numbers in $(0, u_0)$ whose Lebesgue measure is zero, we have

$$\int_S dH(u) = 0. \tag{10}$$

The integral on the left hand side of (10) is the expected number of jumps whose magnitudes m belong to S. It now follows that each $F_k^+(x)$ is absolutely continuous. To complete the proof that $F(x)$ is absolutely continuous, we need the following lemma.

Lemma. *Let* $0 < a_k < 1$ *be constants and let* $\prod_1^{\infty}(1 - a_k) > 0$. *If the* d.f. $G_k(x)$ *are absolutely continuous for all* k *and if the infinite convolution*

$$G(x) = \prod_k {}^* [a_k + (1 - a_k) G_k(x)]$$

exists, $G(x)$ *is absolutely continuous.*

Proof. For any d.f. let us denote the corresponding probability measure by the same symbol. Given any $\varepsilon > 0$, we can choose an N such that $z_p = \prod_{N+1}^{N+p}(1 - a_k) > > 1 - \varepsilon$ for all $p > 1$. Since the infinite convolution $\prod_{N+1}^{\infty} {}^* [a_k + (1 - a_k) G_k(x)]$ exists, it follows that the family $\{G^{(p)} : p > 1\}$ of measures, where

$$G^{(p)} = \prod_{N+1}^{N+p} {}^* [a_k + (1 - a_k) G_k],$$

has compact closure★. Moreover, the expansion of $G^{(p)}$ reveals that

$$G^{(p)} = z_p \left(\prod_{N+1}^{N+p} {}^* G_k\right) + (1 - z_p) \cdot L_p,$$

★ in the usual topology.

where L_p is some probability measure. Consequently, for any set E on the line,

$$\left(\prod_{N+1}^{N+p} {}^* G_k\right)(E) \leqq \frac{1}{1-\varepsilon} G^{(p)}(E),$$

from which we may conclude that the family $\left\{\prod_{N+1}^{N+p} {}^* G_k : p > 1\right\}$ has compact closure. Thus there exists a sequence $p_1, p_2, \ldots \to \infty$ and a d.f. $\tilde{G}(x)$ such that

$$\prod_{N+1}^{N+p_j} {}^* G_k \Rightarrow \tilde{G}.$$

Since each $G_k(x)$ is absolutely continuous, so is $\tilde{G}(x)$. The equation

$$G^{(p_j)} = z_{p_j}\left(\prod_{N+1}^{N+p_j} {}^* G_k\right) + (1 - z_{p_j}) L_{p_j}$$

enables us to infer that $L_{p_j} \Rightarrow \tilde{L}$, where $\tilde{L}$ is some probability measure (note that $z_{p_j} \to \tilde{z} = \prod_{N+1}^{\infty} (1 - a_k)$ with $\tilde{z} \geqq 1 - \varepsilon$). Consequently,

$$\prod_{N+1}^{\infty} {}^* [a_k + (1 - a_k) G_k] = \tilde{z} \cdot \tilde{G} + (1 - \tilde{z}) \cdot \tilde{L},$$

and hence

$$G = \tilde{z} \cdot \tilde{G}_1 + (1 - \tilde{z}) \tilde{L}_1, \tag{11}$$

where $\tilde{G}_1 = \prod_{1}^{N} {}^* [a_k + (1 - a_k) G_k] * \tilde{G}$ and

$$\tilde{L}_1 = \prod_{1}^{N} {}^* [a_k + (1 - a_k) G_k] * \tilde{L}.$$

Since $\tilde{G}$ is absolutely continuous, so is $\tilde{G}_1$. Thus the inequality $\tilde{z} \geqq 1 - \varepsilon$ and the relation (11) imply that for each ε the absolutely continuous component of G has total mass $\geqq 1 - \varepsilon$. This shows that G must be absolutely continuous and completes the proof of the lemma.

The theorem now follows from (8) and the lemma. In fact, by the lemma, $\prod_{1}^{\infty} {}^* [a_k + (1 - a_k) F_k^+(x)]$ is absolutely continuous and hence so is $F(x)$.

Corollary. *If $F(x)$ is a nondegenerate L-distribution function, then it is absolutely continuous.*

Proof. As shown by the first author ([*4*], Lemmas 1 and 3), the assumptions of the preceding theorem are satisfied for a nondegenerate L-distribution function, so that the corollary follows at once.

This corollary strengthens a previous result of the first author [*3*] to the effect that a nondegenerate L-distribution function is continuous.

We remark that the problem of finding necessary and sufficient conditions for the absolute continuity of an i.d.d.f. with $\sigma = 0$ is still open. The absolute continuity of $H(u)$ is not necessary. An example of an absolutely continuous i.d.d.f. with a purely discrete $H(u)$ has been given by Hartman and Wintner ([*5*],

p. 295). In constructing that example, they made use of a theorem due to themselves ([5], p. 286), according to which the relation

$$\left[\int_{-\infty}^{0-} + \int_{0+}^{\infty} |u|^{2-\lambda} dH(u)\right] = \infty . \tag{12}$$

where $\lambda < 2$, implies that $F(x)$ is not only absolutely continuous but has at every x derivatives of any order. Obviously, (12) may be satisfied for $H(u)$ purely discrete. We, however, remark that unboundedness and absolute continuity of $H(u)$ is less restrictive than relation (12). Indeed, take in formula (1) $\gamma = \lambda = 0$ and

$$H(u) = \begin{cases} 0 & (u < 0), \\ 2\log u & (0 < u \leqq 1), \\ 0 & (u > 1). \end{cases}$$

This is the H function of an L-distribution function (see KUBIK [7]) and thus, by our Corollary, that d.f. is absolutely continuous. However, (12) does not hold for any $\lambda < 2$, since for every such λ, we have

$$2\int_0^1 u^{1-\lambda} du < \infty .$$

Our next remark is that an example of a singular i.d.d.f. has been constructed by HARTMAN and WINTNER ([5], p. 288).

We finally remark that if $\sigma = 0$, the necessary and sufficient condition for the continuity of an i.d.d.f., due to HARTMAN and WINTNER and to BLUM and ROSENBLATT, is equivalent to the unboundedness of $H(u)$.

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