Z. Wahrscheinlichkeitstheorie verw. Geb. 28, 113–128 (1974)

Multiple Points for a Process in R^2 with Stable Components

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Section 1: Introduction

The object of this paper is to investigate the existence and Hausdorff dimension of the set of multiple points of the sample paths of a type of Markov process in R^2 with stationary independent increments. To define the process, let $X_{\alpha_i,1}(t) \equiv X_i(t)$ denote a stable process of index α_i in R^1 for $i=1,2$; assume the two processes independent and let $X(t) \equiv (X_1(t), X_2(t))$. We call $X(t)$ a process in R^2 with stable components. If the α_i exceed 1, each component of $X(t)$ is point recurrent but $X(t)$ can be shown (see [6] or [10]) to be transient. If $\alpha_1 = \alpha_2$, $X(t)$ is stable. Hence we assume that $1 < \alpha_2 < \alpha_1 \leqq 2$. The situation for $\alpha_2 \leqq 1$ or for a process with stable components in R^3 is discussed briefly in Section 6.

Taylor [12] has studied the existence and Hausdorff dimension of multiple points for symmetric stable processes of index α in R^n. Our basic argument will resemble his, although significant modifications are necessary. After stating the necessary preliminaries in Section 2 we use some potential theory in Section 3 to obtain estimates of various hitting probabilities with respect to $X(t)$ and then proceed to obtain conditions on k, α_1 and α_2 for existence (Section 4) of points of multiplicity k. In Section 5 we compute the Hausdorff dimension of the multiple points. The problem of multiple points for processes with more than 2 stable components or for ones with a component in a higher dimension remains open, as does the problem for multiple points (or at least double points) for a general process with stationary and independent increments.

Section 2: Preliminaries

The n-dimensional characteristic function of a stable process $X_{\alpha,n}(t)$ of index $\alpha \neq 1$ in R^n has the form $\exp[t\psi(y)]$, where

$$\psi(y) = i(a, y) - \delta |y|^\alpha \int_{S_n} w_\alpha(y, \theta)\, \mu(d\theta),$$

with $a \in R^n$, $\delta > 0$,

$$w_\alpha(y, \theta) = [1 - i \operatorname{sgn}(y, \theta) \tan \pi\alpha/2]\, |(y/|y|, \theta)|^\alpha,$$

and μ a probability measure on the surface of the unit sphere S_n in R^n [8]. We assume $a=0$, $\delta=1$, and that μ is not supported by a proper subspace of R^n. If μ is uniform, the process is said to be symmetric.

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* Research supported in part by the National Science Foundation, Grant GP-33908X, at Case Western Reserve University.

When speaking of a stable process $X_{\alpha,n}$ we will always write the two subscripts to indicate index α and dimension n except when the process is actually a component of a process with stable components. When the latter occurs, we use a single subscript i to indicate the i-th component. The symbol $X(t)$ will be understood to refer to the process (X_1, X_2) as defined in Section 1. We will use the notation and methods of [6] to determine various hitting probabilities in Section 3. To do this we use the facts that stable densities $p_{\alpha,n}(t)$ of the type being considered are positive, continuous, bounded in x for each fixed t, and satisfy the scaling property [11]:

$$p_{\alpha,n}(t,x)=r^{n/\alpha}\, p_{\alpha,n}(rt, r^{1/\alpha}x) \quad \text{for } r>0. \tag{2.1}$$

The density, $p(t,x)$, of $X(t)$ will have the form:

$$p(t,x)=p_{\alpha_1,1}(t,x_1)\, p_{\alpha_2,1}(t,x_2), \quad \text{where } x=(x_1,x_2). \tag{2.2}$$

It will be assumed that all processes being considered have been defined so as to have sample functions $X(t,\omega)$ which are right continuous and have left limits everywhere [1]. In addition, we assume that the strong Markov property holds and that $X(0)=0$ with probability one.

We define the hitting probability $\Phi(x,E)$ of a planar Borel set E starting from $x\in R^2$ by:

$$\Phi(x,E)=P^x[X(t)\in E \text{ for some } t>0].$$

Often in writing $P^0[\;]$ we simply use $P[\;]$. The set of points in R^2 that the sample path $X(t,\omega)$ hits in the time interval $[a,b]$ is denoted by $L(a,b;\omega)$, where $0\leqq a<b\leqq\infty$. Likewise, for any positive integer $k\geqq 2$, we denote by $L^k(a,b;\omega)$ the set of points hit k times in the time interval $[a,b]$. Sometimes we use $E_k(\omega)$ to denote the set $L^k(0,\infty;\omega)$ of k-multiple points of the path $X(t,\omega)$. Positive constants whose values remain fixed throughout the discussion will be introduced in order and denoted by $c_1,\ldots,c_{35}$. The letter c will be used as a positive constant which can vary in size from statement to statement or line to line.

Section 3: Potential Theory and Hitting Probabilities

Taylor [13] gives a brief background of the potential theory which we will need. If we let $p_1(t,x_1)$ be the density of $X_1(t)$ and $p_2(t,x_2)$ that of $X_2(t)$, the density of $X(t)$ is their product and will be denoted by $p(t,x)$, where $x=(x_1,x_2)$. $U(y)$, the kernel of the process, is given by

$$U(y)=\int_0^\infty p(t,y)\,dt \tag{3.1}$$

and converges for all $y\neq(0,0)$ in R^2.

If we let μ be any measure defined on Borel subsets of compact sets E in the plane, the potential at x of the measure μ on E is $W_\mu(x)=\int_E U(y-x)\,\mu(dy)$. The capacity of E is zero iff W_μ is unbounded for every μ for which $\mu(E)>0$. If there are some μ such that W_μ is bounded, we define the capacity of E with respect to $X(t)$ by:

$$C(E)=\sup\{\mu(E)\colon W_\mu(x)\leqq 1 \text{ for all } x\}.$$

When E is compact, this supremum is actually attained for a measure ν, called the capacitory measure on E.

Finally, we denote the hitting probability of a compact set E starting from x by:

$$\Phi(x, E) = P^x\{X(t) \in E \text{ for some } t > 0\}.$$

Hitting probabilities are then given in terms of the kernel and the capacitory measure:

$$\Phi(x, E) = \int_E U(y-x)\, \nu(dy).$$

Our method will be to obtain bounds on the kernel, the capacity of rectangular regions, and thereby upon hitting probabilities of rectangles. Some of the results are interesting in themselves, but since they will be used to prove later theorems, they will be stated as lemmas.

Lemma 3.1. *Let $y=(y_1, y_2)$ and $U(y)$ be the kernal as defined by* (3.1). *Then positive constants $c_1, \dots, c_4$, independent of y, can be found such that:*

$$\frac{c_1}{|y_2|^{1-\alpha_2+\alpha_2/\alpha_1}} \leqq U(y) \leqq \frac{c_2}{|y_2|^{1-\alpha_2+\alpha_2/\alpha_1}} \quad \text{if } |y_1|^{\alpha_1} \leqq |y_2|^{\alpha_2}$$

$$\frac{c_3}{|y_1|^{1-\alpha_1+\alpha_1/\alpha_2}} \leqq U(y) \leqq \frac{c_4}{|y_1|^{1-\alpha_1+\alpha_1/\alpha_2}} \quad \text{if } |y_2|^{\alpha_2} \leqq |y_1|^{\alpha_1}.$$

Before proceeding to the proof, observe that once the lemma is established, the two inequalities on the right can be made to hold in the entire plane by using $|y_2|^{-1} \leqq |y_1|^{-\alpha_1/\alpha_2}$ in the first inequality and $|y_1|^{-1} \leqq |y_2|^{-\alpha_2/\alpha_1}$ in the second.

Proof. The lower bound estimates are derived (p. 265) in [6], and the upper bounds are proven (Lemma 3.1) in [7].

Lemma 3.2. *Let $R_{a,b}$ be a rectangle centered at* (0, 0) *with sides of length $2a$ ($2b$) parallel to the y_1 (y_2) axis. For positive $\delta \leqq 1$ denote by $R(\delta)$ the resulting rectangle when $a=\delta^{\alpha_2}$ and $b=\delta^{\alpha_1}$, and denote by $S(\delta)$ the square which has $a=b=\delta$. Then positive constants c_5 and c_6 whose values do not depend upon a or b can be found such that:*

$$c_5 \max\{a^{1-\alpha_1+\alpha_1/\alpha_2}, b^{1-\alpha_2+\alpha_2/\alpha_1}\} \leqq C(R_{a,b}) \leqq c_6 \max\{a^{1-\alpha_1+\alpha_1/\alpha_2}, b^{1-\alpha_2+\alpha_2/\alpha_1}\}.$$

In particular we have bounds of the form

$$\delta^{\alpha_1+\alpha_2-\alpha_1\alpha_2} \quad and \quad \delta^{1-\alpha_2+\alpha_2/\alpha_1}$$

for $C(R(\delta))$ and $C(S(\delta))$ respectively.

Proof. As in the proofs of Lemmas 3.1 and 3.3 of [6] we have:

$$1 \geqq C(R_{a,b}) \min_{y \in R_{a,b}} U(y),$$

so that

$$C(R_{a,b}) \leqq \{\min_{y \in R_{a,b}} U(y)\}^{-1} = \max\{c_1^{-1} b^{1-\alpha_2+\alpha_2/\alpha_1}, c_3^{-1} a^{1-\alpha_1+\alpha_1/\alpha_2}\}$$

upon application of Lemma 3.1. To obtain the lower bound, let μ_L denote Lebesgue measure in R^2 and observe that:

$$\int_{R_{a,b}} U(y-x)\,\mu_L(dy) \leqq c_2 \int_{-b}^{b} \int_{-a}^{a} |y_2-x_2|^{\alpha_2-1-\alpha_2/\alpha_1}\, dy_1\, dy_2$$

$$= 2\,a\,c_2 \int_{-b-x_2}^{b-x_2} |u|^{\alpha_2-1-\alpha_2/\alpha_1}\, du = c\,a\,b^{\alpha_2-\alpha_2/\alpha_1},$$

where c is a positive constant independent of a, b and x. The last integration can be justified by using the c_r inequality (p. 155 of [9]) in the cases $0<-b-x_2$ and $b-x_2<0$, and the fact that $\alpha_1>1$ for the case $|x_2|\leqq b$. Now define the measure μ^* on Borel subsets E of $R_{a,b}$ by:

$$\mu^*(E) = c^{-1}\,a^{-1}\,b^{-\alpha_2+\alpha_2/\alpha_1}\cdot \mu_L(E).$$

Then $\int_{R_{a,b}} U(y-x)\,\mu^*(dy) \leqq 1$ for all x and:

$$C(R_{a,b}) \geqq \mu^*(R_{a,b}) = c^{-1}\,b^{1-\alpha_2+\alpha_2/\alpha_1}.$$

In the same manner we can use the fact that $\alpha_2>1$ to show that

$$C(R_{a,b}) \geqq c^{-1}\,a^{1-\alpha_1+\alpha_1/\alpha_2}.$$

Our next lemma gives estimates on the delayed hitting probabilities of the rectangles $R_{a,b}$. For rectangles $R_{a,b}$ as defined in Lemma 3.2, and for $T>0$ and all $x\in R^2$ let

$$Q(x, R_{a,b}, T) = P^x[X(t)\in R_{a,b} \text{ for some } t\geqq T].$$

We can then proceed as in the proof of Lemma 3.2 of [6] to establish

Lemma 3.3. *Let $Q(x, R_{a,b}, T)$ be as given above. Then positive constants c_7 and c_8, independent of x, a, b, and T, can be found such that:*

(i) $Q(x, R_{a,b}, T) \leqq c_7\, C(R_{a,b})\, T^{1-1/\alpha_1-1/\alpha_2}$,

(ii) $Q(x, R_{a,b}, T) \geqq c_8\, C(R_{a,b})\, T^{1-1/\alpha_1-1/\alpha_2}$ if $x\in R_{a,b}$ and $T\geqq \max\{(2a)^{\alpha_1}, (2b)^{\alpha_2}\}$.

We can now use both parts of Lemma 3.3 to reason as in the proof of Lemma 4.2 of [6] to obtain

Lemma 3.4. *Choose $T_1 \geqq \max\{(2a)^{\alpha_1}, (2b)^{\alpha_2}\}$ and $x\in R_{a,b}$. Then positive constants $c_9>1$ and c_{10}, independent of a, b and x, can be found such that when $T_2\geqq c_9 T_1$:*

$$P^x[X(t)\in R_{a,b} \textit{ for some } t\in[T_1, T_2]] \geqq c_{10}\, C(R_{a,b})\, T_1^{1-1/\alpha_1-1/\alpha_2}.$$

Various estimates of hitting probabilities of rectangles can be developed depending upon the location of the rectangle and starting point, but we only require

Lemma 3.5. *Suppose that $t\notin R_{a,b}$ and that $|x|+d<1$, where d is the length of the diagonal of $R_{a,b}$. Then positive constants c_{11}, c_{12} and c_{13} (independent of a, b and x) can be found such that:*

(i) $P^x[X(t)\in R_{a,b}$ *for some* $t>0] \geqq c_{11}\, C(R_{a,b})\cdot(|x|+d)^{\alpha_2-1-\alpha_2/\alpha_1}$,

(ii) $P^x[X(t)\in R_{a,b}$ *for some* t *in* $[0, T]] \geqq c_{12}\, C(R_{a,b})\cdot(|x|+d)^{\alpha_2-1-\alpha_2/\alpha_1}$ *whenever* $T\geqq c_{13}(|x|+d)^{\alpha_2}$.

Proof. From Lemma 3.1 we obtain the estimate $U(y) \geqq c_1 |y|^{\alpha_2 - 1 - \alpha_2/\alpha_1}$ if $|y| \leqq 1$. If ν is the capacitory measure on $R_{a,b}$, (i) is a consequence of:

$$\Phi(x, R_{a,b}) = \int_{R_{a,b}} U(y-x)\,\nu(dy) \geqq c_1 \int_{R_{a,b}} (|x|+d)^{\alpha_2-1-\alpha_2/\alpha_1}\,\nu(dy).$$

To prove (ii) we take the difference of the estimates (i) in Lemmas 3.5 and 3.3 and choose c_{13} appropriately.

Section 4: Existence Theorems

We now determine conditions for the existence of k-multiple points for $X(t)$. For any positive integer k a point x in R^2 is said to be a k-multiple point of the sample path $X(t, \omega)$ if there are k distinct times $0 < t_1 < \cdots < t_k$ such that $X(t_i, \omega) = x$ for $i = 1, 2, \ldots, k$. In [12], Taylor showed that symmetric stable processes $X_{\alpha,n}(t)$ of index $\alpha < n$ have (with probability one) k-multiple points if $k(n-\alpha) < n$ and that (with probability one) such points do not exist if $k(n-\alpha) \geqq n$. In Theorems 1 and 2 below we shall see fairly direct proofs for existence and nonexistence, respectively, of k-multiple points except for the critical case $k(\alpha_1 + \alpha_2 - \alpha_1\alpha_2) = \alpha_1 + \alpha_2$; we conclude with a remark as to how the critical case could be handled.

Theorem 1. *$X(t)$ has (with probability one) k-multiple points if* $k(\alpha_1 + \alpha_2 - \alpha_1\alpha_2) < \alpha_1 + \alpha_2$.

Proof. We could follow Taylor's (Section 4 of [12]) argument to prove existence, but we postpone the elaborate argument required and state that existence of k-multiple points is assured under our hypothesis once we show that this same hypothesis implies that the set of k-multiple points has positive Hausdorff dimension. This is done in Theorem 4 of the next section.

Theorem 2. *With probability one, k-multiple points do not exist if* $k(\alpha_1 + \alpha_2 - \alpha_1\alpha_2) > \alpha_1 + \alpha_2$.

Proof. Cover R^2 with abutting rectangles R_i, $i = 1, 2, 3, \ldots$, congruent to $R(\delta)$ of Lemma 3.2 and whose long sides are parallel to the horizontal axis. We shall estimate how many of these rectangles are hit by time $t = 2$, and then show that the probability of $(k-1)$ delayed returns to at least one such rectangle approaches zero as $\delta \to 0$.

The first step is to let $T(\delta, s)$ denote the amount of time the process spends in the rectangle $R(\delta)$ up to time s. $T(\delta, s)$ is a random variable whose expectation we estimate by the methods of Pruitt and Taylor [10] (p. 278). Let $s \geqq \delta^{\alpha_1\alpha_2}$. Then:

$$\begin{aligned} E[T(\delta, s)] &\equiv \int_0^s P[X(t) \in R(\delta)]\,dt \\ &= \int_0^s P[|X_1(1)| \leqq \delta^{\alpha_2} t^{-1/\alpha_1}]\,P[|X_2(1)| \leqq \delta^{\alpha_1} t^{-1/\alpha_2}]\,dt \qquad (4.1) \\ &\geqq \int_0^{\delta^{\alpha_1\alpha_2}} P[|X_1(1)| \leqq 1]\,P[|X_2(1)| \leqq 1]\,dt = c_{14}\,\delta^{\alpha_1\alpha_2}. \end{aligned}$$

If we apply the same reasoning of Pruitt and Taylor [10] in their Lemma 6.1 we can estimate $E[M(\delta, s)]$, where $M(\delta, s)$ denotes the number of rectangles from

$\{R_i\}_{i=1}^{\infty}$ that the process enters by time s. Their methods, along with (4.1) above, give (for $\delta \leqq 1$ and $s \geqq \delta^{\alpha_1 \alpha_2}$):

$$E[M(\delta, s)] \leqq 2 c_{15}\, s \{E[T(\delta/3, s)]\}^{-1} \leqq c_{16}\, s\, \delta^{-\alpha_1 \alpha_2}. \tag{4.2}$$

Define events A_i and B_i (independent) by: $A_i = [X(t)$ enters R_i at some time $t \in [1, 2]]$, $B_i = [X(t)$ enters R_i at $(k-1)$ times: $t_1 < \cdots < t_{k-1}]$, where $t_j \in [2j+1, 2j+2]$, $j = 1, 2, \ldots, k-1$. Let I_i be the indicator function of event A_i. Then: $P[k-1$ returns (during the specified time intervals) to at least one of the $R_i]$

$$\begin{aligned}
&= \sum_i P[A_i \cap B_i] = \sum_i P[A_i]\, P[B_i] \\
&\leqq c\, \delta^{(\alpha_1 + \alpha_2 - \alpha_1 \alpha_2)(k-1)} \sum_i E[I_i] \quad \text{by Lemmas 3.2 and 3.3} \\
&\leqq c\, \delta^{(\alpha_1 + \alpha_2 - \alpha_1 \alpha_2)(k-1)}\, E[M(\delta, 2)] \\
&\leqq c\, \delta^{(\alpha_1 + \alpha_2 - \alpha_1 \alpha_2)(k-1) - \alpha_1 \alpha_2} \quad \text{by (4.2).}
\end{aligned}$$

The hypothesis $k(\alpha_1 + \alpha_2 - \alpha_1 \alpha_2) > \alpha_1 + \alpha_2$ guarantees that this final quantity approaches zero as $\delta \to 0$. Consequently, the event

$$[X(t_j) = X(t) \text{ for some } t \in [1, 2] \text{ and } t_j \in [2j+1, 2j+2],\ j = 1, 2, \ldots, k-1]$$

has probability zero. Now apply the same reasoning to any sequence

$$0 < r_1 < r_2 < \cdots < r_{2k} \leqq 1$$

of $2k$ distinct rational numbers r_j in $[0, 1]$, to conclude that for such a sequence with probability one there are not k time instants $t_1 < \cdots < t_k$ for which

$$t_j \in [r_{2j-1}, r_{2j}], \quad 1 \leqq j \leqq k$$

and:

$$X(t_1, \omega) = X(t_2, \omega) = \cdots = X(t_k, \omega).$$

Since the rationals in $[0, 1]$ are countable, so is the collection $\{K_\alpha\}$ of all sets of $2k$ distinct rationals taken from $[0, 1]$. Thus the K's can be indexed $\{K_\alpha\} = \{K_i\}_{i=1}^{\infty}$. Define the event C_i, $i = 1, 2, 3, \ldots$ by:

$$C_i = [X(t_1) = \cdots = X(t_k);\ t_j \in [r_{2j-1}, r_{2j}];\ K_i = \{r_1, \ldots, r_{2k}\}].$$

Then $\bigcup_{i=1}^{\infty} C_i = [X(t_1) = \cdots = X(t_k)$ for k distinct times in $[0, 1]]$ has probability zero since $P[C_i] = 0$ for each i. Apply the same reasoning to any finite time interval $[0, t]$ to complete the nonexistence proof.

Remark. Our theorems have not taken care of the criterical case

$$k(\alpha_1 + \alpha_2 - \alpha_1 \alpha_2) = \alpha_1 + \alpha_2.$$

It is possible to show non-existence in this case too, but the details are tedious. The methods used by Taylor [12] to settle the critical case for a stable process need to be modified by considering a new type of Hausdorff measure in which only coverings by long thin rectangles of the form used in the proof of Theorem 2 are allowed.

Section 5: Dimension Theorems

In this section we establish bounds for the Hausdorff dimension of the set of multiple points. We use the terminology $L(a,b;\omega)$, $L^k(a,b;\omega)$ and $E_k(\omega)$ of Section 2 and assume throughout this section that $k(\alpha_1+\alpha_2-\alpha_1\alpha_2)<\alpha_1+\alpha_2$. In Theorem 3 we use methods similar to those of Theorem 2 to establish upper bounds for $\dim E_k$, the Hausdorff dimension of E_k, and in Theorem 4 we follow Taylor's [12] argument for the lower bound.

Theorem 3. *Under the above terminology and hypotheses, with probability one* $\dim E_k(\omega)$ *satisfies:*

$$\dim E_k(\omega)\leqq\min\begin{cases}2-k(\alpha_1+\alpha_2-\alpha_1\alpha_2)/\alpha_1 & (=\lambda)\\ 1+\alpha_1/\alpha_2-k(\alpha_1+\alpha_2-\alpha_1\alpha_2)/\alpha_2 & (=\eta).\end{cases}$$

Proof. We first show that $\dim E_k\leqq\eta$ by showing that $\dim E_k\leqq\theta$ whenever $\theta>\eta$. Let such a θ be chosen and cover R^2 with abutting rectangles $\{R_i\}_{i=1}^{\infty}$ congruent to $R(\delta)$ as in the proof of Theorem 2. $d_{i,\delta}$, the diagonal of any such rectangle R_i, is less than $3\delta^{\alpha_2}$ for all i. Define events A_i and B_i as before, so that

$$Q_k(\omega)\equiv\bigcap_{j=1}^{k}L(2j-1,2j;\omega)\subset\bigcup_i{}^{*}R_i,$$

where the union is taken over those indices i for which the event $A_i\cap B_i$ occurs. We now obtain an estimate of $E\left[\sum_{i=1}^{\infty}{}^{*}d_{i,\delta}^{\theta}(\omega)\right]$, the summation extending over the indices in the above union. Let $\mathscr{F}_2$ denote the σ-field of the process up to time $t=2$ and reason exactly as in Theorem 2:

$$\begin{aligned}E\left[\sum_{i=1}^{\infty}{}^{*}d_{i,\delta}^{\theta}(\omega)\right]&=\sum_{i=1}^{\infty}E\left[d_{i,\delta}^{\theta}I_{A_i}\cdot I_{B_i}\right]\leqq c\,\delta^{\alpha_2\theta}\sum_{i=1}^{\infty}E\left[I_{A_i}E\left[I_{B_i}|\mathscr{F}_2\right]\right]\\&\leqq c\,\delta^{\alpha_2\theta+(k-1)(\alpha_1+\alpha_2-\alpha_1\alpha_2)}\cdot E\left[M(\delta,2)\right]\leqq c\,\delta^{\alpha_2(\theta-\eta)}.\end{aligned}$$

This final quantity approaches zero as $\delta\to 0$ whenever $\theta>\eta$. Consequently, with probability one there is a sequence of integers $\{j_n(\omega)\}_{n=1}^{\infty}$ such that $\sum_{i=1}^{\infty}d_{i,1/j_n}^{\theta}\to 0$ as $n\to\infty$. Hence (with probability one) the θ-Hausdorff measure of $Q_k(\omega)$ is zero. Similarly, the set $\bigcap_{j=1}^{k}L(r_{2j-1},r_{2j};\omega)$ has zero θ-Hausdorff measure for any sequence of $2k$ distinct rational r_i, $0<r_1<r_2<\cdots<r_{2k}\leqq 1$. Any point x in $L^k(0,1;\omega)$ is contained in a set of this form. Since the number of such sets is countable, $L^k(0,1;\omega)$ has zero θ-Hausdorff measure with probability one. The same proof applies for any finite time interval $[0,t]$. Consequently $\dim E_k\leqq\eta$.

The proof that $\dim E_k\leqq\lambda$ is very similar to the above proof, so we only outline it. This time we cover R^2 with abutting squares $\{R_i\}_{i=1}^{\infty}$ of side $\delta\leqq 1$. Define A_i and B_i in analogous fashion. The corollary to Lemma 6.1 of Pruitt and Taylor [10] tells us that $E[M(\delta, 2)] \leqq c\delta^{-\rho}$, where $\rho = 1 + \alpha_2 - \alpha_2/\alpha_1$. We then obtain (for $\theta > \lambda$):

$$E\left[\sum_{i=1}^{\infty}{}^{*} d_{i,\delta}^{\theta}(\omega)\right] \leqq c\delta^{\theta}\{C(S(\delta))\}^{k-1} E[M(\delta, 2)] \leqq c\delta^{\theta-\lambda}.$$

Reason as before to show that $\dim E_k \leqq \lambda$.

We conclude this section with an outline of the modifications of Taylor's [12] argument to establish

Theorem 4. *Under the same hypotheses as in Theorem 3, with probability one* $\dim E_k(\omega)$ *satisfies:*

$$\dim E_k(\omega) \geqq \min\begin{cases} 2 - k(\alpha_1 + \alpha_2 - \alpha_1\alpha_2)/\alpha_1 \\ 1 + \alpha_1/\alpha_2 - k(\alpha_1 + \alpha_2 - \alpha_1\alpha_2)/\alpha_2. \end{cases}$$

Proof. For ease of reference we formulate and prove a sequence of lemmas and indicate the correspondence of our lemmas with those of [12]. With the exception of the upper bound estimate of Taylor's Lemma 1 we now have (through our Lemmas 3.3–3.5) the counterparts of his Lemmas 1–5. We now handle this exception with

Lemma 5.1. *Let R_1 and R_2 be two rectangles (with diagonal $d < \frac{1}{2}$) which have the same orientation and shape as $R_{a,b}$, and whose respective centers $e_i = (c_i, d_i)$, $i = 1$ and 2, are distinct. If $c_1 \neq c_2$ ($d_1 \neq d_2$) assume that $|c_1 - c_2| \geqq 10a$ ($|d_1 - d_2| \geqq 10b$). Let $y \in R_2$. Then:*

$$P^y[X(t) \in R_1 \text{ for some } t > 0] \leqq c_{17}\, U(e_1 - e_2) \cdot C(R_{a,b})$$

for some $c_{17} > 0$ which is independent of a, b, e_1 and e_2.

Proof. Let ν be a capacitory measure on R_1. Then:

$$\Phi(y, R_1) = \int_{R_1} U(x-y)\, \nu(dx) \leqq C(R_1) \max_{p_i \in R_i} U(p_1 - p_2).$$

Now for any $p_i \in R_i$, $i = 1$ and 2, we can use the bounds for the kernel (Lemma 3.1) to conclude that:

$$U(p_1 - p_2) \leqq \min\{c_2\, U((0, \tfrac{4}{5}|d_1 - d_2|)),\ c_4\, U((\tfrac{4}{5}|c_1 - c_2|, 0))\} \leqq c_{17}\, U(e_1 - e_2).$$

Since the p_i are selected arbitrarily in the R_i, the proof is complete. $\Phi(y, R_2)$, $y \in R_1$, can be given the same bound as above once we observe (by Lemma 3.1) that $U(e_2 - e_1) \leqq c\, U(e_1 - e_2)$ for some finite and positive c independent of the e_i. This symmetrizing property of the kernel will be used as necessary without further comment.

Our next lemma corresponds to Taylor's Lemma 8. Put $M = \max\{c_9, c_{13}\} > 1$ (see Lemmas 3.4 and 3.5). This means that if $|x| + d < 1$, the estimates of Lemma 3.5(ii) for entry in $[0, M]$ and Lemma 3.4 for reentry in $[1, M]$ are valid in our next proof.

Lemma 5.2. *Let k be a positive integer, and R_1 and R_2 a pair of rectangles which satisfy the hypotheses of Lemma 5.1. Also let $\frac{1}{4} \leqq |e_i| \leqq \frac{1}{2}$, $i = 1$ and 2, and assume that $10d < \min_{i=1,2} |e_i|$. Denote by R_i^* ($i = 1$ and 2) the event that there are k time instants*

$t_1, \ldots, t_k$ *with* $0<t_1\leqq M$; $1\leqq t_j-t_{j-1}\leqq M$, $j=2,3,\ldots,k$ *such that:*

$$X(t_j)\in R_i, \quad j=1,2,\ldots,k.$$

Then positive constants c_{18} *and* c_{19}, *independent of the* R_i, *exist such that:*

(i) $P[R_i^*]\geqq c_{18}[C(R_i)]^k, \quad i=1,2,$

(ii) $P[R_1^*\cap R_2^*]\leqq c_{19}[C(R_i)]^{2k}[U(e_1-e_2)]^k.$

We remark at this stage that our proof of this lemma is close to Taylor's, although we correct a misprint in a combinatorial argument in the proof of (ii). In addition, we shall need to examine carefully the estimate given by μ_2 in several places later in our argument, so we provide the details of the proof.

Proof. A lower bound for the probability that $X(t)$ will hit R_i in $[0, M]$ is given by (ii) of Lemma 3.5. If the process hits R_i in $[0, M]$ let t_1 be the first entry time; t_1 is a stopping time. Restart the process at $X(t_1)$ and use the strong Markov property. The conditional probability of reentry into R_i in $[1, M]$ can be estimated by Lemma 3.4. Repeat the argument $(k-1)$ times to obtain (i).

To prove (ii) we define some probabilities and give estimates on them:

$$\mu_1=\Phi(0, R_1\cup R_2)$$
$$\mu_2=\max\{\sup_{y\in R_1}\Phi(y, R_2);\ \sup_{y\in R_2}\Phi(y, R_1)\}$$
$$\mu_3=\sup_{\text{all } y}P^y[X(t)\in R_1\cup R_2 \text{ for some } t\geqq\tfrac{1}{2}].$$

μ_1 is estimated by means of the bounds on $U(y)$, μ_2 is estimated by Lemma 5.1, and μ_3 by 3.3(i):

$$\mu_1\leqq\sum_{i=1}^{2}\int_{R_i}U(y-0)\,v(dy)\leqq\sum_{i=1}^{2}C(R_i)\max_{y\in R_i}U(y)\leqq c\,C(R_i), \quad i=1,2.$$
$$\mu_2\leqq c_{17}\,U(e_1-e_2)\cdot C(R_i), \quad i=1,2.$$
$$\mu_3\leqq 4c_7\,C(R_i), \quad i=1,2.$$

Now let $\omega\in R_1^*\cap R_2^*$, and N_k denote the number of ways of selecting k integers out of $2k$. There must be at least $2k$ distinct times $t_1<\cdots<t_{2k}$ for which $X(t_j,\omega)\in R_1\cup R_2$. We may assume that the times corresponding to any two hits of a given rectangle differ by at least one time unit. Thus, for $j=2,3,\ldots,2k-1$ we must have $t_{j+1}-t_j\geqq\frac{1}{2}$ or $t_j-t_{j-1}\geqq\frac{1}{2}$. This implies that there must be at least $(k-1)$ (Taylor writes k here instead of $(k-1)$, though he later corrects himself) integers j for which $t_j-t_{j-1}\geqq\frac{1}{2}$.

The t_i are stopping times; hence the product of probabilities for any one of the N_k sequences of times must contain at least $(k-1)$ factors of $2\mu_3$ (the 2 accounts for the fact that each factor may be inserted in two different ways); the first factor is always μ_1; for each of the k remaining t_i-t_{i-1} there is a factor of at most $\mu_2+\mu_3$. Therefore:

$$P[R_1^*\cap R_2^*]\leqq N_k\,\mu_1(2\mu_3)^{k-1}(\mu_2+\mu_3)^k, \tag{5.1}$$

from which (ii) follows by use of the estimates upon the μ's and the fact that $\mu_2+\mu_3\leqq c\,C(R_i)\,U(e_1-e_2)$ for $i=1,2$.

Before stating the next lemma, we introduce some terminology. We will be using rectangular grids of $r_1 r_2$ points formed by the points of intersection of r_1 equally spaced horizontal lines and r_2 equally spaced vertical lines. The points $\{x_i\}$ of such a grid will be said to be numbered by columns starting from the lower left if starting with the left-hand column we number consecutively upward from the bottom and proceed column by column to the right. Thus, the lower left grid point is x_1 and the upper right is $x_{r_1 r_2}$. For convenience of notation in the discussion which follows, we regard, for $\zeta>0$ and integral r, r^ζ as $[r^\zeta]$, where $[\,]$ denotes the greatest integer function, whenever such an expression must be integral valued.

Next, let r be a positive integer. Choose a positive number $L<1$ so that a rectangle R with sides L^{α_1} parallel to the y_2 axis and of length L^{α_2} parallel to y_1 can be placed somewhere in the annulus $A=\{y: \frac{1}{4}\leqq|y|\leqq\frac{1}{2}\}$. Form the grid G of $r^{\alpha_1} r^{\alpha_2}$ points by locating r^{α_j} equally spaced points along the sides of R which are of length L^{α_j}, $j=1$ and 2, and connecting pairs of opposite points by horizontal or vertical lines. Number the points of $G=\{x_v: 1\leqq v\leqq r^{\alpha_1} r^{\alpha_2}\}$ by columns starting from the lower left and let rectangles R_v congruent to $R(\delta)$ have their centers at x_v, $1\leqq v\leqq r^{\alpha_1} r^{\alpha_2}$. Assume that the rectangles satisfy the spacing requirements:

$$10\,\delta^{\alpha_i}\leqq(L/r)^{\alpha_i}, \qquad i=1,2.$$

Thus, the sides of the rectangles are smaller by a factor of at least 5 than the horizontal and vertical spacing between their centers. Such a grid of rectangles will be said to be spaced according to α_1 and α_2. Observe that any pair of rectangles from such a grid satisfy the conditions of Lemma 5.2 for all large r.

Our technique for finding a lower bound for dim E_k will involve finding those γ for which an independent symmetric stable process $X_{\gamma,2}(t)$ of index γ in R^2 hits the set E_k and then using the following result of Taylor [12]:

Proposition 5.1. *Let A be an analytic set in R^2. Then for any point x,*

$$\dim A=2-\inf\{\gamma: \Phi_{\gamma,2}(x,A)>0\}$$

where $\Phi_{\gamma,2}(x,A)$ denotes the probability that a symmetric stable process of index γ in R^2 starting at x will hit A.

For processes of the type given in this proposition we can use the results of [13] (pp. 1235 and 1237) and of our own Lemma 3.2 (when $\alpha_1=\alpha_2=\gamma>1$) to estimate the kernel, $U_\gamma(\,)$, and capacity, $C_\gamma(R_{a,b})$, for various $R_{a,b}$ with respect to $X_{\gamma,2}$:

$$c_5\,\delta^{\alpha_2(2-\gamma)}\leqq C_\gamma(R(\delta))\leqq c_6\,\delta^{\alpha_2(2-\gamma)} \quad \text{when } \gamma>1,$$
$$C_\gamma(S(\delta))=c_{20}\,\delta^{2-\gamma} \quad\text{and}\quad U_\gamma(x)=c_{21}\,|x|^{\gamma-2},$$

for positive constants c_{20} and c_{21} independent of x and δ.

Our final two Lemmas, 5.3 and 5.4, are the counterparts of Taylor's Lemma 15 in the respective cases when $k(\alpha_1+\alpha_2-\alpha_1\alpha_2)$ lies in the interval $[\alpha_1,\alpha_1+\alpha_2)$ and when it is in the interval $(0,\alpha_1)$. Once these lemmas are established the proof of our Theorem 4 can be completed according to the methods of Taylor's paper and we shall know the value of dim E_k.

Lemma 5.3. *Let r be a large positive integer, assume that $k(\alpha_1+\alpha_2-\alpha_1\alpha_2)$ lies in the interval $[\alpha_1, \alpha_1+\alpha_2)$ and suppose that for some $\varepsilon>0$: (ε quite small)*

$$2-\gamma=1+\alpha_1/\alpha_2-k(\alpha_1+\alpha_2-\alpha_1\alpha_2)/\alpha_2-\varepsilon \quad (<1).$$

Form a grid of $r^{\alpha_1}r^{\alpha_2}$ rectangles congruent to $R(\delta)$ spaced according to α_1 and α_2. Let $\delta=c_{22}r^{-\mu}$, where

$$\mu=(\alpha_1+\alpha_2)/[k(\alpha_1+\alpha_2-\alpha_1\alpha_2)+\alpha_2(2-\gamma)] \quad (>1)$$

and $c_{22}>0$ will be chosen later.

Let E_ν, $1\leqq\nu\leqq r^{\alpha_1}r^{\alpha_2}$ be the event that there are time instants $0<t_1\leqq N$, $1\leqq t_j-t_{j-1}\leqq N$ $(2\leqq j\leqq k)$, and $0<t_{k+1}\leqq N$ such that:

$$X_{\gamma,2}(t_{k+1},\omega)\in R_\nu \quad \text{and} \quad X(t_j,\omega)\in R_\nu \quad (1\leqq j\leqq k).$$

Then c_{22} and c_{23}, positive and independent of r, can be chosen such that

$$P\left[\bigcup_{\nu=1}^{r^{\alpha_1}r^{\alpha_2}} E_\nu\right]\geqq c_{23} \quad \text{for all large } r.$$

(In the above, $X_{\gamma,2}$ and X are assumed independent and defined on the same basic probability space, and N is chosen large enough to ensure that the required estimates are valid for each process.)

Proof. We first make four observations:

(i) $\mu>1$, so that the $R(\delta)$ satisfy the spacing requirements:

$$10\delta^{\alpha_i}=10(c_{22}r^{-\mu})^{\alpha_i}\leqq(L/r)^{\alpha_i}, \quad i=1 \text{ and } 2.$$

(ii) The estimates for the capacity $C(R(\delta))$ of the R_ν for the (X_1, X_2) process and $C_\gamma(R(\delta))$ for the $X_{\gamma,2}$ process give:

$$\{C(R(\delta))\}^k C_\gamma(R(\delta))\sim r^{-(\alpha_1+\alpha_2)}.$$

(iii) If $\theta<1$ and r is a large positive integer, a positive constant c_{24} (independent of r) exists such that:

$$\sum_{j=1}^{r}(1/j)^\theta\leqq c_{24}r^{1-\theta}.$$

(iv) $P\left[\bigcup_{\nu=1}^{r^{\alpha_1}r^{\alpha_2}} E_\nu\right]\geqq r^{\alpha_1+\alpha_2}P[E_1]-\sum^* P[E_\nu\cap E_{\nu'}]-\sum^{**} P[E_\nu\cap E_{\nu'}]$, where $\sum^{**}$ is taken over distinct ν and ν' for which the corresponding grid points x_ν and $x_{\nu'}$ lie on the same vertical line, and Σ^* is taken over all remaining pairs of distinct ν and ν'. Denote these sums by S^{**} and S^* respectively.

Since the rectangles R_ν satisfy the conditions of Lemma 5.2 we can use both parts of that lemma and (ii) above to write:

$$r^{\alpha_1+\alpha_2}P[E_1]-S^*\geqq c_{25}-c_{26}r^{-2(\alpha_1+\alpha_2)}\sum^*\{U(x_\nu-x_{\nu'})\}^k U_\gamma(x_\nu-x_{\nu'}) \quad (5.2)$$

where the summation is over the pairs of distinct x_ν and $x_{\nu'}$ which are not on the same vertical line. Moreover,

$$S^{**}\leqq c_{27}[C(R(\delta))]^{2k} C_\gamma(R(\delta))\sum^{**}\{U(x_\nu-x_{\nu'})\}^k \quad (5.3)$$

where the summation is over the x_ν and $x_{\nu'}$ which lie on the same vertical line. Thus, in (5.3) we are bounding μ_2 (of Lemma 5.2) above by 1 so that the estimate of (5.1) for $X_{\gamma,2}$ to hit R_1 and R_2 each once is simply $c\,C_\gamma(R(\delta))$. Failure to do this can result in a μ_2 estimate (for the $X_{\gamma,2}$ process) which exceeds one. Our objective now is twofold:

(a) To show that the summation (with we henceforth denote by $\sum^*$) in (5.2) is bounded above by $c_{28}\,r^{2(\alpha_1+\alpha_2)}$ for some $c_{28}>0$ which does not depend upon r and to then select c_{22} (in the formation of $\delta=c_{22}\,r^{-\mu}$) so that the desired constant c_{23} can be shown to exist.

(b) To show that the bound upon S^{**} goes to zero as $r\to+\infty$.

Objective (b) is more easily accomplished than (a), so we do (b) first when $k(\,)\equiv k(\alpha_1+\alpha_2-\alpha_1\alpha_2)\in(\alpha_1,\alpha_1+\alpha_2)$ and then state how the case $k(\,)=\alpha_1$ can be handled. Under the first hypothesis we have:

$$\begin{aligned} S^{**} &\leqq c_{27}\left[C(R(\delta))\right]^{2k} C_\gamma(R(\delta))\, r^{\alpha_2} \sum_{j=1}^{r^{\alpha_1}-1} \sum_{i=1}^{r^{\alpha_1}-j} \{U((0, i(L/r)^{\alpha_1}))\} \\ &\leqq c\, r^{-(\alpha_1+\alpha_2)}\, r^{(1-\mu)k(\,)}\, r^{\alpha_2} \sum_{j=1}^{r^{\alpha_1}-1} \sum_{i=1}^{r^{\alpha_1}-j} i^{-k(\,)/\alpha_1} \\ &\leqq c\, r^{(1-\mu)k(\,)} \to 0 \quad \text{as } r\to\infty \text{ since } \mu>1. \end{aligned}$$

When $k(\,)=\alpha_1$ the summation on i diverges (as $r\to\infty$) to give a $\ell n(r^{\alpha_1}-j)$ term and Stirling's formula can be used on the resulting $\ell n(r^{\alpha_1}-1)!$ factor to complete the proof of (b).

We now consider the $\sum^*$ factor of (5.2):

$$\sum{}^* \leqq 2\, r^{\alpha_1+\alpha_2} \sum_{\nu>r^{\alpha_1}} \{U(x_\nu-x_1)\}^k\, U_\gamma(x_\nu-x_1).$$

Hence we can regard the grid G as being repositioned with its sides along the coordinate axes, x_1 at (0, 0) and $x_{r^{\alpha_1+\alpha_2}}$ at $(L^{\alpha_2}, L^{\alpha_1})$, and that we are required to sum $\{U(z_\nu)\}^k\, U_\gamma(z_\nu)$ over those ν such that $Z_\nu\equiv x_\nu-x_1$ does not lie on the vertical (y_2) coordinate axis. In the first quadrant of the (y_1, y_2) plane the line $y_2=L^{\alpha_1-\alpha_2}\,y_1$ and the curve $y_2^{\alpha_2}=y_1^{\alpha_1}$ intersect at (0, 0) and $(L^{\alpha_2}, L^{\alpha_1})$ and divide the rectangle which forms the repositioned grid into three regions which we number (from bottom to top) by I, III and II respectively. Our sums over these regions will be denoted by $S_{\mathrm{I}}, S_{\mathrm{III}}, S_{\mathrm{II}}$ and we will be finished once we show each sum bounded above by $c\,r^{\alpha_1+\alpha_2}$. We now do this.

Region I contains the y_1 coordinate axis, and along the vertical line through $(j(L/r)^{\alpha_2}, 0)$ there are at most $(j^{\alpha_1/\alpha_2}+1)$ grid points which lie in I, $j=1, 2, \ldots, r^{\alpha_2}$. If we estimate $U_\gamma(y_1, y_2)$ by $U_\gamma(y_1, 0)$ in I we then have (we suppress the argument of U_γ when it is the same as that of U):

$$\begin{aligned} S_{\mathrm{I}} &\leqq 2\sum_{j=1}^{r^{\alpha_2}} j^{\alpha_1/\alpha_2}\{U(j(L/r)^{\alpha_2}, 0)\}^k\, U_\gamma(\;,\;) \\ &\leqq c\, r^{k(\,)+\alpha_2(2-\gamma)} \sum_{j=1}^{r^{\alpha_2}} (1/j)^{k(\,)/\alpha_2+(2-\gamma)-\alpha_1/\alpha_2} \leqq c\, r^{\alpha_1+\alpha_2} \end{aligned}$$

by using the defining relationship for $(2-\gamma)$ to justify use of observation (iii).

Region II contains the y_2 axis, and along the horizontal line through $(0, j(L/r)^{\alpha_1})$ there are at most $j\,r^{\alpha_2-\alpha_1}$ grid points which lie in II, $r^{\alpha_1-\alpha_2} \leqq j \leqq r^{\alpha_1}$. To see this, determine the points of intersection of the grid lines with the line $y_2 = L^{\alpha_1-\alpha_2} y_1$. If we estimate $U_\gamma(y_1, y_2)$ by $U_\gamma(0, y_2)$ in II we then have:

$$S_{\mathrm{II}} \leqq \sum_{j=r^{\alpha_1-\alpha_2}}^{r^{\alpha_1}} j\,r^{\alpha_2-\alpha_1} \{U(0, j(L/r)^{\alpha_1})\}^k\, U_\gamma(\;,\;)$$

$$\leqq c\, r^{\alpha_2-\alpha_1+k(\,)+\alpha_1(2-\gamma)} \sum_{j=1}^{r^{\alpha_1}} (1/j)^{k(\,)/\alpha_1+(2-\gamma)-1} \leqq c\, r^{\alpha_1+\alpha_2}$$

by reasoning as before and using the fact that $k(\,) \geqq \alpha_1$.

Region III is slightly more complicated. Along horizontal lines in III we estimate $U(y_1, y_2)$ by $U(0, y_2)$ and $U_\gamma(y_1, y_2)$ by $U_\gamma(y_1, 0)$. Hence

$$S_{\mathrm{III}} \leqq \sum_{j=1}^{r^{\alpha_1}} \sum_{i=1}^{j^{\alpha_2/\alpha_1}} \{U(i(L/r)^{\alpha_2}, j(L/r)^{\alpha_1})\}^k\, U_\gamma(\;,\;)$$

$$\leqq r^{k(\,)+\alpha_2(2-\gamma)} \sum_{j=1}^{r^{\alpha_1}} (1/j)^{k(\,)/\alpha_1} \sum_{i=1}^{j^{\alpha_2/\alpha_1}} (1/i)^{2-\gamma} \leqq c\, r^{\alpha_1+\alpha_2}$$

by noting that $(2-\gamma) < 1$ and applying observation (iii) to both summations.

Finally, note that a positive power of the constant c_{22} used to define δ enters as a factor of the c_{25} in (5.2) and as a squared factor in c_{26}. Hence the derived c_{23} can be found and Lemma 5.3 is proven. The conclusion of Theorem 4 when $k(\,) \in [\alpha_1, \alpha_1+\alpha_2)$ can now be established by using Taylor's argument (Section 7 of [12]) and noting the relationship between the two functions $f_1(x) = 2 - x/\alpha_1$ and $f_2(x) = 1 + \alpha_1/\alpha_2 - x/\alpha_2$ $0 < x < \alpha_1 + \alpha_2$ on the interval $[\alpha_1, \alpha_1+\alpha_2)$. We now complete the proof of Theorem 4 by handling the case $k(\,) \in (0, \alpha_1)$ in our final lemma.

Lemma 5.4. *Let r be a large positive integer, assume that $k(\,)$ lies in the interval $(0, \alpha_1)$ and suppose that for some $\varepsilon > 0$: (ε quite small)*

$$2 - \gamma = 2 - k(\,)/\alpha_1 - \varepsilon \qquad (>1).$$

Form a grid of r^2 squares of side δ centered at the points of intersection of r equally spaced vertical lines along the L^{α_2} side and r equally spaced horizontal lines along the L^{α_1} side of a rectangle of the type used in Lemma 5.3. Number the centers of the squares by columns starting from the lower left and let $\delta = c_{29} r^{-\mu}$, where

$$\mu = 2\alpha_1 / [k(\,) + \alpha_1(2-\gamma)] \qquad (>1)$$

and $c_{29} > 0$ will be chosen later.

Let E_ν, $1 \leqq \nu \leqq r^2$ be the event that there are time instants $0 < t_1 \leqq N$,

$$1 \leqq t_j - t_{j-1} \leqq N \qquad (2 \leqq j \leqq k),$$

and $0 < t_{k+1} \leqq N$ such that

$$X_\gamma(t_{k+1}, \omega) \in R_\nu \quad \text{and} \quad X(t_j, \omega) \in R_\nu \qquad (1 \leqq j \leqq k).$$

Then c_{29} and c_{30}, positive and independent of r, can be chosen such that

$$P\left[\bigcup_{\nu=1}^{r^2} E_\nu\right] \geqq c_{30}$$

for all large r.

Proof. We again make some observations:

(i°) $\mu > 1$, so that the squares satisfy the requirements

$$10\delta = 10(c_{29} r^{-\mu}) \leqq L^{\alpha_i}/r, \qquad i = 1 \text{ and } 2.$$

(ii°) $[C(S(\delta))]^k C_\gamma(S(\delta)) \sim \delta^{k(\,)/\alpha_1 + (2-\gamma)}$.

(iii°) If $\theta > 1$ and r is a large positive integer, a positive constant c_{31} (independent of r) exists such that:

$$\sum_{j=r}^{\infty} (1/j)^\theta < c_{31} r^{1-\theta}.$$

(iv°) $P\left[\bigcup_{v=1}^{r^2} E_v\right] \geqq r^2 P[E_1] - \sum^{\mathrm{o}} P[E_v \cap E_{v'}] - \sum^{\mathrm{oo}} P[E_v \cap E_{v'}]$, where $\sum^{\mathrm{oo}}$ is taken over distinct v and v' for which the corresponding grid points x_v and $x_{v'}$ lie on the same horizontal line, and $\sum^{\mathrm{o}}$ is taken over all remaining pairs of distinct v and v'. Denote these sums by S^{oo} and S^{o} respectively.

Now argue as in (5.2) to establish

$$r^2 P[E_1] \geqq c_{32} - c_{33} r^{-2} \sum^{\mathrm{o}} \{U(x_v - x_{v'})\}^k U_\gamma(x_v - x_{v'}) \tag{5.2°}$$

where the summation is over distinct x_v and $x_{v'}$ which are not on the same horizontal line. Moreover,

$$S^{\mathrm{oo}} \leqq c_{34} [C(S(\delta))]^k [C_\gamma(S(\delta))]^2 \sum^{\mathrm{oo}} U_\gamma(x_v - x_{v'}) \tag{5.3°}$$

where the summation is over distinct x_v and $x_{v'}$ on the same horizontal line. This time we are bounding μ_2 above by 1 so that the estimate of (5.1) for $X(t)$ to hit R_1 and R_2 k times apiece is simply $c[C(S(\delta))]^k$. Again we have a twofold objective:

(a°) Show the summation (denoted by $\sum^{\mathrm{o}}$) in (5.2°) is bounded above by $c_{35} r^2$ and select c_{29} appropriately.

(b°) Show that the bound upon S^{oo} goes to zero as $r \to +\infty$.

First consider (b°). By (iii°) above and (5.3°) we have:

$$\begin{aligned} S^{\mathrm{oo}} &\leqq c(1/r)^{2+\mu(2-\gamma)} r \sum_{j=1}^{r-1} \sum_{i=1}^{r-j} U_\gamma((i L^{\alpha_2}/r, 0)) \\ &\leqq c\, r^{(2-\gamma)(1-\mu)} \to 0 \qquad \text{as } r \to +\infty \ \ (\text{since } \mu > 1 \text{ and } 2-\gamma > 1). \end{aligned}$$

We now consider the $\sum^{\mathrm{o}}$ factor of (5.2°):

$$\sum^{\mathrm{o}} \leqq 2r^2 \sum \{U(x_v - x_1)\}^k U_\gamma(x_v - x_1) \qquad (v \neq j r + 1,\ 0 \leqq j \leqq r-1).$$

Reposition the grid as before, form the three regions I, III and II, and consider the sum of $\{U(z_v)\}^k U_\gamma(z_v)$ for z_v lying in these regions (denoted by $S^{\mathrm{o}}_{\mathrm{I}}$, $S^{\mathrm{o}}_{\mathrm{III}}$ and $S^{\mathrm{o}}_{\mathrm{II}}$ respectively). Once we show that each of these sums is bounded above by $c\, r^2$, the lemma (and hence the theorem) will be established. In regions I and II we use observation (ii) of our previous lemma, while III requires (ii°) of the present lemma. To use (ii) in I we also require the condition $k(\,) < \alpha_1$. The desired bounds follow from:

$$S^{\mathrm{o}}_{\mathrm{I}} \leqq c\, r^{1-\alpha_1/\alpha_2} \sum_{j=1}^{r} j^{\alpha_1/\alpha_2} \{U((j L^{\alpha_2}/r, 0))\}^k U_\gamma(\,),$$

$$S^{\mathrm{o}}_{\mathrm{II}} \leqq c \sum_{j=1}^{r} j \{U((0, j L^{\alpha_1}/r))\}^k U_\gamma(\,),$$

and

$$
\begin{aligned}
S_{\mathrm{III}}^{\circ} &\leqq \sum_{j=1}^{r} \sum_{i=j}^{r} \{U((i\,L^{\alpha_2}/r, j\,L^{\alpha_1}/r))\}^k\, U_\gamma(\ ,\ ) \\
&\leqq r^{k(\ )/\alpha_1+(2-\gamma)} \sum_{j=1}^{r} (1/j)^{k(\ )/\alpha_1} \sum_{i=j}^{r} (1/i)^{2-\gamma} \\
&\leqq r^{k(\ )/\alpha_1+(2-\gamma)} \sum_{j=1}^{r} (1/j)^{k(\ )/\alpha_1+(2-\gamma)-1}.
\end{aligned}
$$

Section 6: Remarks

(i) In [4] we have recently shown that for fixed Borel sets $E \subset [0,1]$ and for $X(t)$ of the type being considered here we have (with probability one):

$$
\dim X(E) = \begin{cases} \alpha_1 \dim E & \text{for } 0 \leqq \dim E \leqq 1/\alpha_1 \\ 1+\alpha_2 \dim E - \alpha_2/\alpha_1 & \text{for } 1/\alpha_1 \leqq \dim E \leqq 1 \end{cases}
$$

where $X(E)$ denotes the set of points in R^2 hit at some time $t \in E$. Intuitively, we can think of using up the X_1 component on sets E of small dimension (dim $X_1(E) = \alpha_1 \dim E$ when $\dim E \leqq 1/\alpha_1$ according to the results of Blumenthal and Getoor [2] in the stable case) and then using up the X_2 component.

Theorems 3 and 4 show us hat:

$$
\dim E_k = \begin{cases} 2-k(\alpha_1+\alpha_2-\alpha_1\,\alpha_2)/\alpha_1 & \text{when } 0 < k(\alpha_1+\alpha_2-\alpha_1\,\alpha_2) < \alpha_1 \\ 1+\alpha_1/\alpha_2 - k(\ )/\alpha_2 & \text{when } \alpha_1 \leqq k(\ ) < \alpha_1+\alpha_2. \end{cases}
$$

For stable processes $X_{\alpha,2}$ of index $\alpha > 1$ in the plane Taylor's result is: $\dim E_k = 2-k(2-\alpha)$. Thus, we are again led to think (intuitively) of successively using up the two components. In fact, this is what led to the discovery of our $\dim E_k$ results.

(ii) The proofs of our dimension theorems required considerable effort to uncover, and it did not seem possible to avoid using the two kinds of grids and rectangles. This is no doubt a reflection of some of the basic properties of the process.

(iii) If we consider the quantity $k(\alpha_1+\alpha_2-\alpha_1\,\alpha_2)$ we can determine a maximum multiplicity possible. For the processes being considered double points must exist, but it is possible (by choosing α_1 and α_2 close to 1) that triple points do not exist. Moreover, given an integer $k > 2$, it is possible (by choosing α_1 and α_2 near 2 for large k) to have points of multiplicity k but that $(k+1)$-multiple points will fail to exist.

(iv) The problem of multiple points in R^3 when one of the components is planar or if 3 independent one-dimensional components are present remains unsolved. Earlier work on these processes ([4, 6, 10]) suggests that the present case is the most interesting one, but Fristedt's [3] extension of Taylor's [12] result suggests that the problem may involve some detailed calculations. We conjecture that there are basically two cases to consider, depending upon whether $\alpha_1 > d_1$ (dimension of X_1) or $\alpha_1 \leqq d_1$. Taylor's work in the stable case ([12] and [13]) requires each of the stable components to be type A, and to have stable index greater than $\frac{1}{2}$ when the component is linear. When $\alpha_1 > d_1$ we suspect the results to be identical to those obtained in the present paper. When $\alpha_1 \leqq d_1$ we predict that the behaviour of the process is similar to that of X_1. Analogous results in the cases $\alpha_1 > d_1$ and $\alpha_1 \leqq d_1$ were obtained in the previous studies.

(v) The above conjecture has been found to be true for (X_1, X_2) in R^2 when $\frac{1}{2} < \alpha_2 \leqq 1 < \alpha_1$ and X_2 is type A (and in addition symmetric if $\alpha_2 = 1$). One must only check that the estimates of Lemma 3.1 for the kernel and of Lemma 3.2 for the capacity of the rectangles actually used in the argument hold. In Lemma 3.1 we can estimate the kernel in the same way as before. The lower bound poses no difficulty, while the upper bound as done in Lemma 3.1 of [7] does seem to require $\alpha_2 \geqq \frac{1}{2}$ (which we assume anyway). In Lemma 3.2 we thus have the same upper bounds upon $C(R_{a,b})$ as before. The lower bounds for $C(R(\delta))$ and $C(S(\delta))$ which are obtained by only assuming $\alpha_1 > 1$, namely $c^{-1} b^{1-\alpha_2+\alpha_2/\alpha_1}$, are the same as the upper bounds for the capacity of these two types of rectangles. Hence it suffices in all of our theorems to allow $\frac{1}{2} < \alpha_2 \leqq 1 < \alpha_1$ when the process is planar.

This research was begun as a part of the author's doctoral dissertation under the direction of Professor W.E. Pruitt at the University of Minnesota. The original results were incomplete. Recent communications with Professor Pruitt and with S.J. Taylor of Westfield College have provided many helpful ideas in establishing the present results. The contributions of Professors Pruitt and Taylor to this work are gratefully acknowledged.

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(Received March 8, 1970; in revised form February 24, 1973)