Z. Wahrscheinlichkeitstheorie verw. Geb. 28, 139–164 (1974)

Balayage and Multiplicative Functionals

R. K. Getoor and M. J. Sharpe*

1. Introduction

The original purpose of this paper was to extend the last exit decomposition obtained in [5] to general semigroups (Q_t) subordinate to (P_t). It is well known that such a subordinate semigroup is generated by a multiplicative functional (m_t) and we regard (m_t) rather than (Q_t) as the basic datum. The decomposition is contained in (6.17). When $m_t = 1_{[0, D)}(t)$ where D is the hitting time of a finely closed nearly Borel set the decomposition (6.17) reduces to the last exit decomposition of [5], but we actually have more detailed information about the ingredients of the decomposition here than we had in [5]. (Somewhat more generally, if D is the debut of a Markov set M as defined in [7], see also Section 3 of this paper, then (6.17) reduces to the last exit decomposition given by Meyer in [7].)

In the course of proving our decomposition theorem it was natural to extend the notion of balayage of an additive functional on a set F to balayage on (or via) a multiplicative functional m. This seems to be of some interest in its own right and we have developed these ideas in Section 3 and 4 in more detail and generality than would have been strictly necessary for the proof of (6.17). In this development we have been greatly influenced by Meyer's approach in [7] to the basic results of our paper [5]. Also Azéma [11] has studied balayage in a general setting and given many applications to Markov processes some of which are related to our work.

In Section 2 we collect some elementary facts about subsets of $\mathbf{R}^+ = [0, \infty)$ and certain functions on $\mathbf{R}^+$. These ideas unify and simplify some of the basic properties of random sets. In particular some of the results of Section VI-1 of [3] may be given alternate proofs using these ideas. However, we do not pursue this point here. Sections 3, 4 and 5 contain the main results of the paper. In Section 6 these results are applied to prove the decomposition (6.17). Finally there is an appendix that contains a useful characterization of the well measurable processes over a Markov process that is used in Section 3. Undoubtedly this characterization is known to the experts, but we have been unable to find it in the literature and so it is, perhaps, worthwhile to set it down explicitly here.

Throughout this paper we shall work with a Markov process satisfying the "right" hypotheses. In this paper we shall often, but not always, omit the phrase "almost surely" when it is obviously required. In particular equality of random variables will mean equality almost surely. On the other hand equality of processes (or subsets of $\mathbf{R}^+ \times \Omega$) will mean P^μ indistinguishability for all finite initial measures μ.

* This research was supported in part by NSF grant GP-30737.

2. Preliminaries

This section contains no probability theory. Rather it contains some elementary facts about functions on $\mathbf{R}^+ = [0, \infty)$ that will be useful in later sections. In particular certain notations are established that will be used throughout this paper. If f is a function on $\mathbf{R}^+$ or $[0, \infty]$, we shall sometimes write f_t for $f(t)$ as is customary in probability theory. In this section the letters t, s, u, v always stand for arbitrary elements of $\mathbf{R}^+$ unless stated otherwise.

(2.1) *Definition.* An increasing function S defined on $\mathbf{R}^+$ and taking values in $[0, \infty]$ is called an announcing function provided $S_t \geqq t$ for all t and S is constant on every interval of the form $[t, S_t)$.

It is immediate from the definition (consider the cases $u < v$ and $v \leqq u < S_v$) that

$$(2.2) \qquad u < S_v \quad \text{implies} \quad S_u \leqq S_v$$

whenever S is an announcing function.

Here are two simple examples of announcing functions that motivate the terminology. Let M be a subset of $\mathbf{R}^+$ and define

$$S_t^1 = \inf\{u > t : u \in M\}; \qquad S_t^2 = \inf\{u \geqq t : u \in M\}.$$

(Here and in the sequel we adopt the conventions that the infimum of the empty set is infinity and the supremum of the empty set is zero.)

Given an announcing function S we define two functions D and d on $\mathbf{R}^+$ as follows:

$$(2.3) \qquad \begin{aligned} D_t &= S_{t+} = \inf_{u > t} S_u \\ d_t &= S_{t-} = \sup_{u < t} S_u \quad \text{if} \quad t > 0; \ d_0 = 0. \end{aligned}$$

We make the convention that if f is a real function defined at 0 then $f_{0-} = f_0$. Thus $d_{0-} = d_0 = 0$. Obviously $D_t \geqq t$, $d_t \geqq t$, $d_t \leqq S_t \leqq D_t$, and $d_t = D_t$ except for countably many t. Moreover $d_t = D_{t-}$ if $t > 0$ and $D_t = d_{t+}$ for all t. Clearly D is right continuous and d is left continuous.

(2.4) **Proposition.** (i) *Both D and d are announcing functions.* (ii) $S_t = D_t$ *if* $S_t > t$. (iii) $d(S_t) = S_t$ *and* $d(D_t) = D_t$. (iv) *If* $t > 0$, *then either* $S_t = D_t$ *or* $S_t = d_t$.

Proof. If $v < D_u$, then $v < S_{u+\varepsilon}$ for all $\varepsilon > 0$ and so by (2.2), $S_v \leqq S_{u+\varepsilon}$ for all $\varepsilon > 0$. Hence $v < D_u$ implies $S_v \leqq D_u$. As a result $S_v = D_u$ if $u < v < D_u$. Thus S is constant on (u, D_u) and therefore D is constant on $[u, D_u)$. Since $D_u \geqq u$, D is an announcing function. Similarly if $d_u > u$ (which implies $u > 0$), then $S_{u-\varepsilon} > u$ for all sufficiently small $\varepsilon > 0$ and so d is constant on $(u - \varepsilon, S_{u-\varepsilon})$ for small $\varepsilon > 0$. But this implies that d is constant on $[u, d_u)$ and hence d is an announcing function.

It is immediate from (2.2) and the definition of D that $D_t = S_t$ if $S_t > t$. Also if $v < S_u$ then $S_v \leqq S_u$ by (2.2), and so $d(S_u) = \lim_{v \uparrow S(u)} S(v) \leqq S_u$ if $S_u > 0$. But $d_t \geqq t$ for all t and hence $d(S_u) = S_u$ if $S_u > 0$. But $d_0 = 0$ and thus $d(S_u) = S_u$ in all cases. Since D also satisfies (2.2) and $d_t = D_{t-}$ if $t > 0$, exactly the same argument shows $d(D_t) = D_t$. Finally if $t > 0$ and $S_t > d_t$, then $S_t > t$ and so $S_t = D_t$ by the first sentence of this paragraph. Consequently if $t > 0$ either $S_t = d_t$ or $S_t = D_t$, and this completes the proof of (2.4).

We now define two more increasing functions L and l on $\mathbf{R}^+$ as follows:

$$(2.5)\qquad \begin{aligned} L_t &= \inf\{u: D_u > t\} = \sup\{u: D_u \leqq t\} \\ l_t &= \inf\{u: D_u \geqq t\} = \sup\{u: D_u < t\}. \end{aligned}$$

Note $l_0 = L_0 = 0$. It is standard and easy to check that L is right continuous, l is left continuous, $L_t = l_{t+}$, and $l_t = L_{t-}$ for $t > 0$. Moreover it is obvious that $l_t \leqq L_t \leqq t$. In addition, since the intervals $\{u: D_u > t\}$, $\{u: S_u > t\}$, and $\{u: d_u > t\}$ differ by at most a singleton (the same statement holds if "$>$" is replaced by "$\geqq$" throughout) it is immediate that

$$(2.6)\qquad \begin{aligned} L_t &= \inf\{u: d_u > t\} = \inf\{u: S_u > t\} \\ l_t &= \inf\{u: d_u \geqq t\} = \inf\{u: S_u \geqq t\}, \end{aligned}$$

and the corresponding statements involving suprema are also valid.

(2.7) **Proposition.** (i) *$D_t \geqq u$ if and only if $l_u \leqq t$.* (ii) *$L_t \geqq u$ if and only if $d_u \leqq t$. In particular, D is the right continuous inverse of both L and l, while d is the left continuous inverse of both L and l.*

Proof. If $D_t \geqq u$, then $l_u = \sup\{v: D_v < u\} \leqq t$. Conversely if $l_u \leqq t$ then $D_{t+\varepsilon} \geqq u$ for all $\varepsilon > 0$, and hence $D_t \geqq u$. This proves (i) and a similar argument applied to d and L proves (ii). The last sentence in (2.7) is an immediate consequence of (i) and (ii) and the definitions.

Given an announcing function S and with d, D, l, and L as above we define a subset M of $\mathbf{R}^{++} = (0, \infty)$ as follows:

$$(2.8)\qquad M = \{t > 0: L_t = t\}.$$

Since $L_t \leqq t$ and $d_t \geqq t$, it follows from (2.7 ii) that

$$(2.9)\qquad M = \{t > 0: d_t = t\}.$$

The right continuity of L and left continuity of d together with (2.8) and (2.9) show that M is closed (in $\mathbf{R}^{++}$). It follows from (2.4 iii) and $d_t = D_{t-}$ that if $t > 0$, then

$$(2.10)\qquad D_t \in M, \quad S_t \in M, \quad \text{and} \quad d_t \in M \quad \text{provided they are finite.}$$

But $t \in M$ implies $d_t \in M$ and so

$$(2.11)\qquad M = \{d_t < \infty: t > 0\} = \{D_t < \infty: t > 0\}^- = \{S_t < \infty: t > 0\}^-$$

where the bar denotes closure in $\mathbf{R}^{++}$. Next observe that for $t > 0$, $M \cap [t, \infty) = \{u \geqq t: d_u = u\}$ has infimum d_t since $d_t \geqq t$ and $d_t \in M$. Hence

$$(2.12)\qquad \begin{aligned} d_t &= \inf\{u \geqq t: u \in M\}, \qquad t > 0 \\ D_t &= \inf\{u > t: u \in M\}. \end{aligned}$$

The last assertion is a consequence of the first and $D_t = d_{t+} = \inf_{u > t} d_u$.

(2.13) **Proposition.** (i) *$L(L_t) = L_t$; L is constant on $[L_t, t]$, $M = \{L_t: L_t > 0\}$, and $L_t = \sup\{u \leqq t: u \in M\}$.* (ii) *$l_t = \sup\{u < t: u \in M\}$ and $\{l_t: l_t > 0\}$ is dense in M.*

Proof. From (2.6), (2.7 ii), and (2.10) we obtain

$$\begin{aligned} L(L_t) &= \sup\{u>0: d_u \leqq L_t\} \\ &= \sup\{u>0: d(d_u)\leqq t\} = \sup\{u>0: d_u \leqq t\} = L_t. \end{aligned}$$

Combining this with (2.8) we see that $M=\{L_t: L_t>0\}$. If $L_t \leqq u \leqq t$ then $L_t = L(L_t) \leqq L_u \leqq L_t$, and hence L is constant on $[L_t, t]$. But these facts imply that for $t>0$, $\sup\{u\leqq t: u\in M\} = \sup\{L_u \leqq t: L_u>0\} = \sup\{L_u \leqq L_t;\ L_u>0\} = L_t$, while $L_0 = 0 = \sup\{u\leqq 0: u\in M\}$. This establishes (i). Now for $t>0$,

$$l_t = \sup_{u<t} L_u = \sup\{u<t: u\in M\}$$

by (i), and (ii) is now obvious.

Since M is closed in $\mathbf{R}^{++}$, $M^c = \mathbf{R}^{++} - M$ is a countable union of open intervals. Following Meyer [7] we let $M^{\rightarrow}$ denote the set of all *strictly positive* left endpoints and $M^{\leftarrow}$ the set of all right endpoints of these contiguous intervals. We emphasize that $M^{\rightarrow}$ and $M^{\leftarrow}$ are subsets of $\mathbf{R}^{++}$. It is immediate from (2.13) and (2.12) that

$$\text{(2.14)} \qquad \begin{aligned} M^{\rightarrow} &= \{t>0: d_t = t < D_t\} = \{t>0: L_t = t < D_t\} \\ M^{\leftarrow} &= \{t>0: l_t < t = L_t\} = \{t>0: l_t < t = d_t\}. \end{aligned}$$

In particular, $M^{\rightarrow} \cap M^{\leftarrow} = \{t>0: l_t < d_t = t < D_t\} = \{t>0: l_t < L_t = t < D_t\}$. From (2.4 iv) and (2.12) it follows that

$$\text{(2.15)} \qquad M^c = \bigcup_{t>0} (t, D_t) = \bigcup_{t>0} (t, d_t) = \bigcup_{t>0} (t, S_t),$$

and in each case the union may be restricted to rational $t>0$, or to rational $t\in M^c$.

(2.16) *Remark.* If M is a subset of $(0,\infty)$ which is closed in $(0,\infty)$, and one defines S_t^1 and S_t^2 as in the paragraph following (2.2), it is easy to check that in each case, the corresponding functions are given by $D_t = S_t^1$, $d_t = S_t^2$, and that $M = \{t: L_t = t\}$. That is, one recovers precisely the same set M.

3. Homogeneous Sets and Measures

In this paper we shall work with the canonical right continuous realization $(\Omega, \mathscr{F}, \mathscr{F}_t, X_t, \theta_t, P^\mu)$ of a semigroup (P_t) on E that satisfies the hypotheses of the right, that is, (HD1) and (HD2) of [9]. We assume that E is a Borel subset of a compact metric space. As is usual for such processes, the burial point Δ is adjoined to E in such a way that Δ is an isolated point in $E_\Delta = E \cup \{\Delta\}$.

Recall from [4] that a process $Y=(Y_t)$ is said to be well measurable (previsible) if for each initial measure μ there exists a process Y^μ that is well measurable (previsible) over $(\Omega, \mathscr{F}_t^\mu, P^\mu)$ such that Y and Y^μ are P^μ indistinguishable. A process $Y=(Y_s)$ is called *homogeneous* provided $Y_s \circ \theta_t = Y_{s+t}$ for $t\geqq 0$ and $s>0$. A subset M of $\mathbf{R}^{++}\times\Omega$ is *homogeneous* provided its indicator function is a homogeneous process. This is equivalent to the statement that $(t, \theta_s\,\omega)\in M$ if and only if $(t+s, \omega)\in M$ for all $t>0$, $s\geqq 0$, and $\omega\in\Omega$. We shall say that a subset M of $\mathbf{R}^{++}\times\Omega$ is a *Markov set* provided that M is progressively measurable (relative to the family $(\mathscr{F}_t)$), homogeneous, and closed. The statement that M is closed means that for each ω, the ω-section $M(\omega)$ of M is closed in $\mathbf{R}^{++}$. It is well known [3] that if M is closed and progressively measurable, then it is well measurable.

Let M be a homogeneous set. If ω_Δ is the unique element of Ω such that $X_t(\omega_\Delta)=\Delta$ for all $t\geqq 0$, then $\theta_t\,\omega_\Delta=\omega_\Delta$ for all $t\geqq 0$. As a result if $s>0$, $t+s\in M(\omega_\Delta)$ if and only if $s\in M(\theta_t\,\omega_\Delta)=M(\omega_\Delta)$, and so $M(\omega_\Delta)$ is either empty or $\mathbf{R}^{++}$. Now if $\zeta(\omega)<\infty$, $t+\zeta(\omega)\in M(\omega)$ if and only if $t\in M(\theta_{\zeta(\omega)}\,\omega)=M(\omega_\Delta)$ since $\theta_{\zeta(\omega)}\,\omega=\omega_\Delta$ for all ω. Thus for all ω, $M(\omega)\cap(\zeta(\omega),\infty)$ is either empty or all of $(\zeta(\omega),\infty)$ according as $M(\omega_\Delta)$ is empty or $\mathbf{R}^{++}$.

If M is a Markov set, define for $t\geqq 0$, $D_t(\omega)=\inf\{u>t\colon u\in M(\omega)\}$ and set $D(\omega)=D_0(\omega)$. Then it is easy to check that D is an exact terminal time that is *perfect* in the sense that

$$t+D(\theta_t\,\omega)=D(\omega)\quad\text{if}\quad t<D(\omega) \tag{3.1}$$

without any exceptional points. (Note that this usage of "perfect" differs from that in [2] where it is only assumed that (3.1) holds for all $t\geqq 0$ and $\omega\in\Omega_0$ where $P^x(\Omega_0)=1$ for all x. Later on when discussing multiplicative or additive functionals we shall use "perfect" in the above sense, that is, the defining relationship holds identically in ω without exception.) Clearly $D_t=t+D\circ\theta_t$ and for each ω, $t\to D_t(\omega)$ is a right continuous announcing function as defined in Section 2. Moreover for each fixed t, D_t is a stopping time. If $d_t(\omega)$, $l_t(\omega)$, and $L_t(\omega)$ are defined as in Section 2, then for each ω, $M(\omega)$ is the closed subset of $\mathbf{R}^{++}$ defined from the announcing function $t\to D_t(\omega)$. See (2.16) especially.

In dealing with multiplicative functionals it turns out to be necessary to proceed slightly more generally. To this end let S be a perfect terminal time that is *not assumed to be exact*. Define, $S_t(\omega)=t+S(\theta_t\,\omega)$. Here and in what follows t, u, v are arbitrary points of $\mathbf{R}^{+}$ unless stated otherwise. It is immediate that if $t<S\circ\theta_u$ then $t+u+S\circ\theta_t\circ\theta_u=u+S\circ\theta_u$ and so $S_{u+t}=S_u$ if $S_u>u+t$. Consequently $t\to S_t(\omega)$ is an announcing function for each ω. Defining for each ω the functions $D_t(\omega)$, $d_t(\omega)$, $L_t(\omega)$, and $l_t(\omega)$ as in Section 2 one easily checks that the following shift properties hold identically in t, u, and ω.

(3.2)
(i) $u+S_t(\theta_u\,\omega)=S_{t+u}(\omega)$;
(ii) $u+D_t(\theta_u\,\omega)=D_{t+u}(\omega)$;
(iii) $u+d_t(\theta_u\,\omega)=d_{t+u}(\omega),\quad t>0$;
(iv) $L_t(\theta_u\,\omega)=(L_{t+u}(\omega)-u)^+$;
(v) $l_t(\theta_u\,\omega)=(l_{t+u}(\omega)-u)^+;\quad t>0$.

Clearly $D=D_0$ is a perfect exact terminal time which, in fact, is precisely the exact regularization of S. Of course, $D(\omega)=S(\omega)$ if $S(\omega)>0$. Plainly each D_t and d_t is a stopping time. Hence by (2.7), (L_t) and (l_t) are adapted processes. Let $M(\omega)$ be the closed subset of $\mathbf{R}^{++}$ defined by the announcing function $t\to S_t(\omega)$ as in (2.8). Define $M\subset\mathbf{R}^{++}\times\Omega$ by

$$M=\{(t,\omega)\colon t\in M(\omega)\}=\{(t,\omega)\colon t>0,\ L_t(\omega)=t\}, \tag{3.3}$$

so that for each ω, $M(\omega)$ is the ω-section of M. Since $\{L_t\}$ is adapted and right continuous, M is well measurable, and using (3.2iv) one readily checks that M is homogeneous. Therefore M is a Markov set.

Following Meyer [7] we define

$$F=\{x\in E\colon P^x(D=0)=1\}$$
$$(3.4)\qquad \rho_F=\{(t,\omega)\colon t>0,\ X_t(\omega)\in F\}.$$

Thus F is the set of regular points for D and is a finely closed nearly Borel set. Obviously ρ_F is a well measurable homogeneous set. As usual $\bar{\rho}_F$ is the subset of $\mathbf{R}^{++}\times\Omega$ whose ω section is the closure of $\rho_F(\omega)$ in $\mathbf{R}^{++}$ for each ω. Then $\bar{\rho}_F$ is a Markov set, and Meyer [7], Proposition 1, shows that $\bar{\rho}_F(\omega)\subset M'(\omega)$ almost surely where $M'(\omega)$ denotes the derived set of $M(\omega)$ for each ω.

As in Section 2 we let $M^{\rightarrow}(\omega)$ denote the set of strictly positive left endpoints of the intervals contiguous to $M(\omega)$. It follows at once from (2.14) and (3.2) that $M^{\rightarrow}$ is a progressively measurable homogeneous set (not necessarily closed). (The fact that $M^{\rightarrow}$ is progressively measurable is also proved in Dellacherie [3].) The next result is of basic importance. It is proved in Meyer [7].

(3.5) **Proposition.** *One has $M^{\rightarrow}-\rho_F=M-\rho_F$, and consequently $M^{\rightarrow}-\rho_F$ is well measurable.*

Proof. Let $\varphi(x)=E^x\{e^{-D}\}$. Then φ is 1-excessive and hence nearly Borel. Obviously $E-F=\bigcup_{k\geqq 1}\{\varphi\leqq 1-1/k\}$. Fix β with $0<\beta<1$ and let

$$T=\inf\{t>0\colon \varphi(X_t)\leqq\beta,\ t\in M\}.$$

Then $T\geqq D$ and $\varphi(X_T)\leqq\beta$ on $\{T<\infty\}$. Thus if $\varphi(x)\leqq\beta$, $E^x(e^{-T})\leqq E^x(e^{-D})\leqq\beta$, and, in particular, by Blumenthal's zero-one law $P^x(T=0)=0$. Let $T^1=T$ and $T^{n+1}=T^n+T\circ\theta_{T^n}$. Then for all $x\in E$ one has

$$E^x\{e^{-T^{n+1}}\}=E^x\{e^{-T^n}E^{X(T^n)}(e^{-T});\ T^n<\infty\}\leqq\beta E^x(e^{-T^n}),$$

and so $\lim T^n=\infty$. But this implies that

$$M-\{(t,\omega)\colon\varphi(X_t)\leqq\beta\}=\{(t,\omega)\colon t>0,\ t=T^n(\omega)\ \text{for some } n\},$$

and this, in turn, implies that $M-\rho_F$ is a countable union of graphs of stopping times. If R is a stopping time such that $[R]\subset M-\rho_F$, then $X_R\notin F$ on $\{R<\infty\}$ and so for all x

$$P^x[D\circ\theta_R=0,\ R<\infty]=E^x\{P^{X(R)}(D=0);\ R<\infty\}=0.$$

Hence $[R]\subset M^{\rightarrow}$ and as a result $M-\rho_F=M^{\rightarrow}-\rho_F$, completing the proof of (3.5).

Once again following Meyer [7] we set $M_w^{\rightarrow}=M^{\rightarrow}-\rho_F$ and $M_\pi^{\rightarrow}=M^{\rightarrow}\cap\rho_F$. Then $M_w^{\rightarrow}$ is a well measurable homogeneous set which is a countable union of graphs of stopping times, while $M_\pi^{\rightarrow}$ is a progressively measurable homogeneous set all of whose ω-sections are countable and such that $M_\pi^{\rightarrow}$ contains the graph of no stopping time. This last assertion is an obvious consequence of the strong Markov property.

We next turn to the study of homogeneous measures. These should be viewed as a generalization of additive functionals.

(3.6) *Definition.* A function $\kappa(B, \omega)$ defined for $\omega \in \Omega$ and $B \in \mathscr{B}^{++}$ (the Borel sets of $\mathbf{R}^{++}$) and taking values in $\mathbf{R}^{+} = [0, \infty]$ is called a homogeneous measure if

(i) For each $B \in \mathscr{B}^{++}$, $\omega \to \kappa(B, \omega)$ is $\mathscr{F}$ measurable.

(ii) For each ω, $\kappa(\cdot, \omega)$ is a countable sum of finite measures carried by $(0, \zeta(\omega))$.

(iii) $\kappa(B, \theta_t \omega) = \kappa(B + t, \omega)$ identically in $t \geq 0$, $B \in \mathscr{B}^{++}$ and $\omega \in \Omega$.

More explicitly κ is a homogeneous measure on $\mathbf{R}^{++}$, but this is the only type of homogeneous measure that we shall consider in this paper. It is convenient to extend $\kappa(\cdot, \omega)$ to $\mathbf{R}^{+}$ by setting $\kappa(\{0\}, \omega) = 0$. This extension is *not* homogeneous on $\mathbf{R}^{+}$, that is, (iii) does not hold for Borel sets B of $\mathbf{R}^{+}$ which contain zero.

As is standard we shall sometimes suppress the ω in our notation and write $\kappa(B)$ for $\kappa(B, \cdot)$. If $\kappa((0, t], \omega) < \infty$ for all $t < \zeta(\omega)$, $\omega \in \Omega$, then $A_t(\omega) = \kappa((0, t], \omega)$ is a perfect raw additive functional.

(3.7) *Definition.* A homogeneous measure, κ, is called well measurable if there exists a sequence $\{K_t^n\}$ of increasing right continuous processes, adapted to $(\mathscr{F}_t)$, with $E^x(K_\infty^n) < \infty$ for all $x \in E$, $n \geq 1$, and $\kappa(dt, \omega) = \sum dK_t^n(\omega)$ for all $\omega \in \Omega$.

If $A_t(\omega) = \kappa((0, t], \omega)$ is finite, then it is clear that κ is well measurable according to (3.7) if and only if the right continuous process A is adapted.

It is immediate that if κ is a well measurable homogeneous measure and Z is a positive measurable process with well measurable projection Z^w, then

$$E^\mu \int_0^\infty Z_t \kappa(dt) = E^\mu \int_0^\infty Z_t^w \kappa(dt) \tag{3.8}$$

for all initial measures μ.

It is a standard argument to show that if κ is a homogeneous measure, and if there exists a sequence $\{Z^n\}$ of positive well measurable processes such that $(0, \zeta(\omega)) \subset \bigcup_n \{t: Z_t^n(\omega) > 0\}$ for each ω and such that for each n, $\int_0^\infty Z_u^n \kappa(du) < \infty$ and $\int_{(0,t]} Z_u^n \kappa(du)$ is $\mathscr{F}_t$ measurable for all $t \geq 0$, then κ is well measurable. In fact, there exists, under these hypotheses, a strictly positive well measurable process Z such that $\int_0^\infty Z_u \kappa(du) \leq 1$, and such that $t \to \int_{(0,t]} Z_u \kappa(du)$ is adapted.

We come now to the main result of this section. For its statement we fix a perfect terminal time S, not necessarily exact, which satisfies $S \leq \zeta$. Then D_t, d_t, L_t, l_t, and M are defined as above. Note that $S \leq \zeta$ implies $S(\omega_\Delta) = 0$ and consequently $S_t(\omega_\Delta) = d_t(\omega_\Delta) = t$. Hence by (2.9), $M(\omega_\Delta) = (0, \infty)$, and thus by the discussion in the third paragraph of this section $(\zeta(\omega), \infty) \subset M(\omega)$ for all ω. Since M is closed this implies that $[\zeta(\omega), \infty) \subset M(\omega)$ if $\zeta(\omega) > 0$, that is, if $\omega \neq \omega_\Delta$. Consequently $M^{\rightarrow}(\omega) \subset (0, \zeta(\omega))$ for all ω, $M^{\rightarrow}(\omega_\Delta)$ being empty. Also $M^c(\omega) = \mathbf{R}^{++} - M(\omega) \subset (0, \zeta(\omega))$.

(3.9) **Proposition.** *Let A be a perfect raw (that is, not necessarily adapted) additive functional of (X, S) such that $A_t(\omega) < \infty$ if $t < S(\omega)$. Then there exists a unique homogeneous measure κ which is carried by M^c and such that $\kappa(dt) = dA_t$ on $[0, S)$. Moreover κ is well measurable if and only if A is adapted. We say that κ extends A.*

Proof. Recall that a raw additive functional A of (X, S) is an increasing right continuous process (A_t) with $A_0=0$ satisfying $A_t=A_{S-}$ if $t \geqq S$ and

$$A_{t+u}=A_t+1_{[0,S)}(t)\, A_u \circ \theta_t.$$

Let $\kappa_0(dt)=dA_t$. Then κ_0 is carried by $(0, S)$. If $t+u<S(\omega)$, then

$$\kappa_0\big((u, t+u], \omega\big)=A_{t+u}(\omega)-A_u(\omega)=A_t(\theta_u \omega)=\kappa_0\big((0, t], \theta_u \omega\big).$$

Hence, for $u<S(\omega)$ and $B \in \mathscr{B}(u, S(\omega))$ – the Borel subsets of $(u, S(\omega))$ – one has

$$\kappa_0(B, \omega)=\kappa_0(B-u, \theta_u \omega). \tag{3.10}$$

For each $t \geqq 0$ define $\kappa_t(\cdot, \omega)$ on (t, ∞) by setting $\kappa_t(B, \omega)=\kappa_0(B-t, \theta_t \omega)$ for $B \in \mathscr{B}(t, \infty)$. This definition is consistent when $t=0$. Since κ_0 is carried by $(0, S)$ it is clear that κ_t is carried by (t, S_t). Moreover for each $t \geqq 0$ and $B \in \mathscr{B}(t, \infty)$, $\omega \to \kappa_t(B, \omega)$ is $\mathscr{F}$ measurable. Suppose that t, u, and ω satisfy $t+u<S_u(\omega)$. Then $S_u(\omega)=S_{t+u}(\omega)$, and so if $B \in \mathscr{B}(t+u, S_{t+u}(\omega))$ one has $B-u \in \mathscr{B}(t, S(\theta_u \omega))$. Consequently, using (3.10) in the second step one obtains for $B \in \mathscr{B}(t+u, S_{t+u}(\omega))$

$$\kappa_u(B, \omega)=\kappa_0(B-u, \theta_u \omega)=\kappa_0(B-u-t, \theta_t \theta_u \omega)=\kappa_{t+u}(B, \omega).$$

Hence $\kappa_u(\cdot, \omega)$ and $\kappa_{t+u}(\cdot, \omega)$ agree on $(u, S_u(\omega)) \cap (t+u, S_{t+u}(\omega))=(t+u, S_u(\omega))$. But $M^c(\omega)=\bigcup (r, S_r(\omega))$ where the union is over all strictly positive rationals (see (2.15)), and so there exists a unique measure $\kappa(\cdot, \omega)$ carried by $M^c(\omega)$ such that for each t, $\kappa(\cdot, \omega)$ and $\kappa_t(\cdot, \omega)$ agree on $(t, S_t(\omega))$ and such that $\omega \to \kappa(B, \omega)$ is $\mathscr{F}$ measurable for all $B \in \mathscr{B}(0, \infty)$. Next observe that if $B \in \mathscr{B}(t+u, S_{t+u}(\omega))$, then $B-t \in \mathscr{B}(u, S_u(\theta_t \omega))$ and therefore

$$\kappa(B-t, \theta_t \omega)=\kappa_u(B-t, \theta_t \omega)=\kappa_0(B-t-u, \theta_{t+u} \omega)=\kappa_{t+u}(B, \omega)=\kappa(B, \omega).$$

Since $\kappa(\cdot, \omega)$ is carried by $M^c(\omega)$ and $M^c(\omega)$ is a countable union of intervals $(r, S_r(\omega))$ it follows that κ satisfies (3.6 iii), and we have already observed that κ satisfies (3.6 i).

Finally let

$$W_t(\omega)=e^{-A_t(\omega)}\, 1_{[0, S(\omega))}(t).$$

Then W is right continuous and $W_t(\omega)>0$ if $t<S(\omega)$. Clearly W is adapted if A is adapted. If (τ_u) is the right continuous inverse of (A_t), then since $A(\tau_t) \geqq t$ if $\tau_t<\infty$

$$\int W_t\, dA_t=\int_{(0, A_S)} W(\tau_t)\, dt \leqq \int e^{-t}\, dt=1. \tag{3.11}$$

Let $\{r_n \colon n \geqq 1\}$ be an enumeration of the strictly positive rationals. Let $Z_t^0(\omega)= 1_M(t, \omega)$ and for $n \geqq 1$

$$Z_t^n(\omega)=1_{M^c}(r_n, \omega)\, 1_{(r_n, S_{r_n}(\omega))}(t)\, W_{t-r_n}(\theta_{r_n} \omega).$$

It is easy to see that each Z^n, $n \geqq 0$ is a positive measurable process which is well measurable if A, and hence W, is adapted. Moreover

$$\bigcup_{n \geqq 0} \{t: Z_t^n(\omega) > 0\} = M \cup \bigcup_{n \geqq 1} (r_n, S_{r_n}(\omega)) = M \cup M^c = \mathbf{R}^{++},$$

and so to complete the proof that κ is a homogeneous measure it suffices to show that $\int Z_u^n \kappa(du)$ is finite for $n \geqq 0$. If, in addition, A is adapted, then κ will be well measurable provided that $\int_{(0,t]} Z_u^n \kappa(du)$ is $\mathscr{F}_t$ measurable for all $n \geqq 0$ and $t \geqq 0$. See the remarks following (3.8). But κ is carried by M^c and so the assertions are clear if $n=0$. Fix $n \geqq 1$ and write r for r_n and Z for Z^n. Then if $r \in M^c(\omega)$ and $t \geqq r$, we have

$$\begin{aligned}
\int_{(0,t]} Z_u \kappa(du) &= \int_{(r,t]} W_{u-r}(\theta_r) 1_{(r,S_r)}(u) \kappa(du) \\
&= \int_{(0,t-r]} W_u(\theta_r) 1_{(0,S(\theta_r))}(u) \kappa(du+r) \\
&= \Big(\int_{(0,t-r]} W_u 1_{(0,S)}(u) \kappa(du) \Big) \circ \theta_r \\
&= \Big(\int_{(0,t-r]} e^{-A_u} 1_{(0,S)}(u)\, dA_u \Big) \circ \theta_r .
\end{aligned}$$

Combining this with (3.11) yields the desired conclusions. This establishes the existence of κ in (3.9) and that κ is well measurable if A is adapted. Clearly A is adapted if κ is well measurable.

For the uniqueness, suppose that ν is any homogeneous measure that agrees with κ on $[0, S)$ and which is carried by M^c. If $B \in \mathscr{B}(t, S_t(\omega))$, then $B - t \in \mathscr{B}(0, S(\theta_t \omega))$ and so

$$\nu(B, \omega) = \nu(B-t, \theta_t \omega) = \kappa(B-t, \theta_t \omega) = \kappa(B, \omega).$$

Therefore ν and κ agree on $(t, S_t(\omega))$ for each t, and hence they agree on M^c. Consequently $\nu = \kappa$ and the proof of (3.9) is complete.

Suppose that A is a raw additive functional of (X, S) with $E^x(A_t) < \infty$ for all x and t. It is known, see [1], [5], or [7], that there exists a unique adapted additive functional $\tilde{A}$ of (X, S) which is the dual well measurable projection of A relative to $(\mathscr{F}^\mu, \mathscr{F}_t^\mu, P^\mu)$ for all initial measures μ. These authors only consider the case of additive functionals of X, but the extension to additive functionals of (X, S) is straightforward. If A has a finite α-potential for some $\alpha \geqq 0$, then $\tilde{A}$ is the unique adapted additive functional of (X, S) with the same α-potential operator as A. See (A-4).

(3.12) **Proposition.** *Let A be a raw additive functional of (X, S) with $E^x(A_t) < \infty$ for all x and t. Let $\tilde{A}$ be its dual well measurable projection and let κ and $\tilde{\kappa}$ be the unique homogeneous measures extending A and $\tilde{A}$. Then $\tilde{\kappa}$ is the dual well measurable projection of κ in the sense that*

$$E^\mu \int Z_t \kappa(dt) = E^\mu \int Z_t \tilde{\kappa}(dt) \tag{3.13}$$

for all positive well measurable Z and all initial measures μ.

Proof. Fix $0 < r < q$ and let $g = g_n$ be the indicator of $\{x: E^x(A_q) \leqq n\}$. Let $\Gamma_t = \Gamma_t^{n,r,q} = g_n(X_r) 1_{(r,S_r)}(t) 1_{(0,q)}(t)$. Then (Γ_t) is a well measurable process. Let f

be a bounded positive Borel function. Using the notation of the proof of (3.9) we have

$$\begin{aligned} E^x \int f(X_t)\, \Gamma_t\, \kappa(dt) &\leqq E^x\left\{g(X_r) \int_r^q f(X_t)\, 1_{(r,S_r)}(t)\, \kappa(dt)\right\} \\ &= E^x\left\{g(X_r) \int_r^q f(X_t)\, 1_{(r,S_r)}(t)\, \kappa_0(dt-r, \theta_r)\right\} \\ &= E^x\left\{g(X_r) \left(\int_0^{q-r} f(X_t)\, dA_t\right) \circ \theta_r\right\} \\ &= E^x\left\{g(X_r)\, E^{X(r)} \int_0^{q-r} f(X_t)\, dA_t\right\}. \end{aligned}$$

First of all the last line of the above display is unchanged if we replace A by $\tilde{A}$ and secondly it is bounded by $n\,\|f\|$ because $E^y(A_{q-r}) \leqq E^y(A_q)$ for all y. Since the same calculation is valid starting with $\tilde{\kappa}$ we see that

$$B_u = \int_{(0,u]} f(X_t)\, \Gamma_t\, \kappa(dt); \qquad \tilde{B}_u = \int_{(0,u]} f(X_t)\, \Gamma_t\, \tilde{\kappa}(dt)$$

define right continuous increasing process with $E^x(B_\infty) = E^x(\tilde{B}_\infty) \leqq n\,\|f\|$ for all x. We next claim that for all $0 \leqq v \leqq u$ and x one has

$$E^x\{B_u - B_v | \mathscr{F}_v\} = E^x\{\tilde{B}_u - \tilde{B}_v | \mathscr{F}_v\}. \tag{3.14}$$

Clearly it suffices to prove (3.14) when $u \leqq q$. Suppose first that $v \leqq r$. Then as in the previous calculation

$$\begin{aligned} E^x\{B_u - B_v | \mathscr{F}_v\} &= E^x\left\{\int_{(v,u]} f(X_t)\, \Gamma_t\, \kappa(dt) | \mathscr{F}_v\right\} \\ &= E^x\left\{g(X_r) \left(\int_0^{u-r} f(X_t)\, dA_t\right) \circ \theta_r | \mathscr{F}_v\right\} \\ &= E^x\left\{g(X_r)\, E^{X(r)} \int_0^{u-r} f(X_t)\, dA_t | \mathscr{F}_v\right\} \end{aligned}$$

since $v \leqq r$. But this calculation is valid with B replaced by $\tilde{B}$, and, since the last line is unchanged if one replaces A by $\tilde{A}$, it follows that (3.14) holds if $v \leqq r$. If $v > r$, $(v,u] \cap (r, S_r) = (v,u] \cap (v, S_v)$ if $v < S_r$ and is empty if $v \geqq S_r$. Thus if $v > r$

$$\begin{aligned} E^x\{B_u - B_v | \mathscr{F}_v\} &= E^x\left\{g(X_r) \int_{(v,u]} f(X_t)\, 1_{(v,S_v)}(t)\, \kappa(dt);\, S_r > v | \mathscr{F}_v\right\} \\ &= g(X_r)\, 1_{\{S_r > v\}}\, E^x\left\{\left(\int_0^{u-v} f(X_t)\, dA_t\right) \circ \theta_v | \mathscr{F}_v\right\} \\ &= g(X_r)\, 1_{\{S_r > v\}}\, E^{X(v)} \int_0^{u-v} f(X_t)\, dA_t \end{aligned}$$

which establishes (3.14).

It follows from (3.14) and Meyer's integration lemma (VII-T17) of [6] that

$$E^x \int_0^\infty Y_t \, dB_t = E^x \int_0^\infty Y_t \, d\tilde{B}_t$$

for all previsible bounded processes Y. Combining this with Proposition (A.1) of the Appendix and the definitions of B and $\tilde{B}$ we find

$$E^x \int_0^\infty Z_t \Gamma_t \kappa(dt) = E^x \int_0^\infty Z_t \Gamma_t \tilde{\kappa}(dt) \tag{3.15}$$

for all bounded well measurable processes Z and that the expressions in (3.15) are bounded in x by $n\|Z\|$. For fixed n, $0<r<q$, define stopping times $R_{n,r}=r$ if $g_n(X_r)=1$, $R_{n,r}=\infty$ if $g_n(X_r)=0$, and $S_{n,r,q}=\min(R_{n,r}+S\circ\theta_{R_{n,r}}, q)$. Recalling the definition of Γ we see that $\Gamma^{n,r,q}$ is the indicator of the stochastic interval $((R_{n,r}, S_{n,r,q}))$. For fixed x define measures on the well measurable subsets of $\mathbf{R}^+ \times \Omega$ by

$$\nu^x(Z) = E^x \int Z_t \kappa(dt); \qquad \tilde{\nu}^x(Z) = E^x \int Z_t \tilde{\kappa}(dt).$$

Then ν^x and $\tilde{\nu}^x$ are carried by M^c and (3.15) states their restrictions to $((R_{n,r}, S_{n,r,q}))$ are equal and finite. But M^c is the union of the stochastic intervals $((R_{n,r}, S_{n,r,q}))$ over $n \geqq 1$ and over rationals r, q with $0<r<q$. Consequently $\nu^x = \tilde{\nu}^x$ and since x is arbitrary this establishes (3.12).

(3.16) *Remark.* A similar, but simpler, argument establishes the following previsible analogue of (3.12). Let A be as in the statement of (3.12) and let $\tilde{\tilde{A}}$ be its dual previsible projection. Let κ and $\tilde{\tilde{\kappa}}$ be the unique homogeneous measures extending A and $\tilde{\tilde{A}}$. Then $\tilde{\tilde{\kappa}}$ is the dual previsible projection of κ in the sense that

$$E^\mu \int Y_t \kappa(dt) = E^\mu \int Y_t \tilde{\tilde{\kappa}}(dt)$$

for all positive previsible Y and all initial measures μ.

4. Multiplicative Functionals and Balayage

Let $m=(m_t)$ be an exact multiplicative functional of X with $0 \leqq m_t(\omega) \leqq 1$ for all $t \geqq 0$ and $\omega \in \Omega$. We assume that $t \to m_t(\omega)$ is right continuous and decreasing on $[0, \infty)$ for each $\omega \in \Omega$. In view of the recent work of Walsh [10], we may also assume without loss of generality that

$$m_{t+s}(\omega) = m_t(\omega)\, m_s(\theta_t \omega) \qquad \text{for all } t,\ s \geqq 0,\ \omega \in \Omega; \tag{4.1}$$

(4.2) $s \to m_{t-s}(\theta_s \omega)$ is right continuous and increasing on $[0, t)$ for all $t>0$, $\omega \in \Omega$;

since any exact right continuous multiplicative functional is equivalent to one satisfying (4.1) and (4.2). In addition we assume that $m_t(\omega)=0$ for all $t \geqq \zeta(\omega)$. Again this is no loss of generality since (m_t) is equivalent to $(m_t 1_{[0,\zeta)}(t))$. We fix once and for all a multiplicative functional m satisfying these properties. We define $S(\omega)=\inf\{t: m_t(\omega)=0\}$. Then S is a perfect terminal time with $S \leqq \zeta$. However, S need not be exact. Let $E_m=\{x: P^x(m_0=1)=1\}$. Since m is exact E_m is finely open and nearly Borel. Finally we let $(Q_t)_{t \geqq 0}$ and $(V^\alpha)_{\alpha \geqq 0}$ be the semigroup and resolvent generated by m, that is

$$Q_t f(x) = E^x(f(X_t)\, m_t); \qquad V^\alpha f(x) = E^x \int_0^\infty e^{-\alpha t} f(X_t)\, m_t\, dt \tag{4.3}$$

provided f is nonnegative or bounded (V^0 may not exist for bounded f).

The function appearing in (4.2) will play an important role in our development and so we introduce a special notation. We define

$$\begin{aligned} m(t, s, \omega) &= m_{s-t}(\theta_t \omega) \quad \text{if} \quad 0 \leqq t < s \\ m(t, s, \omega) &= 1 \quad \text{if} \quad 0 \leqq s \leqq t. \end{aligned} \tag{4.4}$$

Thus for each $s > 0$ and $\omega \in \Omega$, $t \to m(t, s, \omega)$ is an increasing right continuous function defined on $\mathbf{R}^+ = [0, \infty)$ which is identically one on $[s, \infty)$. Also for each $t \geqq 0$ and $\omega \in \Omega$, $s \to m(t, s, \omega)$ is decreasing and right continuous on (t, ∞). The following identities are immediate consequences of (4.1)

$$m(r, t, \omega)\, m(t, s, \omega) = m(r, s, \omega) \quad \text{if} \quad 0 \leqq r \leqq t \leqq s, \tag{4.5}$$

$$m(t, s, \theta_r \omega) = m(t+r, s+r, \omega) \quad \text{if} \quad 0 \leqq t \leqq s \text{ and } r \geqq 0. \tag{4.6}$$

We shall sometimes suppress the ω and write $m(t, s) = m(t, s, \omega)$.

It will also be convenient to define

$$\begin{aligned} m^*(t, s, \omega) &= \lim_{r \uparrow s} m(t, r, \omega) = m(t, s-, \omega) \quad \text{if} \quad 0 \leqq t < s \\ m^*(t, s, \omega) &= 1 \quad \text{if} \quad 0 \leqq s \leqq t. \end{aligned} \tag{4.7}$$

One easily verifies that for each $s > 0$ and $\omega \in \Omega$, $t \to m^*(t, s, \omega)$ is an increasing right continuous function defined on $\mathbf{R}^+$ which is identically one on $[s, \infty)$. Also for each $t \geqq 0$ and $\omega \in \Omega$, $s \to m^*(t, s, \omega)$ is decreasing and left continuous on (t, ∞). Clearly m^* satisfies (4.6) and

$$m(r, t, \omega)\, m^*(t, s, \omega) = m^*(r, s, \omega) \quad \text{if} \quad 0 \leqq r \leqq t < s. \tag{4.8}$$

For each $\omega \in \Omega$ and $s > 0$ we define measures $\gamma(\cdot, s, \omega)$ and $\gamma^*(\cdot, s, \omega)$ on $\mathscr{B}^+ = \mathscr{B}(\mathbf{R}^+)$ by

$$\gamma(dt, s, \omega) = d_t m(t, s, \omega); \qquad \gamma^*(dt, s, \omega) = d_t m^*(t, s, \omega). \tag{4.9}$$

Clearly $\gamma(\cdot, s, \omega)$ and $\gamma^*(\cdot, s, \omega)$ are carried by $(0, s]$ and have total mass $1 - m_s(\omega)$ and $1 - m_{s-}(\omega)$ respectively. Moreover γ and γ^* are kernels in the sense that for each $B \in \mathscr{B}^+$, $(s, \omega) \to \gamma(B, s, \omega)$ and $(s, \omega) \to \gamma^*(B, s, \omega)$ are $\mathscr{B}^{++} \otimes \mathscr{F}$ measurable. Also note that for fixed B and s, $\omega \to \gamma(B, s, \omega)$ is $\mathscr{F}_s$ measurable and

$$\omega \to \gamma^*(B, s, \omega)$$

is $\mathscr{F}_{s-}$ measurable. It follows from (4.6) that for $u \geqq 0$

$$\gamma(B, s, \theta_u \omega) = \gamma(B+u, s+u, \omega); \; \gamma^*(B, s, \theta_u \omega) = \gamma^*(B+u, s+u, \omega). \tag{4.10}$$

As usual we shall sometimes write $\gamma(dt, s)$ for $\gamma(dt, s, \omega)$ and similarly for γ^*.

We now define starting from the perfect terminal time S the processes S_t, D_t, d_t, L_t, and l_t, and the Markov set M as in Section 3.

(4.11) **Lemma.** (i) *For each* $s>0$ *and* $\omega\in\Omega$, $\gamma(\cdot, s, \omega)$ *is carried by* $[L_s(\omega), s]$ *and* $\gamma^*(\cdot, s, \omega)$ *is carried by* $[l_s(\omega), s]$. (*ii*) *For each* $\omega\in\Omega$, $\gamma^*(\cdot, s, \omega)=\gamma(\cdot, s, \omega)$ *for all but countably many* s.

Proof. For (i) it suffices to prove that $m(t, s, \omega)=0$ if $t<L_s(\omega)$ and that $m^*(t, s, \omega)=0$ if $t<l_s(\omega)$. By (2.13) if $t<L_s(\omega)$, then $t<\sup\{u\leqq s\colon u\in M(\omega)\}$ and so by (2.12), $D_t(\omega)\leqq s$. Since $S_t\leqq D_t$ this implies that $S(\theta_t\omega)\leqq s-t$ and so $m(t, s, \omega)=m_{s-t}(\theta_t\omega)=0$. On the other hand if $t<l_s(\omega)$ then $t<L_{s-\varepsilon}(\omega)$ for sufficiently small $\varepsilon>0$. Therefore $m(t, s-\varepsilon, \omega)=0$ and hence $m^*(t, s, \omega)=0$.

For (ii) we fix ω and suppress it in our notation. We emphasize, however, that the sets C_t and C defined below depend on ω. Let $C_t=\{s\colon s>t, m(t, s)\neq m^*(t, s)\}$. Clearly C_t is a countable subset of (t, ∞). It follows from (4.5) and (4.8) that if $t<r$, $C_t\cap(r, \infty)\subset C_r$. Let $C=\bigcup C_r$ where the union is over all positive rationals r. Then C is countable. If $s\in C_t$, then $s>t$ and so there exists a rational r with $t<r<s$. Hence $s\in C_r$ and so $C_t\subset C$ for all t. Consequently if $s\notin C$, $m(t, s)=m^*(t, s)$ for all $t<s$, and since $m(s, s)=m^*(s, s)=1$ this establishes (ii).

(4.12) *Example.* Let T be a perfect exact terminal time with $T\leqq\zeta$; for example, the hitting time of a nearly Borel subset of E. Then $m_t=1_{[0, T)}(t)$ is a multiplicative functional satisfying our hypotheses. In this important situation the various objects introduced above may be expressed in a more explicit form. Thus $S=D=T$ and $D_t=S_t=t+T\circ\theta_t$, while if $t<s$

$$m(t, s)=1_{[0, T\circ\theta_t)}(s-t)=1_{[t, D_t)}(s).$$

Using (2.5), (2.12), and the fact that M is closed it is easy to check that if $t<s$, then $s<D_t$ if and only if $L_s\leqq t$. Hence for $t<s$, $m(t, s)=1_{[L(s), s)}(t)$. It follows from this and $m(s, s)=1$, that

$$\gamma(dt, s)=\varepsilon_{L(s)}(dt)\, 1_{\{L(s)>0\}}$$

where ε_u is unit mass at u. Similarly if $t<s$, $m^*(t, s)=1_{(t, D(t)]}(s)=1_{[l(s), s)}(t)$ and $\gamma^*(dt, s)=\varepsilon_{l(s)}(dt)\, 1_{\{l(s)>0\}}$. We shall return to this example later.

We shall now investigate the kernels γ and γ^* in more detail. If t, s, and ω satisfy $0\leqq t<S(\omega)$ and $t\leqq s$, then

$$\begin{aligned} m(t, s, \omega)&=m_s(\omega)/m_t(\omega)=m_s(\omega)\left[\int_{(0,t]} d\left(\frac{1}{m_u(\omega)}\right)+1\right] \\ &=m_s(\omega)\left[\int_{(0,t]}\frac{-dm_u(\omega)}{m_u(\omega)\, m_{u-}(\omega)}+1\right]. \end{aligned}\tag{4.13}$$

But $A_t=-\int_{(0,t]}(m_{u-})^{-1}\, dm_u$ defines an (adapted) additive functional of (X, S) that is finite on $[0, S)$ and so by (3.9) there exists a unique well measurable homogeneous measure κ carried by M^c which extends A. Consequently from (4.13) we find that if $0<s<S(\omega)$, then on $[0, s]$

$$\begin{aligned} \gamma(dt, s, \omega)&=[m_s(\omega)/m_t(\omega)]\, \kappa(dt) \\ &=m(t, s, \omega)\, \kappa(dt). \end{aligned}\tag{4.14}$$

Also from (4.13) we find that if $0<s\leqq S(\omega)$, then on $[0,s)$

$$\gamma^*(dt,s,\omega)=[m_{s-}(\omega)/m_t(\omega)]\,\kappa(dt) \\ =m^*(t,s,\omega)\,\kappa(dt). \tag{4.15}$$

Note, however, that if $m_{s-}(\omega)>0$, in particular, if $s<S(\omega)$, then

$$\lim_{t\uparrow s} m^*(t,s,\omega)=\lim_{t\uparrow s}[m_{s-}(\omega)/m_t(\omega)]=1$$

and so $\gamma^*(\cdot,s,\omega)$ does not charge $\{s\}$.

(4.16) **Proposition.** *For each $s>0$ and $\omega\in\Omega$, $\gamma(dt,s,\omega)=m(t,s,\omega)\,\kappa(dt)$ on $(L_s(\omega),s]$ and $\gamma^*(dt,s,\omega)=m^*(t,s,\omega)\,\kappa(dt)$ on $(l_s(\omega),s)$. Moreover, if $l_s(\omega)<s$ then either $\gamma^*(\cdot,s,\omega)$ does not charge $\{s\}$ or $\gamma^*(\cdot,s,\omega)=\varepsilon_s$. In particular the first alternative holds if $s\in M^c(\omega)$.*

Proof. Fix $s>0$ and $\omega\in\Omega$. We shall first establish the assertion about γ. If $s\in M(\omega)$, then $L_s(\omega)=s$ and the assertion is obvious in this case. If $s\in M^c(\omega)$, then $s<\zeta(\omega)$. In this case the interval $(L_s(\omega),s]\subset M^c(\omega)$ and it can be covered by countably many intervals of the form $(u,S_u(\omega))$ with $u<s<S_u(\omega)$. Thus it suffices to show that the measures $\gamma(dt,s,\omega)$ and $m(t,s,\omega)\,\kappa(dt,\omega)$ agree on $(u,s]$ whenever $u<s<S_u(\omega)$. Suppose $B\in\mathscr{B}(u,s]$. Then $B-u\in\mathscr{B}(0,s-u]$ and since $s-u<S(\theta_u\omega)$ we see from (4.14) that

$$\gamma(B-u,s-u,\theta_u\omega)=\int_{B-u} m(t,s-u,\theta_u\omega)\,\kappa(dt,\theta_u\omega). \tag{4.17}$$

Now $\gamma(B-u,s-u,\theta_u\omega)=\gamma(B,s,\omega)$ by (4.10), while by (4.6) and the homogeneity of κ the right side of (4.17) reduces to $\int_B m(t,s,\omega)\,\kappa(dt,\omega)$. This establishes the desired conclusion about γ.

We turn next to γ^*. If $l_s(\omega)=s$ there is nothing to prove. If $l_s(\omega)<s$ then the interval $(l_s(\omega),s)\subset M^c(\omega)$ and it can be covered by countably many intervals of the form $(u,S_u(\omega))$ with $u<s\leqq S_u(\omega)$. Thus to establish the first assertion about γ^* it suffices to show that $\gamma^*(dt,s,\omega)$ and $m^*(t,s,\omega)\,\kappa(dt)$ agree on any interval (u,s) with $u<s\leqq S_u(\omega)$. This follows from (4.15) by the above argument. Finally fix s and ω with $l_s(\omega)<s$. If $l_s(\omega)<r<t<s$, then $S_r(\omega)\geqq s$ or $S(\theta_r\omega)\geqq s-r>t-r$. Consequently $m(r,t,\omega)=m_{t-r}(\theta_r\omega)>0$, and so from (4.8), $m^*(t,s,\omega)=m^*(r,s,\omega)/m(r,t,\omega)$. Letting $t\uparrow s$ we see that $m^*(s-,s,\omega)$ is either one or zero according as $m^*(r,s,\omega)>0$ or $m^*(r,s,\omega)=0$. In the first case $\gamma^*(\cdot,s,\omega)$ does not charge $\{s\}$ and in the second it is ε_s. If, in addition, $s\in M^c(\omega)$, then $s-r<S(\theta_r\omega)$ and so $m^*(r,s,\omega)\geqq m_{s-r}(\theta_r\omega)>0$ and the first alternative holds. This completes the proof of (4.16).

Let $\nu(\cdot,\omega)$ be the σ-finite measure on $\mathbf{R}^{++}$ that puts mass one at each point of $M^{\rightarrow}(\omega)$. Since $M^{\rightarrow}$ is homogeneous it is evident that ν is a homegeneous measure. However, in general ν is not well measurable in the sense of (3.7). From (4.11), (4.16), and the fact that $m(t,s)=0$ if $t<L_s$ we obtain the following decomposition of γ:

$$\gamma(dt)=m(t,s)\big(\kappa(dt)+\nu(dt)\big)\,1_{\{L_s<s\}}+\varepsilon_s(dt)\,1_{\{L_s=s\}} \quad \text{on } [0,s]. \tag{4.18}$$

Similarly on $[0, s]$

$$\begin{aligned}\gamma^*(dt, s) = {} & 1_{[l_s, s)}(t)\, m^*(t, s)\big(\kappa(dt) + \nu(dt)\big) \\ & + \big[1_{\{l_s < s\}}\big(1 - m^*(s-, s)\big) + 1_{\{l_s = s\}}\big]\, \varepsilon_s(dt).\end{aligned} \tag{4.19}$$

We are now going to define two "transport" operations associated with the multiplicative functional m. These operations amount to mapping the measure ε_s onto the measures $\gamma(\cdot, s)$ and $\gamma^*(\cdot, s)$ respectively. More explicitly let $\lambda(B, \omega)$ be a kernel defined for $B \in \mathscr{B}^+$ and $\omega \in \Omega$ such that for each $B \in \mathscr{B}^+$, $\omega \to \lambda(B, \omega)$ is $\mathscr{F}$ measurable and for each $\omega \in \Omega$, $\lambda(\cdot, \omega)$ is a positive measure carried by $(0, \zeta(\omega))$. We then define its *transport via m*, $\bar{\lambda}(B, \omega)$ and its *$*$-transport via m*, $\bar{\lambda}^*(B, \omega)$ by the formulas

$$\begin{aligned}\bar{\lambda}(B, \omega) &= \int \gamma(B, s, \omega)\, \lambda(ds, \omega) \\ \bar{\lambda}^*(B, \omega) &= \int \gamma^*(B, s, \omega)\, \lambda(ds, \omega).\end{aligned} \tag{4.20}$$

Clearly $\bar{\lambda}$ and $\bar{\lambda}^*$ are again kernels. Since the integration in (4.20) extends only over the interval $(0, \zeta(\omega))$, and since $\gamma(\cdot, s)$ and $\gamma^*(\cdot, s)$ are carried by $[0, s]$ it follows that $\bar{\lambda}(\cdot, \omega)$ and $\bar{\lambda}^*(\cdot, \omega)$ are carried by $(0, \zeta(\omega))$. More generally for any $\alpha \geqq 0$ define the kernels γ^α and $\gamma^{*,\alpha}$ by

$$\begin{aligned}\gamma^\alpha(dt, s, \omega) &= e^{-\alpha(s-t)}\, \gamma(dt, s, \omega) \\ \gamma^{*,\alpha}(dt, s, \omega) &= e^{-\alpha(s-t)}\, \gamma^*(dt, s, \omega).\end{aligned} \tag{4.21}$$

Obviously γ^α and $\gamma^{*,\alpha}$ also satisfy (4.10), and $\gamma^\alpha(\cdot, s, \omega)$ is carried by $(0, s] \cap [L_s, s]$ while $\gamma^{*,\alpha}(\cdot, s, \omega)$ is carried by $(0, s] \cap [l_s, s]$. The *α-transport via m*, $\bar{\lambda}^\alpha$ and the *α-$*$-transport via m*, $\bar{\lambda}^{*,\alpha}$ of a kernel λ are defined by

$$\begin{aligned}\bar{\lambda}^\alpha(B, \omega) &= \int \gamma^\alpha(B, s, \omega)\, \lambda(ds, \omega) \\ \bar{\lambda}^{*,\alpha}(B, \omega) &= \int \gamma^{*,\alpha}(B, s, \omega)\, \lambda(ds, \omega).\end{aligned} \tag{4.22}$$

It follows from (4.11) that if $\lambda(\cdot, \omega)$ is diffuse, then $\bar{\lambda}^\alpha(\cdot, \omega) = \bar{\lambda}^{*,\alpha}(\cdot, \omega)$.

It is helpful to examine these operations in the special situation of Example (4.12). In that situation we find

$$\begin{aligned}\bar{\lambda}(B, s) &= \int \varepsilon_{L(s)}(B)\, 1_{\{L(s) > 0\}}\, \lambda(ds) \\ \bar{\lambda}^*(B, s) &= \int \varepsilon_{l(s)}(B)\, 1_{\{l(s) > 0\}}\, \lambda(ds).\end{aligned}$$

Therefore it is the $*$-transport operation that corresponds to the transport operation used in our paper [5] and by Meyer [7]. The difference between these two operations in the situation of Example (4.12) is that the mass at a right endpoint of a contiguous interval is moved to the left endpoint by the $*$-transport operation, whereas it is left unmoved by the transport operation.

We return now to the general situation.

(4.23) **Proposition.** *If λ is a homogeneous measure, then $\bar{\lambda}^\alpha$ and $\bar{\lambda}^{*,\alpha}$ are homogeneous measures for each $\alpha \geqq 0$.*

Proof. If $B \in \mathscr{B}(0, \infty)$ and $u \geqq 0$, we have

$$\bar{\lambda}^\alpha(B, \theta_u \omega) = \int_{(0,\infty)} \gamma^\alpha(B, s, \theta_u \omega)\, \lambda(ds, \theta_u \omega)$$
$$= \int_{(0,\infty)} \gamma^\alpha(B+u, s+u, \omega)\, \lambda(ds+u, \omega)$$

because of the homogeneity of λ and the fact that γ^α also satisfies (4.10). Therefore

$$\bar{\lambda}^\alpha(B, \theta_u \omega) = \int_{(u,\infty)} \gamma^\alpha(B+u, s, \omega)\, \lambda(ds, \omega)$$
$$= \int \gamma^\alpha(B+u, s, \omega)\, \lambda(ds, \omega) = \bar{\lambda}^\alpha(B+u, \omega),$$

where the second equality results from the facts that $B+u \subset (u, \infty)$ if $B \subset \mathbf{R}^{++}$ and that $\gamma^\alpha(\cdot, s, \omega)$ is carried by $(0, s]$. Since $\gamma(\mathbf{R}^+, s, \omega) \leqq 1$ it is immediate that $\bar{\lambda}^\alpha$ is a countable sum of finite measures whenever λ is such a sum. Hence $\bar{\lambda}^\alpha$ is a homogeneous measure. The argument for $\bar{\lambda}^{*,\alpha}$ is exactly the same.

If A is a raw additive functional with finite α-potential, we define its α-transport via m, $\bar{A}^\alpha$, as follows. Let $\lambda(dt) = dA_t$ and then put $\bar{A}^\alpha_t = \bar{\lambda}^\alpha((0, t])$. Similarly, the α-$*$-transport via m of A is defined by $\bar{A}^{*,\alpha}_t = \bar{\lambda}^{*,\alpha}((0, t])$. It follows from (4.11) that $\bar{A}^\alpha = \bar{A}^{*,\alpha}$ if A is continuous. In view of (4.23) it is evident that $\bar{A}^\alpha$ and $\bar{A}^{*,\alpha}$ are raw additive functionals.

Recall that with a multiplicative functional m one associates the operators P^α_m defined by

$$P^\alpha_m f(x) = -E^x \int e^{-\alpha t} f(X_t)\, dm_t \quad \text{if } x \in E_m$$
$$= f(x) \quad \text{if } x \notin E_m. \tag{4.24}$$

In particular, if $m_t = 1_{[0,T)}(t)$ where T is a terminal time, then $P^\alpha_m = P^\alpha_T$. If A is a raw additive functional with a finite α-potential we define an operator V^α_A as follows:

$$V^\alpha_A f(x) = E^x \int e^{-\alpha t} f(X_t)\, m_t\, dA_t. \tag{4.25}$$

As usual we write $v^\alpha_A = V^\alpha_A 1$. The results contained in the next proposition are routine calculations whose verification we leave to the reader.

(4.26) **Proposition.** *Let A be a raw additive functional with a finite α-potential. If $f \geqq 0$, then $V^\alpha_A f$ is $\alpha-(X, m)$ excessive, that is, α-excessive for the semigroup (Q_t) defined in* (4.3). *If $U^\alpha_A f$ is finite, then*

$$P^\alpha_m U^\alpha_A f(x) = E^x \int e^{-\alpha t}(1 - m_{t-}) f(X_t)\, dA_t, \tag{4.27}$$

and both $P^\alpha_m U^\alpha_A f$ and $U^\alpha_A f - V^\alpha_A f$ are α-excessive. In particular, if A is continuous $U^\alpha_A f - V^\alpha_A f = P^\alpha_m U^\alpha_A f$.

(4.28) **Proposition.** *Let A be a raw additive functional with finite α-potential u^α_A. Then $\bar{A}^\alpha$ and $\bar{A}^{*,\alpha}$ are raw additive functionals with finite α-potentials given by $u^\alpha_A - v^\alpha_A$ and $P^\alpha_m u^\alpha_A$ respectively.*

Proof. Because the total mass of $\gamma(\cdot, s)$ is $1 - m_s$ we have

$$E^x \int e^{-\alpha t}\, d\bar{A}^\alpha_t = E^x \int e^{-\alpha t} \int \gamma^\alpha(dt, s)\, dA_s$$
$$= E^x \int e^{-\alpha s}(1 - m_s)\, dA_s$$
$$= u^\alpha_A(x) - v^\alpha_A(x).$$

Similarly since the total mass of $\gamma^*(\cdot, s)$ is $1-m_{s-}$ we have

$$\begin{aligned} E^x \int e^{-\alpha t}\, d\bar{A}_t^{*,\alpha} &= E^x \int e^{-\alpha s}(1-m_{s-})\, dA_s \\ &= P_m^\alpha u_A^\alpha(x) \end{aligned}$$

by (4.27).

It is clear from (4.28) and the discussion following (4.22) that the $*$-transport operation corresponds to the ordinary balayage operation. In spite of this the transport operation seems to be somewhat simpler and to arise more naturally in our applications. Therefore, in the remainder of this paper we shall treat only the transport operation and leave it to the interested reader to write down the analogous results for the $*$-transport operation. We plan to discuss these operations more systematically in a future publication. Of course, when applied to continuous raw additive functionals the two operations are the same.

The decomposition (4.18) of γ leads to a useful decomposition of $\bar{A}^\alpha$. We fix a raw additive functional A with a finite α-potential and define

$$H^\alpha = H^{\alpha, A} = \int e^{-\alpha t} m_t\, dA_t = \int_{(0,S)} e^{-\alpha t} m_t\, dA_t. \tag{4.29}$$

Then $E^x(H^\alpha) = v_A^\alpha(x)$.

(4.30) **Proposition.** *Using the above notation we have*

$$\bar{A}_t^\alpha = \int_{M \cap (0,t]} dA_s + \int_{(0,t]} \Delta A_s\, \kappa(ds) + \int_{(0,t]} H^\alpha \circ \theta_s \big(\kappa(ds) + \nu(ds)\big).$$

Proof. We write

$$\bar{A}_t^\alpha = \int_M \gamma^\alpha((0,t], s)\, dA_s + \int_{M^c} \gamma^\alpha((0,t], s)\, dA_s. \tag{4.31}$$

If $s \in M$, then $\gamma^\alpha(dt, s) = e^{-\alpha(s-t)} \gamma(dt, s) = \varepsilon_s(dt)$ by (4.18) and (4.21). Consequently the first term on the right side of (4.31) is

$$\int_M \varepsilon_s((0,t])\, dA_s = \int_{M \cap (0,t]} dA_s.$$

If $s \in M^c$, then $\gamma(dt, s) = m t, s)\, 1_{[0,s]}(t)\big(\kappa(dt) + \nu(dt)\big)$ by (4.18), and hence

$$\gamma^\alpha(dt, s) = e^{-\alpha(s-t)} m(t,s)\, 1_{[0,s]}(t)\big(\kappa(dt) + \nu(dt)\big).$$

Therefore

$$\begin{aligned} \int_{M^c} \gamma^\alpha((0,t], s)\, dA_s &= \int_{M^c} \int_{(0,t]} \gamma^\alpha(du, s)\, dA_s \\ &= \int_{M^c} \int_{(0,t]} e^{-\alpha(s-u)} m(u,s)\, 1_{[0,s]}(u)\big(\kappa(du) + \nu(du)\big)\, dA_s \\ &= \int_{(0,t]} \int_{M^c} 1_{[u,\infty)}(s)\, m(u,s)\, e^{-\alpha(s-u)}\, dA_s \big(\kappa(du) + \nu(du)\big). \end{aligned}$$

But the integral over s equals

$$I_{M^c}(u)\, \Delta A_u + \int_{M^c \cap (u,\infty)} e^{-\alpha(s-u)} m(u,s)\, dA_s.$$

Since $s \in M^c(\omega)$ and $s > u$ if and only if $s-u \in M^c(\theta_u \omega)$, the second term in the last display is equal to

$$\int_{M^c(\theta_u)} e^{-\alpha s} m(u, u+s)\, dA_{u+s} = \int_{M^c(\theta_u)} e^{-\alpha s} m_s(\theta_u)\, dA_s(\theta_u)$$
$$= \int_{(0, S \circ \theta_u)} e^{-\alpha s} m_s(\theta_u)\, dA_s(\theta_u) = H^\alpha \circ \theta_u.$$

Combining these results with the fact that ν is carried by $M^{\rightarrow} \subset M$ yields (4.30).

5. Dual Well Measurable Projections

The assumptions and basic data are the same as in Section 4. In particular we fix a raw additive functional A with a finite α-potential and let $\bar{A}^\alpha$ denote its α-transport via m. We are going to calculate the dual well measurable projection of $\bar{A}^\alpha$ which we shall denote by $\tilde{A}^\alpha$. As a first step we shall show that we may pass to the dual well measurable projection of A which for notational convenience we shall denote by B. Thus B is an (adapted) additive functional such that each x

$$E^x \int Z_t\, dA_t = E^x \int Z_t\, dB_t$$

for all positive well measurable Z. Let $\bar{B}^\alpha$ denote the α-transport via m of B and let $\tilde{B}^\alpha$ denote the dual well measurable projection of $\bar{B}^\alpha$.

(5.1) **Proposition.** $\tilde{A}^\alpha = \tilde{B}^\alpha$.

Proof. If Z is a positive well measurable process, then using (4.22)

$$E^x \int Z_t\, d\bar{A}_t^\alpha = E^x \int Z_t \int \gamma^\alpha(dt, s)\, dA_s$$
$$= E^x \int \gamma^\alpha(Z, s)\, dA_s$$

where $\gamma^\alpha(Z, s) = \int Z_t\, \gamma(dt, s)$. Since we also have

$$E^x \int Z_t\, d\bar{B}_t^\alpha = E^x \int \gamma^\alpha(Z, s)\, dB_s,$$

Proposition (5.1) is an immediate consequence of the following lemma which is of some interest in its own right.

(5.2) **Lemma.** *If Z is a bounded (or positive) well measurable process, then $s \to \gamma(Z, s)$ is well measurable.*

Proof. Since stochastic intervals of the form $[[0, R))$ with R a bounded stopping time form a π-system generating the well measurable sets, it suffices to show that $\gamma([[0, R)), s) = m(R-, s)$ is well measurable whenever R is a bounded stopping time. Since $\gamma(\cdot, s)$ does not charge $\{0\}$ we may also assume that R is strictly positive. One checks easily that for each fixed t, $s \to m(t-, s)$ is right continuous on $\mathbf{R}^+$ and identically equal to one on $[0, t)$. Thus $s \to m(R-, s)$ is right continuous and (5.2) will follow provided this process is adapted. However, it is a routine matter to show that $1_{\{R \geqq \varepsilon\}} m(R-\varepsilon, s)$ is $\mathscr{F}_{s+\varepsilon}$ measurable for each fixed s and $\varepsilon > 0$. Letting $\varepsilon \downarrow 0$ and using the right continuity of the family $(\mathscr{F}_t)$ shows that $m(R-, s)$ is $\mathscr{F}_s$ measurable for each s completing the proof of Lemma 5.2.

(5.3) *Remark.* A similar, but simpler, argument shows that $s \to \gamma^*(Z, s)$ is previsible whenever Z is a bounded previsible process.

As a result of Proposition 5.2, in computing $\tilde{A}^\alpha$ we may assume that A itself is adapted. Consequently *in the remainder of this section we suppose that A is an (adapted) additive functional with finite α-potential.* Our starting point is the decomposition (4.30) of $\bar{A}^\alpha$. We first decompose the measure v appearing in (4.30) as $v=v_w+v_\pi$ where $v_w(\cdot,\omega)$ puts unit mass at each point of $M_w^{\rightarrow}(\omega)$ and $v_\pi(\cdot,\omega)$ puts unit mass at each point of $M_\pi^{\rightarrow}(\omega)$. Since $M_w^{\rightarrow}$ and $M_\pi^{\rightarrow}$ are homogeneous sets, v_w and v_π are homogeneous measures. Moreover, it is easy to check using the fact that $M_w^{\rightarrow}$ is a countable union of graphs of stopping times that v_w is well measurable in the sense of (3.7). We now write (4.30) as follows:

$$\begin{aligned}\bar{A}_t^\alpha = & \int_{(0,t]\cap M} dA_s + \int_{(0,t]} \Delta A_s\, \kappa(ds) \\ & + \int_{(0,t]} H^\alpha\circ\theta_s\big(\kappa(ds)+v_w(ds)\big) + \int_{(0,t]} H^\alpha\circ\theta_s\, v_\pi(ds).\end{aligned} \tag{5.4}$$

Since A is adapted and κ is well measurable the first two terms on the right side of (5.4) are right continuous and adapted. On the other hand the well measurable projection of the process $(H^\alpha\circ\theta_s)$ is obviously $\big(v_A^\alpha(X_s)\big)$ where the notation is that of Section 4. But $\kappa+v_w$ is well measurable and so the dual well measurable of the third term on the right side of (5.4) is $\int_{(0,t]} v_A^\alpha(X_s)\big(\kappa(ds)+v_w(ds)\big)$. Let

$$J_t^\alpha = \int_{(0,t]} H^\alpha\circ\theta_s\, v_\pi(ds) = \sum_{s\leq t,\, s\in M_\pi^{\rightarrow}(\omega)} H^\alpha\circ\theta_s \tag{5.5}$$

and let $\tilde{J}^\alpha$ denote the dual well measurable projection of J. Then

$$\tilde{A}_t^\alpha = \int_{(0,t]\cap M} dA_s + \int_{(0,t]} \Delta A_s\,\kappa(ds) + \int_{(0,t]} v_A^\alpha(X_s)\big(\kappa(ds)+v_w(ds)\big) + \tilde{J}_t^\alpha, \tag{5.6}$$

and it remains to find an explicit expression for $\tilde{J}^\alpha$. Since $M_\pi^{\rightarrow}$ contains the graph of no stopping time, it is clear that $\tilde{J}^\alpha$ is continuous.

The dual well measurable projections of functionals of the form (5.5) have been calculated by Meyer [7]. However, his results are somewhat complicated to describe as they involve an auxiliary measure on function space, and, since we do not need the general results in this paper, we shall merely refer the reader to [7]. However, in this connection note that H^α is $\mathscr{F}_{S-}$ measurable and consequently $H^\alpha\circ k_D = H^\alpha$ where (k_t) are the "killing" operators on Ω. In the next section we shall need an explicit form for $\tilde{J}^\alpha$ in the special case $A_t=\int_0^t h(X_s)\,ds$ and shall give a direct argument for such functionals.

Finally it is also of interest to consider the dual previsible projection of $\tilde{A}^\alpha$. Since $\tilde{J}^\alpha$ is continuous this amounts to taking the dual previsible projections of the first three terms on the right side of (5.6). We shall not enter into this here. The interested reader should consult [5].

6. The Decomposition Theorem

In this section we shall obtain a decomposition of the semigroup (P_t) in terms of the semigroup (Q_t) generated by (m_t) that is analogous to the last exit decomposition obtained in [5]. The notation and assumptions are as in the previous sections.

We need some preliminary discussion of the relationship between the sets F and E_m. Let $F_m = E - E_m = \{x: P^x(m_0 = 0) = 1\}$. Then F and F_m are the sets of points regular for D and S respectively. Thus F and F_m are both finely closed nearly Borel sets, and since $S \leqq D$, $F \subset F_m$.

(6.1) **Proposition.** *$F_m - F$ is semipolar.*

Proof. Let $\varphi(x) = E^x(e^{-D})$ and $\psi(x) = V^1 1(x) = E^x \int e^{-t} m_t\, dt$. Then φ is 1-excessive and ψ is nearly Borel and finely continuous because m is exact. Also $F = \{\varphi = 1\}$ and $F_m = \{\psi = 0\}$. If $0 < \beta < 1$ the argument used in the proof of Proposition 3.5 shows that $\{x: \varphi(x) \leqq \beta, \psi(x) = 0\}$ is (totally) thin establishing (6.1).

It follows from (6.1) that F and F_m have the same finely perfect kernel. (See [7] for a discussion of the finely perfect kernel without assuming the existence of a reference measure.) Actually all that we need is the much more elementary fact that any continuous additive functional that is carried by F_m is also carried by F. Also observe that since $M(\omega) - \rho_F(\omega)$ is countable the symmetric difference of $M(\omega)$ and $\rho_{F_m}(\omega)$ is countable (almost surely).

If h is a bounded positive Borel function we define

$$A_t = A_t(h) = \int_0^t h(X_s)\, ds. \tag{6.2}$$

Since $M(\omega) - \rho_F(\omega)$ is countable we obtain the following expression for $\bar{A}^\alpha(h)$, the α-transport via m of $A(h)$, from (5.4)

$$d\bar{A}_t^\alpha(h) = 1_F(X_t)\, h(X_t)\, dt + H^\alpha \circ \theta_t(\lambda(dt) + v_\pi(dt)) \tag{6.3}$$

where we have put $\lambda(dt) = \kappa(dt) + v_w(dt)$ and

$$H^\alpha = \int_0^S e^{-\alpha t} m_t\, h(X_t)\, dt = \int_0^\infty e^{-\alpha t} m_t\, h(X_t)\, dt. \tag{6.4}$$

Clearly λ is a well measurable homogeneous measure that is carried by $M^c \cup M_w^{\rightarrow} \subset \rho_F^c$ while v_π is a homogeneous measure carried by $M_\pi^{\rightarrow} \subset \rho_F$. In the present case $v_A^\alpha(x) = V^\alpha h(x)$ where (V^α) is the resolvent defined in (4.3). Therefore (5.6) becomes

$$d\tilde{A}_t^\alpha(h) = 1_F(X_t)\, h(X_t)\, dt + V^\alpha h(X_t)\, \lambda(dt) + d\tilde{J}_t^\alpha(h) \tag{6.5}$$

where $\tilde{J}^\alpha(h)$ is the (continuous) well measurable projection of

$$J_t^\alpha(h) = \int_{(0,t]} H^\alpha \circ \theta_s\, v_\pi(ds) = \sum_{s \leqq t,\, s \in M_\pi^{\rightarrow}} H^\alpha \circ \theta_s. \tag{6.6}$$

Since v_π is carried by ρ_F, it is clear that $\tilde{J}^\alpha(h)$ is a continuous additive functional that is carried by F. Of course, in light of (6.1), we may replace F by F_m in (6.3) and (6.5). Moreover by (4.28) the α-potential of $\bar{A}^\alpha(h)$ and hence also $\tilde{A}^\alpha(h)$ is given by $P_m^\alpha U^\alpha h = U^\alpha h - V^\alpha h$.

(6.7) **Lemma.** *For all α, $\beta > 0$ and bounded positive Borel h,*

(i) *$\tilde{A}^\alpha(h) - \tilde{A}^\beta(h) = (\beta - \alpha)\, \tilde{A}^\alpha(V^\beta h)$ and*

(ii) *$\tilde{J}^\alpha(h) - \tilde{J}^\beta(h) = (\beta - \alpha)\, \tilde{J}^\alpha(V^\beta h)$.*

Proof. It is evident that the first two terms on the right side of (6.5) satisfy this relation and so (i) will follow from (ii). However, both sides of (ii) are continuous additive functionals and so to prove (ii) it suffices to show that both sides of (ii) have the same (finite) $\alpha+\beta$-potential. But using (6.5) again this will follow if we show that $\bar{A}^\alpha(h)-\bar{A}^\beta(h)$ and $(\beta-\alpha)\,\bar{A}^\alpha(V^\beta h)$ have the same $\alpha+\beta$-potential. From the definitions of the various quantities we have

$$\begin{aligned}E^x \int e^{-(\alpha+\beta)t}\, d\bar{A}_t^\alpha(V^\beta h) &= E^x \int e^{-\beta t} \int \gamma(dt,s)\, e^{-\alpha s}\, V^\beta h(X_s)\, ds\\ &= E^x \int \left[\int e^{-\beta t}\gamma(dt,s)\right] e^{-\alpha s}\, V^\beta h(X_s)\, ds.\end{aligned}$$

The process in square brackets is adapted (actually well measurable by (5.3)), and so by VII-T-15 of [6] the last displayed expression becomes

$$\begin{aligned}&E^x \int \left[\int e^{-\beta t}\gamma(dt,s)\right] e^{(\beta-\alpha)s} \int_s^\infty e^{-\beta u}\, m(s,u)\, h(X_u)\, du\, ds\\ &\quad = E^x \int e^{-\beta u}\, h(X_u) \int_0^u e^{(\beta-\alpha)s}\, m(s,u) \left[\int e^{-\beta t}\gamma(dt,s)\right] ds\, du.\end{aligned}$$

It is immediate from (4.5) that if $t\leqq s\leqq u$, then $m(s,u)\,\gamma(dt,s)=1_{(0,s]}(t)\,\gamma(dt,u)$. Since $\gamma(\cdot,s)$ is carried by $(0,s]$ the double integral over s and t becomes

$$\begin{aligned}&\int_0^u e^{(\beta-\alpha)s} \int_{(0,s]} e^{-\beta t}\gamma(dt,u)\, ds\\ &\quad = \int_0^u \int_{[t,u)} e^{(\beta-\alpha)s}\, ds\, e^{-\beta t}\gamma(dt,u)\\ &\quad = (\beta-\alpha)^{-1} \int_0^u \left[e^{(\beta-\alpha)u}-e^{(\beta-\alpha)t}\right] e^{-\beta t}\gamma(dt,u).\end{aligned}$$

Combining these calculations we find

$$\begin{aligned}&(\beta-\alpha)\, E^x \int e^{-(\alpha+\beta)t}\, d\bar{A}_t^\alpha(V^\beta h)\\ &\quad = E^x \int e^{-\alpha u} h(X_u) \int e^{-\beta t}\gamma(dt,u)\, du - E^x \int e^{-\beta u} h(X_u) \int e^{-\alpha t}\gamma(dt,u)\, du\\ &\quad = E^x \int e^{-(\alpha+\beta)t}\, d\bar{A}_t^\alpha(h) - E^x \int e^{-(\alpha+\beta)t}\, d\bar{A}_t^\beta(h),\end{aligned}$$

which establishes (6.7).

We may now apply the results (not the techniques but the results) of [5] or [7] to obtain the following description of $\tilde{J}^\alpha(h)$. For example, the relevant facts in [7] are Theorem 2 and Proposition 4. Note that the particular form of the resolvent (V^α) and semigroup (Q_t) entering in these statements plays no role. We begin by fixing a continuous additive functional K with a bounded one-potential that is carried by F and such that $\tilde{J}^\alpha(h)$ is absolutely continuous with respect to K for all $\alpha>0$ and h. For example, we may take $K=\tilde{J}^1(1)$. Then for each $\alpha>0$ and $x\in F$ there exists a finite measure $\hat{V}^\alpha(x,\cdot)$ carried by E_m such that for each $B\in\mathscr{E}$, $x\to\hat{V}^\alpha(x,B)$ is universally measurable and satisfying for $x\in F$, $\alpha,\beta>0$, and all h

$$\text{(6.8)}\qquad \begin{aligned}&\text{(i)}\quad \hat{V}^\alpha(x,\cdot)-\hat{V}^\beta(x,\cdot)=(\beta-\alpha)\,\hat{V}^\alpha V^\beta(x,\cdot)\\ &\text{(ii)}\quad \tilde{J}_t^\alpha(h)=\int_0^t \hat{V}^\alpha h(X_s)\, dK_s.\end{aligned}$$

Moreover $\hat{V}^\alpha 1(x) \to 0$ as $\alpha \to \infty$ for each x in F, and so by Theorem 6.9 of [4] there exists for each x in F an entrance law $(\hat{Q}_t(x, \cdot))_{t>0}$ for the semigroup (Q_t) such that

$$\hat{V}^\alpha f(x) = \int e^{-\alpha t} \hat{Q}_t f(x)\, dt \tag{6.9}$$

for each bounded universally measurable f. It follows from standard Laplace inversion formulas that $(t, x) \to \hat{Q}_t(x, B)$ is $\mathscr{B}^{++} \otimes \mathscr{E}_F^*$ measurable for each $B \in \mathscr{E}^*$ — here $\mathscr{E}^*$ is the σ-algebra of universally measurable subsets of E and $\mathscr{E}_F^*$ is the trace of $\mathscr{E}^*$ on F. If $K = \tilde{J}^1(1)$, then $\alpha \hat{V}^\alpha 1(x) \leq (1 \vee \alpha)$ for all x in F, and this in turn implies that $\hat{Q}_t 1(x) \leq (1 - e^{-t})^{-1}$ for $t > 0$ and x in F. We emphasize that the kernels $\hat{V}^\alpha$ and $\hat{Q}_t$ depend on the choice of the continuous additive functional K. Following Meyer it is convenient to extend $\hat{Q}_t(x, \cdot)$ and $\hat{V}^\alpha(x, \cdot)$ by setting

$$\hat{Q}_t(x, \cdot) = Q_t(x, \cdot); \; \hat{V}^\alpha(x, \cdot) = V^\alpha(x, \cdot) \tag{6.10}$$

if $x \notin F$. Note that $\hat{Q}_t(x, \cdot) = Q_t(x, \cdot) = 0$ if $x \in F_m - F$ and that for each $x \in E$ and $t > 0$, $\hat{Q}_t(x, \cdot)$ is a finite measure carried by E_m such that $\hat{Q}_{t+s} = \hat{Q}_t \hat{Q}_s$ for t, $s > 0$. A similar statement holds for $\hat{V}^\alpha$.

One may now combine (6.5) and (6.8 ii). Let $k(dt) = \lambda(dt) + dK_t$. Then k is a well measurable homogeneous measure. Since λ is carried by $M^c \cup M_w^{\rightarrow} \subset \rho_F^c$, the purely discontinuous part of k is carried by ρ_F^c. But K is carried by F and so one obtains from (6.5) and (6.8)

$$d\tilde{A}_t^\alpha(h) = 1_F(X_t)\, h(X_t)\, dt + \hat{V}^\alpha h(X_t)\, k(dt), \tag{6.11}$$

and we may replace F by F_m in (6.11) whenever it is convenient.

Let $Z = (Z_t)$ be a bounded positive well measurable process and let $\gamma(Z, s) = \int Z_t \gamma(dt, s)$. Let $h \in b\mathscr{E}^+$. Then on the one hand

$$\begin{aligned} E^x \int Z_t e^{-\alpha t} d\bar{A}_t^\alpha(h) &= E^x \int Z_t e^{-\alpha t} \int \gamma^\alpha(dt, s)\, h(X_s)\, ds \\ &= \int e^{-\alpha s} E^x [\gamma(Z, s)\, h(X_s)]\, ds, \end{aligned}$$

while on the other hand by (6.11)

$$E^x \int Z_t e^{-\alpha t} d\bar{A}_t^\alpha(h) = \int e^{-\alpha t} E^x [Z_t 1_F(X_t)\, h(X_t)]\, dt + E^x \int e^{-\alpha t} Z_t \hat{V}^\alpha h(X_t)\, k(dt).$$

Expressing $\hat{V}^\alpha$ in terms of $\hat{Q}_s$ the last displayed integral becomes

$$\begin{aligned} E^x \int e^{-\alpha t} Z_t \int e^{-\alpha s} \hat{Q}_s h(X_t)\, ds\, k(dt) &= E^x \int Z_t \int_t^\infty e^{-\alpha s} \hat{Q}_{s-t} h(X_t)\, ds\, k(dt) \\ &= E^x \int e^{-\alpha s} \int_{(0,s)} Z_t \hat{Q}_{s-t} h(X_t)\, k(dt)\, ds. \end{aligned}$$

Combining these formulas and inverting the Laplace transform we find that for each x

$$E^x [\gamma(Z, t)\, h(X_t)] = E^x [Z_t 1_F(X_t)\, h(X_t)] + E^x \int_{(0,t)} Z_s \hat{Q}_{t-s} h(X_s)\, k(ds) \tag{6.12}$$

almost everywhere (Lebesgue) in t. As usual we may replace F by F_m (or $1_F(X_t)$ by $1_M(t)$) in (6.12) whenever it is convenient to do so.

We next claim that if $r>0$, Z a positive well measurable process, and h a positive function vanishing on F, then

(6.13) $$E^x\{\int_{(0,r)} Z_t\gamma(dt,s)h(X_s)\}=E^x\int_{(0,r)} Z_t\hat{Q}_{s-t}h(X_t)k(dt)$$

for *all* $s>r$. To this end suppose that h is the restriction to $E-F$ of a bounded positive continuous function and that Z is a bounded positive well measurable process. Applying (6.12) to $1_{(0,r)}(t)Z_t$ we see that (6.13) holds almost everywhere in s on (r,∞). We shall now show that the left side of (6.13) is right continuous in s on (r,∞). Since $\gamma(\cdot,s)$ is carried by $[L_s,s]$ the left side of (6.13) reduces to

(6.14) $$E^x\{\int_{(0,r)} Z_t\gamma(dt,s)h(X_s);\, L_s<r;\, X_s\notin F\}.$$

Fix $s>r$ with $L_s<r$ and $X_s\notin F$. Then $L_s<s$ and so $s\notin M$. Since M is closed there exists an $\varepsilon>0$ such that $[s,s+\varepsilon]\cap M$ is empty. Then if $u\in[s,s+\varepsilon)$, $X_u\notin F$ and so $h(X_u)\to h(X_s)$ as $u\downarrow s$. Also for such a u, $L_u=L_s$ and it follows from (4.5) that $\gamma(dt,u)=m(s,u)\gamma(dt,s)$ on $(0,r)$. Therefore

$$\int_{(0,r)} Z_t\gamma(dt,u)=m(s,u)\int_{(0,r)} Z_t\gamma(dt,s).$$

But $m(s,u)\uparrow m_0\circ\theta_s$ as $u\downarrow s$ and since $s\notin M$, $S\circ\theta_s>0$ or $m_0\circ\theta_s=1$. Hence

$$\int_{(0,r)} Z_t\gamma(dt,u)$$

approaches $\int_{(0,r)} Z_t\gamma(dt,s)$ as $u\downarrow s$. It is now clear that (6.14) and hence the left side of (6.13) is right continuous in s on (r,∞) for such h and Z. This point being established one may now repeat the argument following (3.18) of [5] to obtain (6.13). Consequently (6.13) holds for each $r>0$ and all $s>r$. Now fix $s>0$ and let $r\uparrow s$ in (6.13) to obtain

(6.15) $$E^x\{\int_{(0,s)} Z_t\gamma(dt,s)h(X_s)\}=E^x\int_{(0,s)} Z_t\hat{Q}_{s-t}h(X_t)k(dt)$$

identically in s provided h vanishes on F.

We come now to the main result of this section. Recall that if $x\notin F$, then $\hat{Q}_t(x,\cdot)=Q_t(x,\cdot)$ and so we define $\hat{Q}_0(x,\cdot)=Q_0(x,\cdot)=1_{E_m}(x)\varepsilon_x$ for $x\notin F$. Recall also that if $k(dt)$ charges $\{s\}$, then $X_s\notin F$.

(6.16) **Proposition.** *Let h be a positive function and Z a positive well measurable process. Then*

$$E^x\{\gamma(Z,s)h(X_s)\}=E^x\{\gamma(Z,s)h(X_s);\, X_s\in F_m\}+E^x\int_{(0,s]} Z_t\hat{Q}_{s-t}h(X_t)\,k(dt),$$

for all x and $s>0$ where $F_m=E-E_m$ as before.

An immediate corollary of (6.16) is the following decomposition of the semigroup (P_t).

(6.17) $$P_t h(x)=Q_t h(x)+E^x\{h(X_t);X_t\in F_m\}+E^x\int_{(0,t]}\hat{Q}_{t-s}h(X_s)k(ds).$$

To see this write

$$P_t h(x) = Q_t h(x) + E^x\{(1-m_t) h(X_t)\}$$

and since $1-m_t=\gamma(1,t)$ we may apply (6.16) with $Z=1$ to obtain (6.17) because $m_t=0$ if $X_t \in F_m$.

We turn now to the proof of (6.16). Suppose first that h vanishes on $F_m \supset F$ so that $Q_0 h = h$. From (6.15) we obtain for $s>0$

(6.18) $$E^x\{\gamma(Z,s) h(X_s)\} = E^x\{Z_s \gamma(\{s\},s) h(X_s)\} + E^x \int_{(0,s)} Z_t \hat{Q}_{s-t} h(X_t) k(dt).$$

Now from (4.18), $1_M(s)\gamma(\{s\},s) = 1_M(s)$ because $L_s = s > 0$ if and only if $s \in M$. Since κ is carried by M^c, $1_M(s) k(\{s\}) = v_w(\{s\}) = 1_{M-\rho_F}(s)$ by the definition of v_w and (3.5). But $h = Q_0 h$ vanishes on $F_m \supset F$ and so combining the above observations yields

(6.19) $$E^x\{Z_s \gamma(\{s\},s) h(X_s) 1_M(s)\} = E^x\{Z_s Q_0 h(X_s) 1_M(s) k(\{s\})\}.$$

Again from (4.18), $1_{M^c}(s)\gamma(\{s\},s) = 1_{M^c}(s)\kappa(\{s\})$ since $m(s,s)=1$ and v is carried by M. But $1_{M^c}(s)\kappa(\{s\}) = 1_{M^c}(s) k(\{s\})$ and so

(6.20) $$E^x\{Z_s \gamma(\{s\},s) h(X_s);\, s \in M^c\} = E^x\{Z_s Q_0 h(X_s) k(\{s\});\, s \in M^c\}.$$

Combining (6.18), (6.19), and (6.20) we obtain (6.16) in the special case in which h vanishes on F_m. Writing $h = h 1_{F_m} + h 1_{E_m}$ in the general case and noting that $\hat{Q}_t(h 1_{E_m}) = \hat{Q}_t h$ completes the proof of (6.16).

It is possible to draw a number of interesting corollaries from (6.16) as in [5] or [7]. However, we shall not pursue this here. We hope to return to this question in a subsequent publication.

Appendix

In this appendix we shall prove that in the framework of Markov processes the well measurable processes are generated by the previsible processes and processes of the form $(t,\omega) \to f \circ X_t(\omega)$. Without doubt this is known to experts in the field and is quite close in spirit to the fact that $\mathscr{F}_T = \mathscr{F}_{T-} \vee \sigma(X_T)$ for all stopping times. However, we have been unable to find the result in the literature and since it is quite useful, we shall record it here.

As in the previous sections X is a Markov process satisfying the hypotheses of the right and we adopt the usual notations and conventions. Several authors have recently established the following result. If $Z=(Z_t)$ is a bounded measurable (with respect to $\mathscr{B}(\mathbf{R}^+) \otimes \mathscr{F}$) process, then there exists a bounded measurable process $w(Z) = (Z_t^w)$ which is the well measurable projection of Z for the family $(\mathscr{F}_t^\mu)$ for each P^μ. See, for example, [1] or [7]. Let $\mathscr{A}^\mu$ be the σ-algebra on $\mathbf{R}^+ \times \Omega$ generated by processes of the form $(t,\omega) \to Y_t(\omega) f \circ X_t(\omega)$ where Y is $(\mathscr{F}_t)$ adapted and continuous and f is a bounded nearly Borel measurable function, and also all processes that are P^μ indistinguishable from zero. Clearly $\mathscr{A}^\mu$ is the same if we require that f be Borel (or even continuous) rather than nearly Borel. Here is our result.

(A.1) **Proposition.** *Let Z be a bounded $\mathscr{B}(\mathbf{R}^+) \otimes \mathscr{F}^\mu$ measurable process and $w(Z)$ be its well measurable projection on the family $(\mathscr{F}_t^\mu)$ relative to P^μ. Then $w(Z)$ is $\mathscr{A}^\mu$*

measurable. In particular if Z *is well measurable over* $(\Omega, \mathscr{F}_t^\mu, P^\mu)$, *then* Z *is* $\mathscr{A}^\mu$ *measurable.*

Proof. Let $\mathscr{H}$ be the collection of all bounded $\mathscr{B}(\mathbf{R}^+)\otimes\mathscr{F}^\mu$ measurable processes Z such that $w(Z)$ is $\mathscr{A}^\mu$ measurable. Then $\mathscr{H}$ is a vector space containing the constants and closed under uniform limits and also increasing uniformly bounded limits. See [3], p. 98. Hence it suffices to show that $\mathscr{H}$ contains all processes of the form $Z_t(\omega)=1_{[a,b]}(t)\,g(\omega)$ where $g\in b\mathscr{F}^\mu$ and $0\leqq a<b<\infty$. In this case $Z_t^w=1_{[a,b]}(t)\,G_t$ where G_t is a right continuous version of the martingale $E^\mu(g\,|\,\mathscr{F}_t^\mu)$. Suppose first that

$$g=\prod_{i=1}^{n}\int_0^\infty e^{-\alpha_i t} f_i(X_t)\,dt \tag{A.2}$$

where $\alpha_i>0$ and $f_i\in b\mathscr{E}$ for $1\leqq i\leqq n$. Then the computation on p. 113 of [8] shows that $G=(G_t)$ is $\mathscr{A}^\mu$ measurable (actually a finite sum of products of the generators). Hence $w(Z)=Z^w\in\mathscr{A}^\mu$ in this case. But finite linear combinations of elements of the form (A.2) are dense in $L^2(\Omega,\mathscr{F}^\mu,P^\mu)$ – see [8]. Thus if $g\in b(\mathscr{F}^\mu)$ there exists a sequence $\{g_n\}$ with each g_n a finite linear combination of elements of the form (A.2) and such that $g_n\to g$ in $L^2(P^\mu)$. If G_t^n and G_t are right continuous versions of the martingales $E^\mu(g_n\,|\,\mathscr{F}_t^\mu)$ and $E^\mu(g\,|\,\mathscr{F}_t^\mu)$ respectively, then it follows from standard martingale estimates that there exists a subsequence $\{n^k\}$ such that almost surely P^μ, $G_t^{n^k}\to G_t$ uniformly in t. This implies that G is $\mathscr{A}^\mu$ measurable and completes the proof.

(A.3) **Corollary.** *Let* A *and* B *be raw additive functionals with finite potentials and such that* $U_A f=U_B f$ *for all bounded Borel* f – *bounded continuous* f *would suffice. Then* A *and* B *have the same dual well measurable projections.*

Proof. Recall that $U_A f(x)=E^x\int_0^\infty f(X_t)\,dA_t$. Suppose $f\geqq 0$ and bounded Borel. The hypotheses imply that $dA_t^*=f(X_t)\,dA_t$ and $dB_t^*=f(X_t)\,dB_t$ are raw additive functionals with the same finite potentials. It is well known and an easy consequence of Meyer's integration lemma ([6], VII-T 17) that

$$E^\mu\int_0^\infty Y_t f(X_t)\,dA_t=E^\mu\int_0^\infty Y_t f(X_t)\,dB_t$$

whenever $Y=(Y_t)$ is a bounded adapted (left) continuous process. Combining this with (A.1) yields Corollary A.3.

(A.4) *Remark.* The reader should have no difficulty in extending the corollary as follows: Let S be an exact terminal time and let A and B be raw additive functionals of (X, S). If for some fixed $\alpha\geqq 0$, A and B have finite α-potentials and $U_A^\alpha f=U_B^\alpha f$ for all bounded Borel f, then A and B have same dual well measurable projections.

Note Added in Proof. K.L. Chung has pointed out that Dynkin in "On extensions of Markov processes" Theory of Prob. and Appl. **13**, 672–676 (1968), has obtained a decomposition of the resolvent using essentially the same technique as that used in [5]. We take this opportunity to acknowledge Dynkin's priority.

Bibliography

1. Benveniste, A., Jacod, J.: Projection des fonctionnelles additives et représentation des potentiels d'un processus de Markov. To appear
2. Blumenthal, R.M., Getoor, R.K.: Markov Processes and Potential Theory. New York: Academic Press 1968
3. Dellacherie, C.: Capacités et Processus Stochastiques. Berlin-Heidelberg-New York: Springer 1972
4. Getoor, R.K., Sharpe, M.J.: Last exit times and additive functionals. Ann. Probab. **1**, 550–569 (1973)
5. Getoor, R.K., Sharpe, M.J.: Last exit decompositions and distributions. Indiana Univ. Math. J. **23**, 377–404 (1973)
6. Meyer, P.A.: Probability and Potentials. Boston: Ginn (Blaisdell) 1966
7. Meyer, P.A.: Ensembles aléatoires Markoviens homogènes II. To appear in Strasbourg Probability Seminar VIII, 1972–73
8. Meyer, P.A.: Processus de Markov. Vol. **26**, Lecture Notes in Math. Berlin-Heidelberg-New York: Springer 1967
9. Meyer, P.A., Walsh, J.B.: Quelques applications des résolvantes de Ray. Invent. math. **14**, 143–166 (1971)
10. Walsh, J.B.: The perfection of multiplicative functionals. Vol. **258**, Lecture Notes in Math. 215–232. Berlin-Heidelberg-New York: Springer 1972
11. Azéma, J.: Quelques applications de la théorie générale des processus. I. Invent. math. **18**, 293–336 (1972)

R.K. Getoor
M.J. Sharpe
Department of Mathematics
University of California, San Diego
La Jolla, California 92037
USA

(Received September 15, 1973)