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The General Theory of Markov Processes According to Doeblin*

By

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§ 0. Introduction

This paper is based on DOEBLIN's paper [1] cited in the Bibliography. Although this work probably represents his crowning achievement in the theory of Markov processes, it is little known and almost never used, even when it is occasionally included in the references as a collector's item. (For what is generally known as DOEBLIN's theory see [2] and [3, Chapter 5].) The present author gave a course on the material of [1] in the spring of 1951 at Columbia University and the lecture notes were mimeographed for limited circulation. The version presented here is an expanded one over these notes, with a number of new results added, but it treats only that part of his theory which may be called the descriptive foundations, stopping short of the principal limit theorem. One reason for doing so is that the presentation of the latter hard theorem still leaves much to be desired, while the part given here seems to have reached a stage where it assumes a quite independent place in the general theory. It is hoped that the appearance in print of this will encourage further research towards various limit theorems in the general context.

It does not seem necessary to detail the differences between this presentation and DOEBLIN's own, since the latter is easily accessible for the sake of comparison. The curious reader may also consult the notes mentioned above which are closer to the original. I shall therefore limit myself to a few remarks. In §§ 1–2 my work has been mainly that of organization and clarification. Proposition 5 is due to BLACKWELL and Proposition 6 to myself, both of which are given new proofs here. Propositions 18 and 19 summarize some basic properties of a specially important type of space; the resemblance of Proposition 18 to the classical theorem of Cantor's on nested sequences of closed sets is notable. §§ 3–4 contain substantial enlargements. In particular in § 3 the arithmetical study in Propositions 34 to 37 (ending in Proposition 45 in § 4) is new. In § 4, Definitions 10 and 11 as well as Propositions 41 and 43 are new, leading to a more stringent definition of our H which resembles DOEBLIN's D, our D being his ϱ. With the present definition the conjecture "$D = H$" was first proved by H. KESTEN (private communcation 1962) and became Proposition 48. In § 5, the proofs of Propositions 50 and 51 are both simpler than DOEBLIN's original ones, the first due to T. E. HARRIS (private communication 1955).

In the remainder of this section we review briefly a constructive definition of Markov processes in the general case considered here. The reader is supposed to

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have some knowledge of these processes at least in a more limited context. Standard terminology and notation such as in [3] or [4] will be used wherever not specified. The letters i, j, k, l, m, n, ν are positive integers or zero where not specified; the complement of a set is sometimes indicated by the superscript "c".

Let X be an abstract space and $\mathscr{B}$ a Borel field of subsets of X. We are given a function $P(\cdot,\cdot)$ where $x \in X$, $B \in \mathscr{B}$ with the following properties:

(i) for each x, $P(x,\cdot)$ is a probability measure on $\mathscr{B}$;
(ii) for each B, $P(\cdot, B)$ is a $\mathscr{B}$-measurable function of x.

Let furthermore an arbitrary probability measure $P_0(\cdot)$ on $\mathscr{B}$ be given. It is known (see [3; p. 613]) that a probability space $(\Omega, \mathscr{F}, \boldsymbol{P})$ can be constructed to satisfy the following requirements. There exists a sequence of functions $\{\xi_n, n \geqq 0\}$, each of which is from Ω into X and is $(\mathscr{F}, \mathscr{B})$ measurable; that is, $\xi^{-1}(\mathscr{B}) \subset \mathscr{F}$. The measure $\boldsymbol{P}$ is completely determined by $P(\cdot,\cdot)$ and $P_0(\cdot)$ on the Borel subfield $\mathscr{F}_{[0,\infty)}$ generated by $\{\xi_n, n \geqq 0\}$, as follows: for any $B_m \in \mathscr{B}$:

$$\text{(A)} \qquad \boldsymbol{P}\{\xi_m \in B_m, 0 \leqq m \leqq n\} = \int_{B_0} P_0(dx_0) \int_{B_1} P(x_0, dx_1) \cdots \int_{B_m} P(x_{m-1}, dx_m).$$

The sequence $\{\xi_n, n \geqq 0\}$ is a (discrete parameter) Markov process with the stationary transition probability function $P(\cdot,\cdot)$ and the initial distribution P_0.

In the particular case $P_0(\cdot) = \delta(x,\cdot)$ where for every $B \in \mathscr{B}$:

$$\delta(x, B) = \begin{cases} 1 & \text{if } x \in B, \\ 0 & \text{if } x \notin B; \end{cases}$$

the corresponding $\boldsymbol{P}$ restricted to $\mathscr{F}_{[0,\infty)}$ will be denoted by $\boldsymbol{P}_x$, and the corresponding Markov process is said to start from x. The function $\boldsymbol{P}_x\{\Lambda\}$ where $x \in X$, $\Lambda \in \mathscr{F}_{[0,\infty)}$ is useful since it is a well-defined and convenient version of the conditional probability $\boldsymbol{P}\{\Lambda \mid \xi_0 = x\}$.

More generally, let $\mathscr{F}_{[n,\infty)}$ be the Borel subfield of $\mathscr{F}$ generated by $\{\xi_m, m \geqq n\}$, then for each $n \geqq 0$ and each $\Lambda \in \mathscr{F}_{[n,\infty)}$, we have for every ω except a set of $\boldsymbol{P}$-measure zero:

$$\text{(B)} \qquad \boldsymbol{P}\{\Lambda \mid \xi_0(\omega), \ldots, \xi_n(\omega)\} = \boldsymbol{P}_{\xi_n(\omega)}\{\Lambda\},$$

where $\boldsymbol{P}$ is given by (A) with an arbitrary P_0. The Markov property of the process is embodied in the equation (B).

Several cases of $\boldsymbol{P}_x(\Lambda)$ for important sets Λ will now be given with special symbols assigned to them. These will be employed throughout the paper and simple intuitive relations connecting them based on the above interpretations of conditional probabilities will be passed muster.

We write $B^c = X - B$ below:

$P^{(n)}(x, B) = \boldsymbol{P}_x\{\xi_n \in B\}$ for $n \geqq 0$ is obtained by putting $B_0 = \{x\}$, $B_1 = \cdots = B_{n-1} = X$, and $B_n = B$ in formula (A);

$K^{(n)}(x, B) = \boldsymbol{P}_x\{\xi_m \in B^c, 1 \leqq m \leqq n-1; \xi_n \in B\}$ for $n \geqq 1$ is obtained by putting $B_0 = \{x\}$, $B_1 = \cdots = B_{n-1} = B^c$, and $B_n = B$ in formula (A);

$$L(x, B) = \sum_{n=1}^{\infty} K^{(n)}(x, B) = \boldsymbol{P}_x\{\bigcup_{n=1}^{\infty} [\xi_n \in B]\};$$

$$Q(x, B) = 1 - \sum_{n=0}^{\infty} \int_B P^{(n)}(x, dy)[1 - L(y, B)]$$
$$= \lim_{n \to \infty} \int_X P^{(n)}(x, dy) L(y, B)$$
$$= \mathbf{P}_x\{\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} [\xi_n \in B]\}.$$

Note that $P^{(0)}(x, B) = \delta(x, B)$; and $P^{(1)}(x, B) = P(x, B)$.

§ 1. Closedness and essentialness

For an arbitrary set E in $\mathscr{B}$, we define four sets:

$$E^0 = \{x : L(x, E) = 0\},$$
$$E^1 = \{x : L(x, E) = 1\},$$
$$E^f = \{x : Q(x, E) = 0\},$$
$$E^\infty = \{x : Q(x, E) = 1\}.$$

For each E in $\mathscr{B}$, the functions $L(\cdot, E)$ and $Q(\cdot, E)$ are $\mathscr{B}$-measurable; hence each of the four sets above is in $\mathscr{B}$. The complement of E^*, where $*$ stands for any of the symbols 0, 1, f or ∞, will be denoted by E^{*c} rather than $(E^*)^c$.

Definition 1. A nonempty set E in $\mathscr{B}$ such that $P(x, E) = 1$ for every $x \in E$ is called *stochastically closed* (cl.).

Proposition 1. *If $x \in E^0$, then $P(x, E^0 E^c) = 1$. The sets E^0 and $E^0 E^c$ are either both empty or both cl.*

Proof. We have

$$0 = L(x, E) = \int_E P(x, dy) + \int_{E^0 E^c} L(y, E) P(x, dy) + \int_{E^{0c} E^c} L(y, E) P(x, dy).$$

Since the integrands in the first and third integrals above are positive, we have

$$P(x, E \cup E^{0c} E^c) = 0$$

or

$$P(x, E^0 E^c) = 1.$$

It follows that E^0 as well as $E^0 E^c$ is cl. unless empty, and that if E^0 is nonempty then so is $E^0 E^c$.

Proposition 2. *If $x \in E^1$, then $P(x, E^1 \cup E) = 1$. If E is cl., then so is E^1.*

Proof. We have

$$1 = L(x, E) = P(x, E^1) + P(x, E^{1c} E) + \int_{E^{1c} E^c} L(y, E) P(x, dy).$$

Since the integrand in the last integral is less than 1, we have

$$P(x, E^{1c} E^c) = 0$$

or

$$P(x, E^1 \cup E) = 1.$$

If E is cl., then $E \subset E^1$; hence the first assertion implies the second.

Proposition 3. *E^f is cl. or empty.*

Proof. Suppose E^f is nonempty and let $x \in E^f$; then

$$0 = Q(x, E) = \int_X Q(y, E)\, P(x, dy) \geqq \int_{E^{fc}} Q(y, E)\, P(x, dy) \geqq 0$$

Since the integrand in the last integral is positive, we have $P(x, E^{fc}) = 0$ or $P(x, E^f) = 1$.

Proposition 4. *E^∞ is cl. or empty.*

Proof. Suppose E^∞ is nonempty and let $x \in E^\infty$; then

$$1 = Q(x, E) = \int_{E^\infty} Q(y, E)\, P(x, dy) + \int_{E^{\infty c}} Q(y, E)\, P(x, dy).$$

Since the integrand in the last integral is less than 1, we have $P(x, E^{\infty c}) = 0$ or $P(x, E^\infty) = 1$.

Proposition 5. *If $E = \bigcup_n E_n$, then $E^0 = \bigcap_n E_n^0$; where $\{E_n\}$ is an arbitrary sequence of sets in $\mathscr{B}$.*

Proof. Clearly $E^0 \subset E_n^0$ so that $E^0 \subset \bigcap_n E_n^0$. On the other hand, if $x \in \bigcap_n E_n^0$, then for every n we have $L(x, E_n) = 0$; consequently

$$L(x, E) = L(x, \bigcup_n E_n) \leqq \sum_n L(x, E_n) = 0,$$

and so $x \in E^0$. Thus $\bigcap_n E_n^0 \subset E^0$.

Definition 2. A set E in $\mathscr{B}$ such that $Q(x, E) = 0$ for every $x \in X$ is called *inessential (iness.)*; otherwise it is called *essential (ess.)*. An essential set which is the union of denumerably many inessential sets is called *improperly essential (imp. ess.)*; otherwise it is called *absolutely essential (abs. ess.)*.

The next two propositions are basic for the sequel. Proposition 6 was given by BLACKWELL [*5*], and Proposition 7 in the lecture notes mentioned in the Introduction and essentially reproduced in [*4*; p. 19]. Both were proved by simple, direct arguments. For the sake of completeness but variation we give alternative proofs below based on the convergence of martingales.

For any E in $\mathscr{B}$, let

$$\Lambda(E) = \limsup_n \{\xi_n \in E\},$$

and for any E and F in $\mathscr{B}$, let

$$Q(x, E, F) = \boldsymbol{P}_x\{\Lambda(E) \cap \Lambda(F)\}.$$

Proposition 6. *If*

$$\sup_{x \in E} Q(x, F) < 1,$$

then for every $x \in X$ we have

$$Q(x, E, F) = 0$$

Proof. Fix an x as the initial point of the process $\{\xi_n, n \geqq 0\}$. Since $\Lambda(E)$ and $\Lambda(F)$ are invariant sets we have with probability one:

$$\boldsymbol{P}_x\{\Lambda(E) \cap \Lambda(F) \mid \xi_0, \ldots, \xi_n\} = \boldsymbol{P}_x\{\Lambda(E) \cap \Lambda(F) \mid \xi_n\} = Q(\xi_n, E, F).$$

This shows that $\{Q(\xi_n, E, F), n \geqq 0\}$ is a martingale and PAUL LÉVY's zero-or-one law asserts that for almost every ω:

$$\lim_{n\to\infty} Q(\xi_n(\omega), E, F) = \Im_{\Lambda(E)\cap\Lambda(F)} \tag{1}$$

where $\Im_\Lambda$ denotes the indicator function of Λ. Now if $\omega \in \Lambda_E$, then $\xi_n(\omega) \in E$ for infinitely many values of n, and consequently for these values of n we have

$$Q(\xi_n(\omega), E, F) \leqq Q(\xi_n(\omega), F) \leqq \sup_{x\in E} Q(x, F) < 1. \tag{2}$$

It follows from (1) and (2) that $\boldsymbol{P}_x(\Lambda(E) \cap \Lambda(F)) = 0$.

Proposition 7. *If*

$$\inf_{x\in E} L(x, F) > 0,$$

then for every $x \in X$ we have

$$Q(x, E) = Q(x, E, F).$$

Proof. Let

$$M_k = \bigcup_{n=k}^{\infty} \{\xi_n \in F\}$$

so that in the previous notation we have

$$\bigcap_{k=1}^{\infty} M_k = \Lambda(F).$$

We have if $n \geqq k$:

$$\boldsymbol{P}_x\{\Lambda_F | \xi_0, \ldots, \xi_n\} \leqq \boldsymbol{P}_x\{M_{n+1} | \xi_0, \ldots, \xi_n\} \leqq \boldsymbol{P}_x\{M_k | \xi_0, \ldots, \xi_n\}$$

with probability one. Letting $n \to \infty$, then $k \to \infty$, we obtain

$$\Im_{\Lambda(F)} \leqq \lim_{n\to\infty} \boldsymbol{P}_x\{M_{n+1} | \xi_0, \ldots, \xi_n\} \leqq \lim_{k\to\infty} \Im_{M_k} = \Im_{\Lambda(F)}.$$

Since

$$L(\xi_n, F) = \boldsymbol{P}_x\{M_{n+1} | \xi_n\} = \boldsymbol{P}_x\{M_{n+1} | \xi_0, \ldots, \xi_n\}$$

with probability one, we conclude that

$$\overline{\lim_{n\to\infty}} L(\xi_n, F) = \Im_{\Lambda(F)}. \tag{3}$$

If $\omega \in \Lambda(E)$, then $\xi_n(\omega) \in E$ for infinitely many values of n and consequently for these values of n we have

$$L(\xi_n(\omega), F) \geqq \inf_{x\in E} L(x, F) > 0. \tag{4}$$

It follows from (3) and (4) that $\boldsymbol{P}_x(\Lambda(E)) = \boldsymbol{P}_x(\Lambda(E) \cap \Lambda(F))$.

Proposition 8. *If E is ess., and $\inf_{x\in E} L(x, F) > 0$, then F is ess.*

Proof. Since E is ess. there exists an x for which $Q(x, E) > 0$. By Proposition 7,

$$Q(x, F) \geqq Q(x, E, F) = Q(x, E) > 0.$$

Hence F is ess.

Proposition 9. *If E is abs. ess., and $\inf_{x \in E} L(x, F) > 0$, then F is abs. ess.*

Proof. It is sufficient to prove that for any sequence of sets F_k in $\mathscr{B}$ such that $F = \bigcup_{k=1}^{\infty} F_k$, there exists an n_0 such that $\bigcup_{k=1}^{n_0} F_k$ is ess. We note the simple relation:

$$L(x, F) = \lim_{n \to \infty} L(x, \bigcup_{k=1}^{n} F_k).$$

Let $x \in E$, then $L(x, F) = \alpha > 0$; hence there exists a finite $m_0(x)$ such that

$$L(x, \bigcup_{k=1}^{m_0(x)} F_k) > \frac{\alpha}{2} > 0.$$

Let $E_n = \{x \in E : m_0(x) = n\}$, then $E = \bigcup_{n=1}^{\infty} E_n$. Since E is abs. ess. there exists an n_0 such that E_{n_0} is ess. By the definition of E_{n_0} we have

$$\inf_{x \in E_{n_0}} L(x, \bigcup_{k=1}^{n_0} F_k) > \frac{\alpha}{2}.$$

Hence by Proposition 8, $\bigcup_{k=1}^{n_0} F_k$ is ess.

Proposition 10. *For any E in $\mathscr{B}$, if there exists an F in $\mathscr{B}$ such that*

$$\sup_{x \in E} Q(x, F) < 1, \qquad \inf_{x \in E} L(x, F) > 0,$$

then E is iness.

Proof. For every x we have by Propositions 6 and 7:

$$Q(x, E) = Q(x, E, F) = 0$$

Hence E is iness. by definition.

Proposition 11. *If $X - E^0$ is abs. ess., then E is abs. ess.*

Proof. Let

$$E_n = \left\{x : L(x, E) \geqq \frac{1}{n}\right\};$$

then we have

$$X = E^0 \cup \bigcup_{n=1}^{\infty} E_n.$$

If $X - E^0$ is abs. ess., then E_n is abs. ess. for some n, and so E is abs. ess. by Proposition 9.

Proposition 11.1. *If X is abs. ess. and $E^0 = 0$, then E is abs. ess.*

Proposition 12. *For any E in $\mathscr{B}$, $X - (E^0 \cup E^\infty)$ is not abs. ess.*

Proof. Let

$$E_n = \left\{x : Q(x, E) \leqq 1 - \frac{1}{n}, L(x, E) \geqq \frac{1}{n}\right\};$$

then

$$X = E^0 \cup E^\infty \cup \bigcup_{n=1}^{\infty} E_n.$$

Each E_n is iness. by Proposition 10, hence their union $X - (E^0 \cup E_\infty)$ is not abs. ess.

Proposition 12.1. *If E is abs. ess., then for any F in $\mathscr{B}$ the set $E(F^0 \cup F^\infty)$ is abs. ess., hence nonempty.*

Proposition 13. *If E is abs. ess., then EE^∞ is abs. ess., in particular $E^\infty \neq 0$.*

Proof. Applying Proposition 12.1 with $F = E$ we see that $EE^0 \cup EE^\infty$ is abs. ess. But EE^0 is clearly iness., hence EE^∞ is abs. ess.

Proposition 14. *If C is cl., then $X - (C \cup C^0)$ does not contain any cl. set and is not abs. ess.*

Proof. Any cl. set contained in $X - C$ must be contained in C^0, hence $E = X - (C \cup C^0)$ does not contain any cl. set. Since $C \cup C^0$ is cl., any point in E^1 must belong to E; in particular $E^\infty \subset E^1 \subset E$. But E^∞ is cl. if not empty, hence $E^\infty = 0$ by the first assertion. It follows from Proposition 13 that E is not abs. ess.

Proposition 14.1. *If C and D are cl. sets such that $D \subset C$ and $C - D$ does not contain any cl. set, then $C - D$ is not abs. ess. In particular, $C - C(D \cup D^0)$ is not abs. ess.*

§ 2. Decomposability

Definition 3. A cl. set which does not contain two disjoint cl. sets is called *indecomposable (indecomp.)*; otherwise it is called *decomposable (decomp.)*. An indecomposable set which is not properly contained in any indecomposable set is called *maximal indecomposable (max. indecomp.)*

Proposition 15. *If E is indecomp., then $(E^0)^0$ is max. indecomp.*

Proof. Suppose $(E^0)^0$ is decomp.; let C and D be two disjoint cl. sets contained in it. For any $x \in C$ we have $x \notin E^0$ since $E^0(E^0)^0 = 0$; hence $L(x, E) > 0$. Since C is cl. this implies that $CE \neq 0$. Similarly $DE \neq 0$. Thus CE and DE are disjoint cl. sets contained in E and E is decomp. We have thus proved that if E is indecomp., then so is $(E^0)^0$. Now suppose that F is cl. and contains $(E^0)^0$ properly. Let $x \in F - (E^0)^0$, then $L(x, E^0) > 0$. Thus E^0 is nonempty and hence cl. by Proposition 1, and FE^0 is also nonempty and hence cl. The set F contains the disjoint cl. sets E and FE^0 and so is decomp. We have therefore proved that any cl. set properly containing $(E^0)^0$ is decomp. Hence $(E^0)^0$ is max. indecomp.

Proposition 16. *Two max. indecomp. sets are either identical or disjoint.*

Proof. Let E and F be two distinct max. indecomp. sets. Then $E \cup F$ is cl. and contains either of them properly. Hence it is decomp. and contains two disjoint cl. sets C and D. Since E is indecomp. at least one of EC and ED is empty. Suppose EC is empty; then $F \supset C$ and since F also contains EF which is either cl. or empty we must have $EF = 0$ since F is indecomp.

Proposition 17. *If X is indecomp. and E is abs. ess., then $E^0 = 0$.*

Proof. By Proposition 13 we have $E^\infty \neq 0$, hence E^∞ is cl. by Proposition 4. By Proposition 1, E^0 is cl. if not empty. Since $E^0 E^\infty = 0$ and X is indecomp., we must have $E^0 = 0$.

Proposition 18. *If X is abs. ess. and indecomp., then every sequence of cl. sets has a cl. intersection whose complement is not abs. ess.*

Proof. Let $\{C_k\}$ be a finite or infinite sequence of cl. sets. Then $D_n = \bigcap_{k=1}^{n} C_k$ is not empty since X is indecomp. We have

$$X = \bigcup_n D_n^c \cup (\bigcap_n D_n).$$

Each D_n is cl. and D_n^c does not contain any cl. set by the indecomposability of X. Hence D_n^c is not abs. ess. by Proposition 14.1. with $C = X$; and so $\bigcup_n D_n^c$ is not abs. ess. Since X is abs. ess. it follows that $\bigcap_n D_n$ is abs. ess., hence it is nonempty, hence it is cl.

Proposition 18.1. *In an indecomp. space the complement of any cl. set is not abs. ess.*

Proposition 19. *In an abs. ess. and indecomp. space X, an abs. ess. set E is characterized by any one of the following three properties:*

$$E^0 = 0, \quad E^\infty \neq 0, \quad E^f = 0.$$

Proof. The first characterization follows from Propositions 11.1 and 17. Next, each of the three sets E^0, E^∞ and E^f is either cl. or empty, by Propositions 1, 4 and 3. Now at least one of the two sets E^0 and E^∞ is nonempty by Proposition 12.1. Hence exactly one of them is nonempty since $E^0 E^\infty = 0$ and X is indecomp. Thus $E^\infty \neq 0$ is equivalent to $E^0 = 0$ and we have proved the second characterization. Finally, since $E^0 \subset E^f$, $E^f = 0$ implies $E^0 = 0$; on the other hand since $E^f E^\infty = 0$, $E^f \neq 0$ implies $E^\infty = 0$ because of indecomposability. Hence the third characterization is a consequence of the first two.

Remark: Let "$E \in \mathscr{A}$" stand for the proposition "E is abs. ess.", "$\Rightarrow$" for "implies" and "$\not\Rightarrow$" for "does not imply". The following table shows the various relations under different hypotheses regarding the space X; where "$\not\Rightarrow$" stands the required example is trivial from the theory of Markov chains.

Table

Arbitrary X	Abs. ess. X	Indecomp. X	Abs. ess. and indecomp. X
$E \in \mathscr{A} \Rightarrow E^\infty \neq 0$	$E \in \mathscr{A} \not\Rightarrow E^0 = 0$	$E \in \mathscr{A} \Rightarrow E^0 = 0$	
	$E \in \mathscr{A} \not\Rightarrow E^f = 0$	$E \in \mathscr{A} \Rightarrow E^f = 0$	
	$E^\infty \neq 0 \not\Rightarrow E \in \mathscr{A}$	$E^\infty \neq 0 \not\Rightarrow E \in \mathscr{A}$	$E^\infty \neq 0 \Rightarrow E \in \mathscr{A}$
	$E^0 = 0 \Rightarrow E \in \mathscr{A}$	$E^0 = 0 \not\Rightarrow E \in \mathscr{A}$	
	$E^f = 0 \Rightarrow E \in \mathscr{A}$	$E^f = 0 \not\Rightarrow E \in \mathscr{A}$	

Proposition 20. *If X is indecomp. and E is abs. ess., then the series*

$$\sum_{n=0}^{\infty} P^{(n)}(x, E) \tag{5}$$

diverges for every $x \in X$. If X is abs. ess. and the series in (5) *has a positive sum for every $x \in X$, then E is abs. ess.*

Proof. Suppose that the series in (5) converges for some x, then by the Borel-Cantelli lemma: $Q(x, E) = 0$ so that $E^f \neq 0$. If X is indecomp. a glance at the preceding table shows that E is not abs. ess. Next suppose that the series in (5) has a positive sum for every x, then $E^0 = 0$. If X is abs. ess., a glance at the preceding table shows that E is abs. ess.

Remark. The converse to the first assertion in Proposition 20 is false. More precisely, it is possible in an indecomp. X that the series in (5) diverges for every x but E is iness. Consider the following example from Markov chains. The space X consists of $\{y_n, n \geqq 1\}$ and $\{x_{nk}, 1 \leqq k \leqq n, n \geqq 1\}$.

$$P(y_1, y_2) = \frac{1}{2};$$

$$P(y_n, y_{n+1}) = 1 - \frac{1}{n^2}; \qquad P(y_n, y_1) = \frac{1}{n^2}, \qquad n \geqq 2;$$

$$P(y_1, x_{n1}) = p_n = \frac{3}{\pi^2 n^2}, \qquad n \geqq 1;$$

$$P(x_{nk}, x_{n,k+1}) = 1, \qquad 1 \leqq k \leqq n-1;$$

$$P(x_{n,n}, y_1) = 1.$$

It is clear that X forms one nonrecurrent class. Let

$$E = \{x_{nk}, 1 \leqq k \leqq n, n \geqq 1\}.$$

We have

$$P^{(n)}(y_1, E) \geqq \sum_{k=n}^{\infty} p_k = \frac{3}{\pi^2} \sum_{k=n}^{\infty} \frac{1}{k^2},$$

so that the series in (5) diverges for $x = y_1$. Since $L(x, y_1) > 0$ for every x it follows easily that it diverges for every x. To see that E is iness., we verify that

$$\inf_{x \in E} L(x, x_{11}) = L(y_1, x_{11}) = p_1 > 0,$$

$$\sup_{x \in E} Q(x, x_{11}) = Q(y_1, x_{11}) \leqq L(y_1, x_{11}) \leqq 1 - \frac{1}{2} \prod_{n=2}^{\infty} \left(1 - \frac{1}{n^2}\right) < 1.$$

Hence E is iness. by Proposition 10.

Definition 4. A set E in $\mathscr{B}$ such that $Q(x, E^c) < 1$ for some $x \in X$ is called *perpetuable (perp.)*. Equivalently, E in $\mathscr{B}$ is perpetuable if there exists an x in E such that $L(x, E^c) < 1$ (see Proposition 24 below). In the literature a perp. set has been called a "sojourn set"; cf. [*4*; p. 110].

Proposition 21. *If E is perp. then it is ess.*

Proof. We have for every E in $\mathscr{B}$:

$$(6) \qquad Q(x, E) + Q(x, E^c) \geqq 1,$$

hence $Q(x, E^c) < 1$ implies $Q(x, E) > 0$.

Remark. A set E for which there is equality in (6) for every $x \in X$ has been called "almost closed"; cf. [*4*; p. 108].

Proposition 22. *If C is cl. and C^c is ess., then C^c is perp.*

Proof. Since C^c is ess. there exists an x for which $Q(x, C^c) > 0$. Since C is cl. this implies that

$$Q(x, C) \leqq L(x, C) < 1.$$

Hence C^c is perp.

Proposition 23. *Any imp. ess. set is contained in an imp. ess. and perp. set.*

Proof. If X is imp. ess. the proposition is trivial. Now suppose that X is abs. ess. and E is imp. ess. Then $E^0 \neq 0$ by Proposition 11.1, and $E^0 E^c$ is cl. by Proposition 1. Since E is not abs. ess., E^{0c} is not abs. ess. by Proposition 11. Hence

$$E^{0c} \cup E = (E^0 E^c)^c$$

is not abs. ess. It contains E and is perp. by Proposition 22 since it is ess. and its complement is closed.

Proposition 23.1. *If $E \subset C$ where E is imp. ess. and C is cl., then there exists an imp. ess. and perp. F such that $E \subset F \subset C$ and $C - F$ is cl.*

Proposition 24. *If E is perp. then*

$$\inf_{x \in E} L(x, E^c) = 0 .$$

Proof. We have for any x in X and E in $\mathscr{B}$, as a completion of (6):

$$(6^*) \qquad 1 = Q(x, E \cup E^c) = Q(x, E) + Q(x, E^c) - Q(x, E, E^c) .$$

If $\inf_{x \in E} L(x, E^c) > 0$ then by Proposition 7 we have

$$Q(x, E) = Q(x, E, E^c)$$

so that the equation (6*) implies $Q(x, E^c) = 1$ for every $x \in X$. Thus E is not perp.

§ 3. Cycles

The properties of a set $\mathscr{B}$ such as "closed" and "essential" were defined with reference to the basic transition probability function $P(\cdot,\cdot)$. If the latter is replaced by its k^{th} iterate $P^{(k)}(\cdot,\cdot)$ then the corresponding property will be prefixed by "$P^{(k)}$ -". Thus the previously defined concepts are the $P^{(1)}$-versions, with the prefix "$P^{(1)}$ -" omitted from the terminology. The results we have proved so far have their $P^{(k)}$-versions which need no new proofs. In terms of the process, we shall be considering $\{\xi_{nk+r}, n \geqq 0\}$ for a fixed k and some r in lieu of $\{\xi_n, n \geqq 0\}$.

Proposition 25. *A set is $P^{(k)}$-iness., $P^{(k)}$-imp. ess., or $P^{(k)}$-abs. ess. according as it is iness., imp. ess., or abs. ess.*

Proof. If a set is iness., it is clearly $P^{(k)}$-iness. If E is ess. there exists an $x \in X$ such that $Q(x, E) > 0$. Then for each k there exists an r, $1 \leqq r \leqq k$, such that

$$\boldsymbol{P}\{\xi_{nk+r} \in E \text{ for infinitely many values of } n \,|\, \xi_0 = x\} > 0.$$

Hence there exists a $y \in E$ and an integer n_0 such that

$$\boldsymbol{P}\{\xi_{nk+r} \in E \text{ for infinitely many values of } n \,|\, \xi_{n_0 k+r} = y\} > 0.$$

This shows that E is $\boldsymbol{P}^{(k)}$-ess. The other assertions follow easily.

Definition 5. For an arbitrary set E in $\mathscr{B}$ we set

$$\mathfrak{A}(E) = \{x : P(x, E) = 1\} .$$

Let $\mathfrak{A}^0(E) = E$, $\mathfrak{A}^1(E) = \mathfrak{A}(E)$ and define $\mathfrak{A}^j(E)$ for each $j \geqq 1$ by

$$\mathfrak{A}^j(E) = \mathfrak{A}(\mathfrak{A}^{j-1}(E)).$$

$\mathfrak{A}^j(E)$ is called the j^{th} *antecedent* of E.

We have $\mathfrak{A}(E) \in \mathscr{B}$ for any $E \in \mathscr{B}$, since $P(\cdot, E)$ is $\mathscr{B}$-measurable for each $E \in \mathscr{B}$.

Proposition 26. *If* $P^{(k)}(x, E) = 1$ *then we have*

$$P^{(k-j)}(x, \mathfrak{A}^j(E)) = 1, \qquad 1 \leqq j \leqq k.$$

Proof. We have

$$1 = P^{(k)}(x, E) = \Big[\int_{\mathfrak{A}(E)} + \int_{\mathfrak{A}(E)^c} \Big] P(y, E)\, P^{(k-1)}(x, dy),$$

where in the second integral the integrand is less than one. Hence the assertion follows for $j = 1$, and the general case then follows from this by induction on j.

Proposition 27. *We have for each* $j \geqq 0$,

$$\mathfrak{A}^j(E) = \{x : P^{(j)}(x, E) = 1\}.$$

Proof. The assertion is true for $j = 1$ by definition. Assume for the sake of induction that it is true for a certain j, then if $x \in \mathfrak{A}^{j+1}(E) = \mathfrak{A}(\mathfrak{A}^j(E))$ we have $P(x, \mathfrak{A}^j(E)) = 1$ and consequently

$$P^{(j+1)}(x, E) = \int_{\mathfrak{A}^j(E)} P^{(j)}(y, E)\, P(x, dy) = \int_{\mathfrak{A}^j(E)} 1\, P(x, dy) = 1$$

by the induction hypothesis. Hence $\mathfrak{A}^{j+1}(E) \subset \{x : P^{(j+1)}(x, E) = 1\}$ by induction. Conversely, if $P^{(j+1)}(x, E) = 1$ then by Proposition 26 we have $P^{(1)}(x, \mathfrak{A}^j(E)) = 1$, and so by definition $x \in \mathfrak{A}(\mathfrak{A}^j(E)) = \mathfrak{A}^{j+1}(E)$.

Definition 6. A sequence of k sets $\{E_j, 1 \leqq j \leqq k\}$ in $\mathscr{B}$ is said to form a *k-cycle* if

$$E_j \subset \mathfrak{A}(E_{j+1}), \qquad 1 \leqq j \leqq k-1,$$

and

$$E_k \subset \mathfrak{A}(E_1).$$

The union $\bigcup_{j=1}^{k} E_j$ will also be called the *cycle* when no confusion is likely and each E_j, $1 \leqq j \leqq k$, a *member* of the cycle. The cycle is called *clean* if the E_j's are disjoint. Note that in general the members of a cycle need not be distinct.

Proposition 28. *Each member of a k-cycle is* $P^{(k)}$*-cl. and the cycle itself is* $P^{(1)}$*-cl. If* E *is* $P^{(k)}$*-cl., then the sequence* $\mathfrak{A}^{k-j}(E)$, $1 \leqq j \leqq k$, *forms a cycle.*

Proof. It follows from Proposition 27 that $E \subset \mathfrak{A}^k(E)$ if and only if E is $P^{(k)}$-cl. Now if $E \subset F$ then $\mathfrak{A}(E) \subset \mathfrak{A}(F)$. Hence by the definition of a cycle we have

$$E_j \subset \mathfrak{A}(E_{j+1}) \subset \cdots \subset \mathfrak{A}^{k-j}(E_k) \subset \mathfrak{A}^{k-j+1}(E_1) \subset \cdots \subset \mathfrak{A}^k(E_j).$$

Thus each E_j is $P^{(k)}$-cl. Furthermore we have

$$\bigcup_{j=1}^{k} E_j \subset \bigcup_{j=0}^{k-1} \mathfrak{A}(E_{j+1}) = \mathfrak{A}\Big(\bigcup_{j=1}^{k} E_j\Big);$$

hence the cycle is $P^{(1)}$-cl.

If E is $P^{(k)}$-cl., then

$$\mathfrak{A}^0(E) = E \subset \mathfrak{A}^k(E) = \mathfrak{A}(\mathfrak{A}^{k-1}(E)),$$

and

$$\mathfrak{A}^{k-j}(E) = \mathfrak{A}(\mathfrak{A}^{k-j-1}(E)), \qquad 1 \leqq j \leqq k;$$

and consequently $\{\mathfrak{A}^{k-j}(E),\ 1 \leqq j \leqq k\}$ forms a cycle.

Definition 7. The cycle in the second part of Proposition 28 is said to be *generated by* E.

Proposition 29. *If* E *is* $P^{(k)}$-cl. *or* $P^{(k)}$*-indecomp. or* $P^{(k)}$*-max. indecomp., then so is* $\mathfrak{A}^j(E)$ *for each* $j \geqq 0$.

Proof. It is sufficient to prove the assertion for $j = 1$, since the general case then follows by iteration. Let $F = \mathfrak{A}(E)$. If E is $P^{(k)}$-cl., then F is $P^{(k)}$-cl. by the second part of Proposition 28. Next, suppose that E is $P^{(k)}$-cl. and F is $P^{(k)}$-decomp; we are going to show that E is $P^{(k)}$-decomp. There exist disjoint $P^{(k)}$-cl. subsets F_1 and F_2 of F. Define

$$E_n = \mathfrak{A}^{k-1}(F_n), \qquad n = 1, 2.$$

Then E_1 and E_2 are disjoint $P^{(k)}$-cl. sets. If $x \in F_n$, we have

$$1 = P^{(k)}(x, F_n) = \int_E P^{(k-1)}(y, F_n)\, P(x, dy)$$

since $P(x, E) = 1$ by the definition of F. Hence there exists a $y \in E$ with $P^{(k-1)}(y, F_n) = 1$ and consequently $y \in E_n$ by definition. Thus $E \cap E_n \neq 0$ for $n = 1, 2$. Each $E \cap E_n$ is $P^{(k)}$-cl. and so E is $P^{(k)}$-decomp. as was to be shown.

Finally, suppose that E is $P^{(k)}$-max. indecomp. Then F is $P^{(k)}$-indecomp. as just proved. Let $\widetilde{F}$ be $P^{(k)}$-cl. and contain F properly. Define $\widetilde{E} = \mathfrak{A}^{k-1}(\widetilde{F})$. Then $\widetilde{E} \supset E$. If $x \in \widetilde{F}$ then $P(x, \widetilde{E}) = 1$ by Proposition 26. If $x \notin F$ then $P(x, E) < 1$ by the definition of F. Since $\widetilde{F} - F$ is nonempty we see by choosing an x in this difference that $\widetilde{E}$ contains E properly. Hence $\widetilde{E}$ is $P^{(k)}$-decomp. and so must be $\widetilde{F}$ by what has been proved. Therefore F is $P^{(k)}$-max. indecomp.

Notation. If k_1 and k_2 are two positive integers, we write $k_1 | k_2$ if k_1 is a divisor of k_2.

Proposition 30. *Let* $d | k$. *A* $P^{(d)}$*-cl. set is* $P^{(k)}$*-cl. A* $P^{(k)}$*-cl. and* $P^{(k)}$*-indecomp. set is* $P^{(d)}$*-indecomp. A* $P^{(d)}$*-max. indecomp. and* $P^{(k)}$*-indecomp. set is* $P^{(k)}$*-max. indecomp. A* $P^{(d)}$*-cl. and* $P^{(k)}$*-max. indecomp. set is* $P^{(d)}$*-max. indecomp.*

Proof. Without loss of generality we may suppose $d = 1$, since we may consider $P^{(d)}(\cdot,\cdot)$ in lieu of $P(\cdot,\cdot)$ as the basic transition probability function. The first two assertions are trivial.

Let E be $P^{(1)}$-max. decomp. and $P^{(k)}$-indecomp. and let F be a $P^{(k)}$-cl. set which contains E properly. We are going to show that F is $P^{(k)}$-decomp. Let G be the k-cycle generated by F. Then G is $P^{(1)}$-cl. and contain E properly. Hence G contains two disjoint $P^{(1)}$-cl. sets A and B. If $x \in A$ then by the defining property of a cycle we have $P^{(j)}(x, F) = 1$ for some j, $1 \leqq j \leqq k$. Since A is $P^{(1)}$-cl. this implies $A \cap F \neq 0$. By the same token $B \cap F \neq 0$. The two sets $A \cap F$ and $B \cap F$ are disjoint and $P^{(k)}$-cl. Hence F is $P^{(k)}$-decomp. as was to be shown.

To prove the last assertion in Proposition 30, let E be $P^{(1)}$-cl. and $P^{(k)}$-max. indecomp. Then E is $P^{(1)}$-indecomp. by the second assertion in Proposition 30. Let F be $P^{(1)}$-cl. and contain E properly; we are going to show that F is $P^{(1)}$-decomp. Since F is $P^{(k)}$-cl. it must be $P^{(k)}$-decomp. Let A and B be disjoint, $P^{(k)}$-cl. sets contained in F. Since E is $P^{(k)}$-indecomp. at least one of $A \cap E$ and $B \cap E$ is empty. Suppose $A \cap E = 0$ and let C be the k-cycle generated by A. Then $C \cap E = 0$ by the property of a cycle. Hence C and E are disjoint, $P^{(1)}$-cl. sets contained in F and F is $P^{(1)}$-decomp. as was to be shown.

In Propositions 31 *to* 37 *the state space X is assumed to be $P^{(1)}$-indecomp.*

Proposition 31. *There are at most k disjoint $P^{(k)}$-cl. sets.*

Proof. Let B_m, $1 \leqq m \leqq n$, be disjoint, $P^{(k)}$-cl. sets. By Proposition 28, each of them generates a k-cycle C_m which is $P^{(1)}$-cl. Since X is $P^{(1)}$-indecomp. $C = \bigcap_{m=1}^{n} C_m$ is nonempty. Let $x \in C$, then by the property of a k-cycle for each m, $1 \leqq m \leqq n$, there exists an integer j_m, $1 \leqq j_m \leqq k$, such that $P^{(j_m)}(E_m) = 1$. Since the E_m's are disjoint the j_m's must be distinct. Therefore $n \leqq k$.

Proposition 32. *Each $P^{(k)}$-cl. set contains a $P^{(k)}$-indecomp. set and intersects a $P^{(k)}$-max. indecomp. set. The number of distinct $P^{(k)}$-max. indecomp. sets is the maximum number of disjoint $P^{(k)}$-cl. sets.*

Proof. If there were a $P^{(k)}$-cl. set which does not contain any $P^{(k)}$-indecomp. subset then the set itself is $P^{(k)}$-decomp. and hence contains two disjoint $P^{(k)}$-cl. sets each of which is $P^{(k)}$-decomp. Hence by induction there would be an infinite number of disjoint $P^{(k)}$-cl. sets, contradicting Proposition 31. Now by the $P^{(k)}$-version of Proposition 15, each $P^{(k)}$-indecomp. set is contained in a $P^{(k)}$-max. indecomp. set; hence each $P^{(k)}$-cl. set intersects a $P^{(k)}$-max. indecomp. set. Two disjoint $P^{(k)}$-cl. sets cannot intersect the same $P^{(k)}$-max. indecomp. set, proving the last assertion.

Proposition 33. *For each k let $\delta(k)$ be the number of distinct $P^{(k)}$-max. indecomp. sets contained in X; then $\delta(k) \mid k$. These $\delta(k)$ sets form a clean cycle $\{I_i, 1 \leqq i \leqq \delta(k)\}$. $X - \bigcup_{i=1}^{\delta(k)} I_i$ does not contain any $P^{(k)}$-cl. sets and is not abs. ess.*

Proof. By Proposition 32, there exists a $P^{(k)}$-max. indecomp. set I. Set

$$I_i = \mathfrak{A}^i(I), \qquad i \geqq 0.$$

By Proposition 28, $\{I_{k-i}, 1 \leqq i \leqq k\}$ is the k-cycle generated by I. We have $I = I_0 \subset I_k$. But by Proposition 29, each I_i is $P^{(k)}$-max. indecomp. Hence by the $P^{(k)}$-version of Proposition 16, $I_0 = I_k$ and consequently $I_i = I_j$ if $i \equiv j \pmod{k}$. Let d be the least positive integer such that $I_0 = I_d$. Then $I_i \neq I_j$ for $0 \leqq i < j \leqq$ $\leqq d - 1$, for otherwise one would have

$$I_0 = I_k = \mathfrak{A}^{k-i}(I_i) = \mathfrak{A}^{k-i}(I_j) = \mathfrak{A}^{k+j-i}(I_0) = I_{k+j-i} = I_{j-i},$$

contradicting the definition of d. By the $P^{(k)}$-version of Proposition 16, the sets I_i, $1 \leqq i \leqq d$, are disjoint and so form a clean cycle. We have now $I_i = I_j$ if and only if $i \equiv j \pmod{d}$, hence $d \mid k$.

This d is the $\delta(k)$ asserted in the proposition, that is, any $P^{(k)}$-max. indecomp. set is one of the I_i's. To see this let J be such a set. As before, there is an integer e such that $e \mid k$ and $\{\mathfrak{A}^j(J),\ 1 \leqq j \leqq e\}$ is a cycle. Let $D = \bigcup_{j=1}^{e} \mathfrak{A}^j(J)$. Since X is indecomp., $C \cap D \neq 0$ and consequently $\mathfrak{A}^i(I) \cap \mathfrak{A}^j(J) \neq 0$ for some i and j. But then $\mathfrak{A}^i(I) = \mathfrak{A}^j(J)$ because both sets are $P^{(k)}$-max. indecomp. and it follows that

$$J = \mathfrak{A}^k(J) = \mathfrak{A}^{k+i-j}(I).$$

Thus J is one of the I_i's and therefore $d = \delta(k)$.

By Proposition 32, any $P^{(k)}$-cl. set must intersect one of the I_i's. Hence $X - C$ does not contain any $P^{(k)}$-cl. set. Then $X - C$ is not abs. ess. by the $P^{(k)}$-version of Proposition 14 and Proposition 25.

Definition 8. $\delta(k)$ is called *the cyclic index belonging to* k and the $\delta(k)$-cycle described in Proposition 33 is called the *cycle belonging to* k. It is uniquely defined for each k.

Notation. For two positive integers k and k' we denote their least common multiple by $k \vee k'$ and their greatest common divisor by $k \wedge k'$.

Proposition 34. *For arbitrary* k *and* k', *we have*

$$\delta(k \vee k') = \delta(k) \vee \delta(k') \tag{7}$$

$$\delta(k \wedge k') = \delta(k) \wedge \delta(k'). \tag{8}$$

Proof. Let $\{D_i,\ 1 \leqq i \leqq \delta(k)\}$ and $\{E_i,\ 1 \leqq i \leqq \delta(k')\}$ be the cycles belonging to k and k' respectively.

We first show that

$$\delta(k) \geqq k \wedge \delta(k'). \tag{9}$$

Writing $d = k \wedge \delta(k')$ and $\delta(k') = qd$, we set

$$F_r = \bigcup_{m=0}^{q-1} E_{md+r}.$$

The sets $\{F_r,\ 1 \leqq r \leqq d\}$, are clearly disjoint and $P^{(d)}$-cl., hence $P^{(k)}$-cl. It follows from Proposition 32 that there are at least d distinct $P^{(k)}$-max. indecomp. sets; hence $\delta(k) \geqq d$, which is (9).

Next, we show that

$$\text{if } k \mid k' \quad \text{then} \quad \delta(k) \mid \delta(k'). \tag{10}$$

Since X is indecomp., and $\bigcup_{i=1}^{\delta(k)} D_i$ and $\bigcup_{i=1}^{\delta(k')} E_i$ are both $P^{(1)}$-cl., we have $D_i \cap E_i \neq 0$ for some i and j. By relabelling we may suppose that $D_1 \cap E_1 \neq 0$. Define D_i and E_i for all $i \geqq 1$ by setting $D_i = D_j$ if $i \equiv j \pmod{\delta(k)}$ and $E_i = E_j$ if $i \equiv j$ $(\mathrm{mod}\ \delta(k'))$. Then it follows from the properties of cycles that the sets $D_i \cap E_i$, $1 \leqq i \leqq \delta(k) \vee \delta(k')$, are disjoint and $P^{(k \vee k')}$-cl., hence $P^{(k')}$-cl. if $k \mid k'$. Hence

$$\delta(k) \vee \delta(k') \leqq \delta(k')$$

by Proposition 32 and consequently (10) is true.

We can now prove that

$$\text{(11)} \qquad \text{if } k \mid k' \quad \text{then} \quad \delta(k) = k \wedge \delta(k').$$

For $\delta(k) \mid k$ by Proposition 33; together with (10) this implies

$$\delta(k) \leqq k \wedge \delta(k').$$

Together with (9) this implies (11).

Let $k \vee k' = l$, then we have by (11):

$$\text{(12)} \qquad \delta(k) = k \wedge \delta(l), \quad \delta(k') = k' \wedge \delta(l).$$

Since $\delta(l) \mid l$ it is a simple arithmetical fact that

$$\text{(13)} \qquad (k \wedge \delta(l)) \vee (k' \wedge \delta(l)) = (k \vee k') \wedge \delta(l) = l \wedge \delta(l) = \delta(l).$$

Substituting from (12) into (13) we obtain (7).

Finally, let $k \wedge k' = d$; then it follows from (12) that

$$\text{(14)} \qquad \delta(d) = d \wedge (\delta(k) \wedge \delta(k')).$$

Since $\delta(k) \mid k$ and $\delta(k') \mid k'$ by Proposition 33, (14) reduces to (8).

Proposition 35. *We have for an arbitrary k,*

$$\text{(15)} \qquad \delta(\delta(k)) = \delta(k);$$

and the cycle belonging to $\delta(k)$ coincides, member for member, with that belonging to k.

Proof. Writing $d = \delta(k)$, we observe that each I_i in Proposition 33 is $P^{(d)}$-cl. and $P^{(k)}$-max. indecomp. Hence it is $P^{(d)}$-max. indecomp. by the last assertion in Proposition 30. Thus $\delta(d) \geqq d$ and since $\delta(d) \mid d$ we have $\delta(d) = d$. The rest follows.

The equation (15) also follows from (8) if we substitute $\delta(k)$ for k' there and use the fact that $\delta(k) \mid k$.

Proposition 36. *To each prime number p there corresponds an e_p which is either a nonnegative integer or „infinite", such that*

$$\delta(p^n) = p^{\min(n, e_p)}$$

for each $n \geqq 1$.

Proof. For each prime p define $e = e_p$ to be the least nonnegative integer such that $\delta(p^{e+1}) \neq p^{e+1}$, or ∞ if such an integer does not exist. Then $\delta(p^n) = p^n$ for $0 \leqq n < e + 1$. If $e = \infty$ there is nothing more to prove. Suppose now $0 \leqq e < \infty$, then by (10):

$$p^e = \delta(p^e) \mid \delta(p^{e+1}) < p^{e+1},$$

so that $\delta(p^{e+1}) = p^e$. Hence for each $n \geqq e + 1$ we have by (11):

$$p^e = \delta(p^{e+1}) = p^{e+1} \wedge \delta(p^n).$$

It follows that $\delta(p^n) = p^e$ since $\delta(p^n) \mid p^n$.

Proposition 37. *Let*

$$k = \prod_p p^{f_p}$$

be the prime-factorization of k, then

$$\delta(k) = \prod_p p^{\min(f_p, e_p)}$$

where e_p is as given in Proposition 36.

Proof. This is an immediate consequence of Proposition 36 and equation (7).

§ 4. Consequent sets

Definition 9. The set C in $\mathscr{B}$ is called a k^{th} *consequent of* x if $P^{(k)}(x, C) = 1$. The sequence $\{C_k, k \geqq 1\}$ is called a *consequent sequence of* x if for each $k \geqq 1$, C_k is a k^{th} consequent of x.

Proposition 38. *Given a consequent $\{C_k, k \geqq 1\}$ of x, there exists a consequent sequence $\{D_k, k \geqq 1\}$ of x such that $D_k \subset C_k$ and $D_k \subset \mathfrak{A}(D_{k+1})$.*

Proof. Let

$$D_k = \bigcap_{j=0}^{\infty} \mathfrak{A}^j(C_{k+j}).$$

Then $D_k \subset \mathfrak{A}^0(C_k) = C_k$; and

$$\mathfrak{A}(D_{k+1}) = \bigcap_{j=0}^{\infty} \mathfrak{A}^{j+1}(C_{k+1+j}) = \bigcap_{j=1}^{\infty} \mathfrak{A}^j(C_{k+j}) \supset D_k.$$

Since $P^{(k+j)}(x, C_{k+j}) = 1$ for each $j \geqq 0$, we have by Proposition 26,

$$P^{(k)}(x, \mathfrak{A}^j(C_{k+j})) = 1;$$

and consequently

$$P^{(k)}(x, \bigcap_{j=0}^{\infty} \mathfrak{A}^j(C_{k+j})) = 1.$$

This proves that D_k is a k^{th} consequent of x for each $k \geqq 1$.

Definition 10. For each x we define a probability measure $\pi_x(\cdot)$ as follows: for each $E \in \mathscr{B}$,

$$\pi_x(E) = \sum_{n=1}^{\infty} \frac{1}{2^n} P^{(n)}(x, E).$$

It is clear that $\pi_x(\cdot)$ is a probability measure, and that $\pi_x(E) = 0$ if and only if $L(x, E) = 0$, or equivalently if and only if $x \in E^0$.

Definition 11. A k^{th} consequent C of x is called *minimal* if C is minimal with respect to the measure π_x, namely if there does not exist a k^{th} consequent D with $\pi_x(D) < \pi_x(C)$. A *minimal consequent sequence* is one in which each member is minimal.

Proposition 39. *For each x and each consequent sequence $\{C_k, k \geqq 1\}$ of x, there exists a minimal consequent sequence $\{D_k, k \geqq 1\}$ such that $D_k \subset C_k$ for each $k \geqq 1$.*

Proof. There always exists a consequent sequence of x, namely the sequence all members of which are X. Writing for a moment $C \in \mathscr{C}_k(x)$ if C is a k^{th} consequent of x we set

$$a_k = \inf_{C \in \mathscr{C}_k(x)} \pi_x(C).$$

Then there exists a $C_{k,n}$ in $\mathscr{C}_k(x)$ with $\pi_x(C_{k,n}) < a_k + \frac{1}{n}$. Let $D_k = C_k \cap \bigcap_{n=1}^{\infty} C_{k,n}$; then D_k is a k^{th} consequent of x and $\pi_x(C_k) = a_k$. Clearly $\{D_k, k \geqq 1\}$ is a minimal consequent sequence and $D_k \subset C_k$ for each $k \geqq 1$.

Proposition 40. *In an indecomp. space X two minimal k^{th} consequents of a given x differ by a set which is not abs. ess.*

Let C_k and D_k be two minimal k^{th} consequents of x, then $\pi_x(C_k \triangle D_k) = 0$ and so by a previous remark $(C_k \triangle D_k)^0 \neq 0$. Consequently $C_k \triangle D_k$ is not abs. ess. by Proposition 17.

Proposition 41. *Let X be indecomp., x an arbitrary point of X, and $\{C_n, n \geqq 1\}$ an arbitrary consequent sequence of x. There exists a not abs. ess. set F (depending on x) and for each $y \in X - F$ there exists a positive integer $m(y)$ such that $\{C_{m(y)+n}, n \geqq 1\}$ is a consequent sequence of y.*

Proof. We have for each pair of integers m and n with $m < n$:

$$1 = P^{(n)}(x, C_n) = \int_X P^{(n-m)}(y, C_n)\, P^{(m)}(x, dy).$$

Hence there is a set $F_{m,n}$ in $\mathscr{B}$ with $P^{(m)}(x, F_{m,n}) = 0$ and such that if $y \in X - F_{m,n}$ then

$$P^{(n-m)}(y, C_n) = 1.$$

Let $F_m = \bigcup_{n=m+1}^{\infty} F_{m,n}$. Then $P^{(m)}(x, F_m) = 0$; and if $y \in X - F_m$, the above equation holds for every $n \geqq m + 1$. Let $F = \bigcap_{m=1}^{\infty} F_m$, then $F \in \mathscr{B}$ and $P^{(m)}(x, F) = 0$ for every $m \geqq 1$. Consequently $F^0 \neq 0$ and F is not abs. ess. by Proposition 17. If $y \in X - F$, then there exists a positive integer $m = m(y)$ such that $y \in X - F_m$ and $P^{(k)}(y, C_{m+k}) = 1$ for every $k \geqq 1$. This proves the proposition.

In propositions 42 to 48 the space X is assumed to be abs. ess. and indecomp.

Proposition 42. *Let X be abs. ess. and indecomp. For each x there exists a finite positive integer $k(x)$ such that if $\{C_k, k \geqq 1\}$ is any consequent sequence of x then there exist m and n both less than $k(x) + 1$ such that $C_m \cap C_n$ is abs. ess.*

It is sufficient to prove this for a fixed minimal consequence $\{C_k, k \geqq 1\}$. For then the conclusion will remain valid with the same m and n for any consequent sequence of x by Proposition 40. Furthermore we may suppose on account of Propositions 39 and 38 that $C_k \subset \mathfrak{A}(C_{k+1})$. Hence $C = \bigcup_{k=1}^{\infty} C_k$ is cl. and consequently abs. ess. by Proposition 18. Set

$$D_k = C_k - \bigcup_{j=1}^{\infty} (C_k \cap C_{k+j}).$$

If $y \in D_k$ then $P^{(j)}(y, C_{k+j}) = 1$ and hence $P^{(j)}(y, D_k) = 0$ for each $j \geqq 1$, since $D_k \cap C_{k+j} = 0$. Thus $L(y, D_k) = 0$ and $D_k \subset D^0{}_k$. Such a D_k is clearly iness. and consequently $D = \bigcup_{k=1}^{\infty} D_k$ is not abs. ess. But

$$C - D = \bigcup_{k=1}^{\infty} \bigcup_{j=1}^{\infty} (C_k \cap C_{k+j}).$$

It follows that at least one $C_k \cap C_{k+j}$ is abs. ess., as was to be proved.

For each x let $\{C_k(x), k \geqq 1\}$ be a minimal consequent sequence of x and let

$$h(x) = \min\{|m-n| : C_m(x) \cap C_n(x) \text{ is abs. ess.}\},$$
$$h'(x) = \text{g. c. d.}\{|m-n| : C_m(x) \cap C_n(x) \text{ is abs. ess.}\},$$

where "g. c. d." stands for "the greatest common divisor". According to Proposition 42, $h'(x) \leqq h(x) \leqq k(x) < \infty$, and both $h(x)$ and $h'(x)$ are independent of the choice of the minimal consequent sequence.

Proposition 43. *There exist an integer H and a set F_H which is in $\mathscr{B}$ and not abs. ess. such that $h(x)$ is equal to H for all $x \in X - F_H$, and $h(x) \leqq H$ for all $x \in X$.*

Proof. Let x be an arbitrary point and let $\{C_n, n \geqq 1\}$ be a minimal consequent sequence of x such that $C_n \subset \mathfrak{A}^k(C_{n+k})$ for each $n \geqq 1$ and $k \geqq 1$. Such a choice is possible by Proposition 38. Let $h(x) = l$, then by definition there exists an integer j such that

$$C_j \cap C_{j+l} \text{ is abs. ess.}$$

For every y in this intersection, we have by the choice of $\{C_n, n \geqq 1\}$, for each $k \geqq 1$,

$$P^{(k)}(y, C_{j+k} \cap C_{j+k+l}) = 1.$$

Hence by Proposition 9,

$$C_{j+k} \cap C_{j+k+l} \text{ is abs. ess.}, \quad \text{for each } k \geqq 0,$$

or

$$C_n \cap C_{n+l} \text{ is abs. ess.}, \quad \text{for each } n \geqq j. \tag{16}$$

According to Proposition 41, there exists a set F in $\mathscr{B}$ which is not abs. ess. such that if $y \in X - F$, then $\{C_{m(y)+n}, n \geqq 1\}$ is a consequent sequence (but not necessarily minimal) of y for some $m(y) \geqq 1$. Hence we have $h(y) \geqq l$ by the definition of $h(\cdot)$.

We now prove that the function $h(\cdot)$ is bounded on X. For otherwise let $\{x_n, n \geqq 1\}$ be points of X such that $\lim_{n\to\infty} h(x_n) = \infty$. By what we have proved, for each x_n there exists a set F_{x_n} in $\mathscr{B}$ which is not abs. ess. and such that

$$h(y) \geqq h(x_n) \quad \text{if} \quad y \in X - F_{x_n}.$$

Since X is abs. ess., $X - \bigcup_{n=1}^{\infty} F_{x_n}$ is not empty; and if y is in this set, $h(y)$ would be ∞ which is impossible. Hence we may set

$$\max_{x \in X} h(x) = H < \infty.$$

By the argument above, there exists a not abs. ess. set F_H such that $h(x) \geqq H$, hence $h(x) = H$ on $X - F_H$, as was to be proved.

Remark. It has not been shown that the function h is $\mathscr{B}$-measurable, but this information will not be needed below.

Definition 12. The integer H is called the *overlapping index*, and the set $X - F_H$ (in $\mathscr{B}$) the *overlapping core* of the abs. ess. and indecomp. space X.

Proposition 43.1. *For each x in $X - F_H$ there exists an integer $\nu(x)$ such that for an arbitrary consequent sequence $\{C_n, n \geqq 1\}$ of x,*

$$C_n \cap C_{n+H} \text{ is abs. ess. for } n \geqq \nu(x).$$

Proof. This is merely a restatement of (16).

Proposition 44. *For each k, $\delta(k) \mid H$.*

Proof. Consider the cycle $\{I_i, 1 \leqq i \leqq \delta(k)\}$ belonging to k and set $I_i = I_j$ if $i \equiv j \pmod{\delta(k)}$. Then $C = \bigcup_{i=1}^{\delta(k)} I_i$ is abs. ess. by Proposition 33, since X is abs. ess. and $X - C$ is not. If $x \in C \cap (X - F_H)$, then $\{I_i, i \geqq 1\}$ is a consequent sequence of x, and we have by Proposition 43,

$$I_i \cap I_{i+H} \neq 0$$

for some i. But the cycle is clean according to Proposition 33, hence $\delta(k) \mid H$.

Definition 13. Let $D = \max_{k \geqq 1} \delta(k)$; D is called the *maximum cyclic index* and the cycle belonging to D is called the *maximum cycle.*

Proposition 45. *In the notation of Proposition 36, we have*

$$D = \prod_p p^{e_p}$$

where $0 \leqq e_p < \infty$ *for each prime* p *and also* $e_p > 0$ *for only a finite number of values of* p. *Furthermore, we have for each* $k \geqq 1$,

(17) $$\delta(k) = k \wedge D.$$

Proof. This is immediate from Propositions 36, 37 and 44, the last implying that $e_p < \infty$ for each p. A more direct proof of (17) is as follows. Let $\delta(k') = D$, then by (9),

(18) $$\delta(k) \geqq k \wedge D.$$

On the other hand, by (7),

$$\delta(k \vee k') = \delta(k) \vee D;$$

hence $\delta(k) \mid D$ for otherwise one would have $\delta(k \vee k') > D$ which is impossible by the definition of D. Since $\delta(k) \mid k$ it follows that $\delta(k) \mid (k \wedge D)$ and so there must be equality in (18).

Example 1. $X = \{1, 2, 3, 4, 5\}$.

$$P(n, n+1) = 1 \quad \text{for} \quad n = 1, 2, 3;$$
$$P(4, 1) = 1; \quad P(5, 1) = P(5, 2) = \tfrac{1}{2}.$$

Each $\{n\}$, $n = 1, 2, 3, 4$, is $P^{(4)}$-max. indecomp.; $\{1, 3\}$ and $\{2, 4\}$ are $P^{(2)}$-indecomp., but $\{2, 4\}$ is not $P^{(2)}$-max. indecomp. since $\{2, 4, 5\}$ is. This example shows that the cycle belonging to a divisor of k is not necessarily obtained by the obvious grouping from the cycle belonging to k.

Example 2. $X = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

$$P(1, 5) = P(1, 6) = \tfrac{1}{2};$$
$$P(2, 5) = P(2, 6) = P(2, 7) = P(2, 8) = \tfrac{1}{4};$$
$$P(3, 7) = P(4, 8) = P(5, 3) = P(6, 4) = P(7, 1) = P(8, 2) = 1.$$

Here the maximum index $D = 2$ and the maximum cycle is composed of $\{1, 2, 3, 4\}$ and $\{5, 6, 7, 8\}$. It is easily verified that $H = 2$ and $F_H = 0$.

The minimal consequent sequence for $\{6\}$ is

$$\{6\}, \{4\}, \{8\}, \{2\}, \{5, 6, 7, 8\}, \{1, 2, 3, 4\}, \{5, 6, 7, 8\}, \ldots .$$

If we denote this sequence of sets by $\{C_n, n \geqq 0\}$, it is to be noted that $C_1 \cap C_n = 0$ for $n = 2, 3, 4$ but $C_2 \cap C_4 \neq 0$.

Example 3. $X = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

$$P(1, 2) = P(1, 4) = P(3, 4) = P(3, 6) = \tfrac{1}{2};$$
$$P(2, 3) = P(4, 5) = P(5, 6) = P(6, 7) = P(7, 8) = P(8, 1) = 1.$$

The minimal consequent sequence for $\{1\}$ is:

$$\{1\}, \{2, 4\}, \{3, 5\}, \{4, 6\}, \{5, 7\}, \{6, 8\}, \{7, 1\},$$
$$\{2, 4, 8\}, \{1, 3, 5\}, \{2, 4, 6\}, \{3, 5, 7\}, \{4, 6, 8\}, \{5, 7, 1\},$$
$$\{2, 4, 6, 8\}, \{1, 3, 5, 7\}, \{2, 4, 6, 8\}, \ldots .$$

Here in notation similar to the above: $C_1 \cap C_3 \neq 0$, $C_3 \cap C_5 \neq 0$, but $C_1 \cap C_5 = 0$.

Proposition 46. *There exist an integer H' and a set $F'_{H'}$ which is in $\mathscr{B}$ and not abs. ess. such that $h'(x)$ is equal to H' for all $x \in X - F'_{H'}$, and $h'(x) \leqq H'$ for all $x \in X$.*

Proof. Let

$$H' = \max_{x \in X} h'(x);$$

since $h'(x) \leqq h(x)$ for every x, we have $H' \leqq H < \infty$. The rest of the proof is exactly the same as the first part of the proof of Proposition 43.

Remark. In Propositions 43 and 46, we may replace the sets $X - F_H$ and $X - F'_{H'}$ by cl. subsets. For if we set

$$G = (X - F_H) \cap (X - F_H)^0, \tag{19}$$

$$G' = (X - F'_{H'}) \cap (X - F'_{H'})^0, \tag{20}$$

then G and G' are cl. by Proposition 1, and $X - G$ and $X - G'$ are not abs. ess. by Proposition 18.1.

The next proposition is due to S. T. C. Moy.

Proposition 47. $D = H'$.

Proof. Choose any x in $K \cap G'$ where K is the maximum cycle and G' is given in (20), $K \cap G'$ being cl. by Proposition 18. Let $\{C_j(x), j \geqq 1\}$ be a minimal consequent sequence of x. Then $C_m(x) \cap C_n(x) \neq 0$ implies $D \mid (m - n)$; hence $D \mid H'$. On the other hand, let us set for such an x:

$$E_r = \bigcup_{n=0}^{\infty} C_{nH'+r}(x), \quad 0 \leqq r \leqq H' - 1$$

where the C_j's have been chosen to satisfy

$$C_j(x) \subset \mathfrak{A}(C_{j+1}(x)) \tag{21}$$

by Proposition 38. Then each E_r is $P^{(H')}$-cl. and the H' sets $\{E_r, 0 \leqq r \leqq H' - 1\}$ form a H'-cycle. If F denotes the union of the pairwise intersections of the $E'_r s$, F is not abs. ess. by the definition of $H' = h'(x)$. Hence $F^0 \cap F^c$ is cl. by Propositions 19 and 1. The H' sets $F^0 \cap F^c \cap E_r$ are disjoint; their union is nonempty since the union of the E_r's is cl. and X is indecomp., hence each $F^0 \cap F^c \cap E_r$ is nonempty and so $P^{(H')}$-cl. by properties of a cycle. Thus we have by Propositions 32 and 33:

$$D \geqq \delta(H') = H'.$$

Hence $D = H'$.

The next proposition, conjectured by the author, was first proved by H. KESTEN. The version given below, using an essential idea of his, is simpler.

Proposition 48. $H' = H$.

Lemma 1. *Let x be arbitrary and $\{C_n, n \geqq 1\}$ a minimal consequent sequence of x. Suppose that $n \neq m$ and*

$$C_m \cap C_n \text{ is abs. ess.,} \tag{22}$$

then

$$P^{(n)}(x, C_m) > 0. \tag{23}$$

Proof. By Proposition 19, (22) implies that $(C_m \cap C_n)^0 = 0$. It follows that

$$\pi_x(C_m \cap C_n) > 0,$$

and consequently

$$\pi_x(C_n \setminus C_m) < \pi_x(C_n).$$

Since C_n is a minimal n^{th} consequent set of x, $C_n \setminus C_m$ cannot be likewise. Thus

$$P^{(n)}(x, C_n \setminus C_m) < 1$$

which implies (23).

Lemma 2. *Let the hypotheses in Lemma 1 hold for an x in G where G is given by* (19). *Then there exists an $l \geqq 1$ such that*

$$C_{n+H+l} \cap C_{m+l} \text{ is abs. ess.} \tag{24}$$

Proof. We may choose the C_j's to satisfy (21) and furthermore $C_j \subset G$ for every $j \geqq 1$. Let $y \in C_n$ and $\{D_j(y), j \geqq 1\}$ be a minimal consequent sequence of y. Owing to (21) we may suppose that for every y in C_n we have $D_j(y) \subset C_{j+n}$ for every $j \geqq 1$. Since $h(y) = H$, there exists a $j = j(y)$ such that

$$D_j(y) \cap D_{H+j}(y) \text{ is abs. ess.}$$

It follows from Lemma 1 that

$$P^{(j)}(y, C_{n+H+j}) \geqq P^{(j)}(y, D_{H+j}(y)) > 0. \tag{25}$$

If $j < k$, we have by (21)

$$P^{(j)}(y, C_{n+H+j}) \leqq P^{(k)}(y, C_{n+H+k}).$$

Consequently if we set

$$C_{n,k} = \left\{y \in C_n : P^{(k)}(y, C_{n+H+k}) \geqq \frac{1}{k}\right\},$$

then $C_n = \bigcup_{k=1}^{\infty} C_{n,k}$ and in particular

$$C_m \cap C_n = \bigcup_{k=1}^{\infty} [C_m \cap C_{n,k}].$$

The hypothesis (22) then implies the existence of an l such that

$$E = C_m \cap C_{n,l} \text{ is abs. ess.}$$

Let us write also

$$F = C_{m+l} \cap C_{n+H+l}\,.$$

For each $y \in E$, it follows from (21) and the definition of C_{n+l} that

$$P^{(l)}(y, F) = P^{(l)}(y, C_{n+H+l}) \geqq \frac{1}{l}\,.$$

Therefore

$$\inf_{x \in E} L(x, F) \geqq \frac{1}{l} > 0\,.$$

Since E is abs. ess., this implies F is abs. ess. by Proposition 9.

Lemma 3. *Under the same hypotheses as in Lemma 2, we have* $H \mid (m - n)$.

Proof. We may suppose that $n < m$ and

$$m - n = qH + r\,, \quad 0 \leqq q\,, \quad 1 \leqq r \leqq H\,.$$

Applying Lemma 2 successively q times, we infer that there exists an $l \geqq 1$ such that

$$C_{n+qH+l} \cap C_{m+l} \text{ is abs. ess.}$$

By the definition of $H = h(x)$, this implies

$$H \leqq (m + l) - (n + qH + l) = r \leqq H$$

Hence $r = H$ and $H \mid (m - n)$.

Proof of Proposition 48. Choose any x in $G \cap G'$ (see (19) and (20)) which is nonempty since the space is indecomp. Then $h(x) = H$, $h'(x) = H'$. By the definitions, we have $h'(x) \leqq H$. Furthermore it follows from Lemma 3 that $H \mid h'(x)$. Hence $H' = h'(x) = H$.

Proposition 48.1. *For every* $x \in G \cap G'$, *we have*

$$h(x) = h'(x) = H = H' = D\,.$$

§ 5. Decomposition theorems

Proposition 49. *Suppose that X is indecomp. and abs. ess. For each x in X there exists a cl. set C such that: if $E \subset C$ then either E is abs. ess. or* $Q(x, E) = 0$.

Proof. Let $\mathscr{C}$ be the family of cl. sets in X, and set

$$\alpha = \alpha(x) = \inf_{C \in \mathscr{C}} L(x, C)\,. \tag{26}$$

For each n, there exists a cl. set C_n such that

$$L(x, C_n) \leqq \alpha + \frac{1}{n}\,.$$

Let $C = C(x) = \bigcap_{n=1}^{\infty} C_n$. We have

$$L(x, C) \leqq \lim_{n \to \infty} L(x, C_n) \leqq \alpha\,;$$

hence

$$L(x, C) = \alpha$$

by the definition of α. Furthermore $\alpha > 0$ since $C^0 = 0$.

If $E \subset C$ and E is not abs. ess., then either E is iness. and so $Q(y, E) = 0$ for every $y \in X$; or E is imp. ess. In the latter case $E^0 \neq 0$ by Proposition 19, and $E^0 \cap C$ is cl. by indecomposability. Starting from x, if the process $\{\xi_n, n \geqq 0\}$ is in E infinitely often, then it must be in C infinitely often and never in $E^0 \cap C$; it follows that

$$Q(x, E) \leqq L(x, C) - L(x, E^0 \cap C).$$

Both terms on the right side are equal to α by the definition of α and C, hence $Q(x, E) = 0$ as was to be proved.

Let us write, for any x in X and E in $\mathscr{B}$:

$$M(x, E) = 1 - Q(x, E^c). \tag{27}$$

Thus $M(x, E)$ is the probability that the process starting from x ultimately stays in E. In this notation the set E is perp. (Definition 4) if and only if $M(x, E) > 0$ for some x in X.

Proposition 49.1. *Let $\mathscr{N}$ be the family of all sets which are not abs. ess., and $\alpha(x)$ be defined as in* (26), *then for each x*:

$$\sup_{E \in \mathscr{N}} M(x, E) = 1 - \alpha(x).$$

Proposition 50. *Let X be arbitrary, C a cl. subset such that $X - C$ does not contain any cl. set. Then there exists a sequence of disjoint iness. (possibly empty) sets $\{E_i, i \geqq 1\}$ such that*

$$X - C = \bigcup_{i=1}^{\infty} E_i; \tag{28}$$

$$\lim_{n \to \infty} P^{(n)}(x, C) = L(x, C) \tag{29}$$

for each x; and

$$\lim_{n \to \infty} P^{(n)}(x, \bigcup_{i=j+1}^{\infty} E_i) = 1 - L(x, C) \tag{30}$$

for each $j \geqq 0$.

Proof. Let

$$E_i = \left\{x \in X - C : \frac{1}{i} \leqq L(x, C) < \frac{1}{i-1}\right\}$$

for $i \geqq 1$. Since $X - C$ does not contain any cl. set, $C^0 = 0$ and consequently (28) holds. The set E_1 is clearly iness., and each E_i, $i \geqq 2$, is iness. by Proposition 10. Since C is cl., we have

$$\sum_{\nu=1}^{n} K^{(\nu)}(x, C) \leqq P^{(n)}(x, C) \leqq \sum_{\nu=1}^{\infty} K^{(\nu)}(x, C).$$

Letting $n \to \infty$ we obtain (29). Furthermore we have for arbitrary x in X and E in $\mathscr{B}$:

$$\overline{\lim_{n \to \infty}}\, P^{(n)}(x, E) \leqq Q(x, E).$$

Since the union of a finite number of iness. sets is iness., it follows that

$$\overline{\lim_{n \to \infty}}\, P^{(n)}(x, \bigcup_{i=1}^{j} E_i) \leqq Q(x, \bigcup_{i=1}^{j} E_i) = 0. \tag{31}$$

Finally we have

$$(32)\qquad 1 = P^{(n)}(x, X) = P^{(n)}(x, C) + P^{(n)}(x, \bigcup_{i=1}^{j} E_i) + P^{(n)}(x, \bigcup_{i=j+1}^{\infty} E_i)$$

Hence (30) follows from (29), (31) and (32).

Proposition 51. *Let X be arbitrary and φ be a σ-finite measure on $(X, \mathscr{B})$ such that if C is cl. then $\varphi(C) > 0$. Then there exists a set A which is the union of at most a denumerable number of indecomp. sets and such that A^0 does not contain any indecomp. set and is imp. ess. Furthermore $X - A - A^0$ does not contain any cl. set and is not abs. ess.*

Proof. It is well known that from a σ-finite measure one can construct a finite measure which is co-positive, hence we may suppose φ to be finite. Let the family of all indecomp. sets be $\{B_\alpha\}$ and let $A_\alpha = (B_\alpha^0)^0$. Each A_α is max. indecomp. by Proposition 15. Since $\varphi(A_\alpha) > 0$ for each α and $\varphi(X) < \infty$, the family of distinct A_α's is at most denumerable by Proposition 16. We put

$$A = \bigcup_\alpha A_\alpha .$$

The set A^0 is either cl. or empty, and since it is disjoint from A it cannot contain any indecomp. set by the definition of A. By Proposition 14, $X - A - A^0$ does not contain any cl. set and is not abs. ess. It remains to prove that A^0 is imp. ess. if not empty.

The following proof, considerably shorter than DOEBLIN's (cf. my Columbia lecture notes), is due to T. E. HARRIS.

Let $\varphi(A^0) = \lambda > 0$. For each x in A^0 let $\mathscr{C}(x)$ be the family of cl. sets containing x and let

$$\tilde{\varphi}(x) = \inf_{C \in \mathscr{C}(x)} \varphi(C).$$

Observing that any sequence of sets in $\mathscr{C}(x)$ has a cl. intersection since it is nonempty, we deduce by the usual argument the existence of a set C_x in $\mathscr{C}(x)$ such that

$$\tilde{\varphi}(x) = \varphi(C_x) > 0 .$$

For each $n \geqq 1$, let

$$E_n = \left\{x \in A^0 : \tilde{\varphi}(x) \leqq \frac{\lambda}{n}\right\}.$$

If $y \in C_x$, then $\tilde{\varphi}(y) \leqq \varphi(C_x)$. It follows that if $x \in E_n$, then $C_x \subset E_n$ so that E_n is cl. for each $n \geqq 1$. Furthermore $A^0 = E_1 \supset E_2 \supset \ldots$, and $\bigcap_n E_n = 0$. For otherwise $\bigcap_n E_n$ would be cl. and if y were any point in it, $\tilde{\varphi}(y)$ would be zero which is impossible. We have therefore

$$(33)\qquad A^0 = \bigcup_n (A^0 - E_n) .$$

Suppose $A^0 - E_n$ were to contain a cl. set, then it would contain a cl. set with arbitrarily small φ-measure since every cl. subset of A^0 is decomp. In particular it would contain a cl. set F with $\varphi(F) \leqq \lambda/n$. Let $y \in F$, then

$$\tilde{\varphi}(y) \leqq \varphi(F) \leqq \frac{\lambda}{n},$$

which is impossible since $y \in A^0 - E_n$ implies $\tilde{\varphi}(y) > \lambda/n$. Hence for each n, $A^0 - E_n$ does not contain any cl. set and so is not abs. ess. by Proposition 11. It follows that A^0 is not abs. ess. by (33); but since A^0 is cl. it is ess. Thus A^0 is imp. ess. as was to be proved.

Proposition 52. *Let X be indecomp. and abs. ess. and φ be a σ-finite measure on $(X, \mathscr{B})$ such that if A is perp. then $\varphi(A) > 0$. Then we have*

$$X = B \cup C, \quad B \cap C = 0; \tag{34}$$

where B is perp. and imp. ess., C is cl. and every ess. subset E of C is abs. ess. and satisfies the relation

$$C \subset E^\infty. \tag{35}$$

Proof. As in the proof of Proposition 51 we may suppose that φ is a finite measure. Let $\mathscr{S}$ be the family of perp. and imp. ess. sets and let

$$\alpha = \sup_{A \in \mathscr{S}} \varphi(A).$$

We deduce by the usual argument the existence of a set A in $\mathscr{S}$ such that $\varphi(A) = \alpha$. Clearly $X - A$ does not contain any set in $\mathscr{S}$. Now take

$$B = (X - A^0) \cup A, \quad C = X - B = A^0 \cap (X - A).$$

Since $X - A^0$ is not abs. ess. by Proposition 18.1 we have $B \in \mathscr{S}$; C is cl. by Proposition 1. Since C does not contain any set in $\mathscr{S}$, it does not contain any imp. ess. set by Proposition 23.1. Hence any ess. subset E of C is abs. ess. By Proposition 19, E^∞ is cl. and $E^0 = 0$. It follows from Proposition 14.1 that $C - CE^\infty$ is not abs. ess. so it is iness. by what has just been proved. Thus if $x \in C - CE^\infty$, we have by Proposition 7 and the inequality (6):

$$Q(x, E) \geqq Q(x, CE^\infty) \geqq 1 - Q(x, C - CE^\infty) = 1.$$

On the other hand, $Q(x, E) = 1$ if $x \in E^\infty$. Thus $Q(x, E) = 1$ for every $x \in C$, and this is equivalent to (35).

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