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A Representation for Invariant Measures for Transient Markov Chains

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§ 1. Introduction

Let $P=(p(i,j))$ be the matrix of transition probabilities of a temporally homogeneous Markov chain $\{X_n\}$, $n=0,1,2,\ldots$, on a countable state space $\mathscr{S}$ indexed by the non-negative integers $0,1,2,\ldots$.

We write $P^n=(p^{(n)}(i,j))$ for the matrix of n-step transition probabilities

$$p^{(n)}(i,j)=Pr\{X_n=j\,|\,X_0=i\}$$

of the chain $\{X_n\}$, and put $P^0=(\delta_{ij})$. We shall assume that $\{X_n\}$ is *irreducible:* that is, for each pair (i,j) there exists $n>0$ such that $p^{(n)}(i,j)>0$.

Irreducibility is not critical for our results, and is mainly to ease notation: a vital assumption, which we shall always make, is that

$\{X_n\}$ *is a transient Markov chain.*

In terms of transition probabilities, this means (Chung, 1967) that

$$\sum_n p^{(n)}(i,j)<\infty \qquad \text{for each pair } (i,j).$$

We shall call a vector $x=(x(j))$ with $0\leqq x(j)<\infty$ $(j=0,1,\ldots)$ and $x(j)$ not identically zero, an *invariant measure for* P (or for $\{X_n\}$) if $x(0)=1$, and x satisfies the left invariant equations

$$x=xP; \tag{1.1}$$

we shall call a vector $y=(y(j))$ with finite non-negative entries, not identically zero, *subinvariant* for P if $y(0)=1$ and

$$y(j)\geqq\sum_k y(k)\,p(k,j), \qquad j\in\mathscr{S}. \tag{1.2}$$

The results we shall prove depend heavily on the following simple lemma.

Lemma A. *Write* $l^{(0)}(0,j)=0$, *and for* $n\geqq 1$,

$$l^{(n)}(0,j)=Pr\{X_n=j,\,X_r\neq 0\,(r=1,\ldots,n-1)\,|\,X_0=0\},$$

and define the vector $L_0=(L_0(j))$ *by*

$$L_0(0)=1,$$

$$L_0(j)=\sum_{n=0}^{\infty} l^{(n)}(0,j), \qquad j=1,2,\ldots.$$

Then $L_0(j)$ is finite for each j, and L_0 satisfies

$$L_0(0) > \sum_k L_0(k)\, p(k,0) = \sum_n l^{(n)}(0,0) = F_{00}, \tag{1.3}$$

$$L_0(j) = \sum_k L_0(k)\, p(k,j), \quad j \neq 0, \tag{1.4}$$

so that L_0 is strictly subinvariant for P. Moreover, if y is any other subinvariant measure for P,

$$y(j) \geqq L_0(j), \quad j \in \mathcal{S}$$

so that L_0 is the minimal subinvariant measure for P. ||

This lemma remains true even if P is substochastic. Proofs of these results are by now well-known (cf. Vere-Jones, 1967). When $\{X_n\}$ is non-transient, the vector L_0 defined above satisfies (1.3) with equality, and it can then be shown that there is a unique solution to (1.2) which in fact satisfies (1.1), in the recurrent case; hence our assumption of transience. Note that each $L_0(j)$ is finite.

Now define the matrix $P_n = (p_n(i,j))$ by

$$p_n(i,j) = \begin{cases} p(i,j) & (i,j \geqq n), \\ 0, & \text{otherwise.} \end{cases}$$

We shall call P_n the *n-th diagonal submatrix* of P. It is formed from P merely by omitting the first n rows and columns of P. Since P is irreducible, the diagonal submatrices of P are strictly substochastic (that is, have at least one row sum strictly less than unity), and in general they will not be irreducible. We shall call a vector $x_n = (x_n(j))$ with finite non-negative entries, *invariant for P_n* if x_n satisfies

$$x_n(j) = 0 \qquad (j = 0, 1, \ldots, n-1),$$

$$x_n(j) = \sum_k x_n(k)\, p_n(k,j) \qquad (j \geqq n),$$

and $x_n(\alpha) > 0$ for some $\alpha \geqq n$. Since P_n may be reducible, we can no longer demand that $x_n(n) = 1$, since $x_n(n)$ may be zero; to avoid ambiguity, we shall further require that

$$x_n(\alpha(n)) = 1$$

where

$$\alpha(n) = \min\{j \geqq n\colon x_n(j) > 0\}.$$

(This difficulty does not arise when $n = 0$, because if x is invariant for P then we can show that $x(j) > 0$ for all j.)

Of course, if x_n is invariant for P_n, it is also invariant for $P_{n+1}, \ldots, P_{\alpha(n)}$.

In this paper we shall show that P possesses an invariant measure if and only if the diagonal submatrices of P all possess invariant measures of a particular kind, and further that if two transition matrices P and P' share a diagonal submatrix, then there is a 1:1 correspondence between their invariant measures.

We shall then derive a representation for invariant measures in terms of taboo probabilities. It follows that there exists an invariant measure for P if and only if a certain set of auxiliary equations are satisfied.

§ 2. The Diagonal Submatrices of P

Theorem 1. *If P is transient, a necessary and sufficient condition for the existence of an invariant measure x for P is the existence of an invariant measure x_1 for P_1 such that*

$$N_0(x_1)=\sum_{j=0}^{\infty} x_1(j)\, p(j,0)<\infty. \tag{2.1}$$

The equations

$$x_1=[x-L_0]/c_0(x), \tag{2.2}$$

$$x=\frac{1-F_{00}}{N_0(x_1)}\, x_1+L_0 \tag{2.3}$$

with L_0 and F_{00} as defined in Lemma A and

$$c_0(x)=x(\alpha)-L_0(\alpha), \tag{2.4}$$

where

$$\alpha=\min(j\colon x(j)-L_0(j)>0),$$

set up a 1:1 *correspondence between the P-invariant measures x and those P_1-invariant measures x_1 for which $N_0(x_1)<\infty$.*

Proof. (i) Suppose that x is P-invariant. From Lemma A, we have $x(j)\geqq L_0(j)$ $(j=0,1,\ldots)$, and since x is P-invariant whilst L_0 is not, this is a strict inequality for some j; thus $c_0(x)$ is well defined by (2.4), and is positive. Define x_1 by (2.2); by definition $x_1(0)=0$, and $x_1(\alpha)=1$ where $\alpha=\min(j\colon x_1(j)>0)$. For $j\neq 0$, from (1.4),

$$\begin{aligned} x_1(j)&=[x(j)-L_0(j)]/c_0(x)\\ &=\Big[\sum_{k\geqq 0} x(k)\, p(k,j)-\sum_{k\geqq 0} L_0(k)\, p(k,j)\Big]/c_0(x)\\ &=\sum_{k\geqq 1} x_1(k)\, p(k,j). \end{aligned}$$

Hence x_1 is P_1-invariant. The quantity $N_0(x_1)$, when x_1 is given by (2.2), is

$$\begin{aligned} N_0(x_1)&=\sum_0^{\infty} x_1(j)\, p(j,0)\\ &=\left[\sum_0^{\infty} x(j)\, p(j,0)-\sum_0^{\infty} L_0(j)\, p(j,0)\right]\Big/ c_0(x)\\ &=[1-F_{00}]/c_0(x), \end{aligned}$$

and so (2.1) holds, and indeed $N_0(x_1)>0$. Thus (2.2) is a map from the set of P-invariant measures into the set of P_1-invariant measures satisfying (2.1). This map is 1:1, for if

$$[x(j)-L_0(j)]/c_0(x)=[\tilde{x}(j)-L_0(j)]/c_0(\tilde{x}),$$

then either $c_0(x)=c_0(\tilde{x})$, whence $x=\tilde{x}$; or $c_0(x)\neq c_0(\tilde{x})$, and, assuming without loss of generality $c_0(x)<c_0(\tilde{x})$, we have

$$x(j)-\tilde{x}(j)\,\frac{c_0(x)}{c_0(\tilde{x})}=L_0(j)[1-c_0(x)/c_0(\tilde{x})]. \tag{2.5}$$

But the right hand side of (2.5) satisfies (1.2) with strict inequality at $j=0$, whilst the left hand side satisfies (1.2) with equality for all j if x, $\tilde{x}$ are P-invariant. Thus we have a contradiction, and $x=\tilde{x}$.

(ii) Suppose now that x_1 is P_1-invariant, and satisfies (2.1). Let $\alpha \geqq 1$ be the first state such that $x_1(\alpha)>0$. Since P is irreducible, there exists $n>0$ such that

$$ {}_0p^{(n)}(\alpha,0)=Pr\{X_n=0, X_r \neq 0 (r=1,\ldots,n-1) | X_0=\alpha\}>0, $$

and since x_1 is P_1-invariant,

$$ \begin{aligned} \sum_1^\infty x_1(k)\,p(k,0) &= \sum_{k=1}^\infty \left[\sum_{j=1}^\infty x_1(j)\,p_1^{(n-1)}(j,k)\right] p(k,0) \\ &= \sum_1^\infty x_1(j)\,{}_0p^{(n)}(j,0) \\ &> x_1(\alpha)\,{}_0p^{(n)}(\alpha,0) \\ &> 0. \end{aligned} $$

Hence $N_0(x_1)$ is positive, and we can define x (finite and non-negative) by (2.3). Since $x_1(0)=0$, $x(0)=L_0(0)=1$, and for $j \neq 0$,

$$ \begin{aligned} x(j) &= \frac{1-F_{00}}{N_0(x_1)}\,x_1(j)+L_0(j) \\ &= \frac{1-F_{00}}{N_0(x_1)} \sum_{k=0}^\infty x_1(k)\,p(k,j)+\sum_{k=0}^\infty L_0(k)\,p(k,j) \\ &= \sum_{k=0}^\infty x(k)\,p(k,j). \end{aligned} $$

For $j=0$,

$$ \begin{aligned} \sum_{k=0}^\infty x(k)\,p(k,0) &= \frac{1-F_{00}}{N_0(x_1)} \sum_{k=0}^\infty x_1(k)\,p(k,0)+\sum_0^\infty L_0(k)\,p(k,0) \\ &= \frac{1-F_{00}}{N_0(x_1)}\,N_0(x_1)+F_{00} \\ &= 1 = x(0) \end{aligned} $$

and so x is P-invariant.

The mappings (2.2) and (2.3) are mutually inverse, since if x is P-invariant, defining x_1 from x by (2.2) and $\tilde{x}$ from x_1 by (2.3) gives, from the definition of N_0 in (2.1),

$$ \tilde{x}=\frac{1-F_{00}}{N_0([x-L_0]/c_0(x))}\;\frac{x-L_0}{c_0(x)}+L_0=x; $$

whilst defining $\tilde{x}_1$ from x_1 via (2.3) and (2.2) gives

$$ \begin{aligned} \tilde{x}_1 &= \left[\frac{1-F_{00}}{N_0(x_1)}\,x_1+L_0-L_0\right] \Big/ c_0\left(\frac{1-F_{00}}{N_0(x_1)}\,x_1+L_0\right) \\ &= x_1, \end{aligned} $$

since
$$\alpha=\min\left(j\colon \frac{1-F_{00}}{N_0(x_1)}x_1(j)+L_0(j)-L_0(j)>0\right)$$
$$=\min(j\colon x_1(j)>0)$$
and for this α, $x_1(\alpha)=1$ by definition. ||

Example 1 [1]. Let P be the transition matrix of a renewing transient branching process; that is,
$$p_{0j}=\gamma_j, \qquad (\sum_j \gamma_j=1, \gamma_0 \neq 1),$$
and
$$\sum_j p_{ij} z^j=[f(z)]^i \qquad (i=1,2,\ldots,|z|\leqq 1),$$
where $f(z)$ is a probability-generating function. Such a process corresponds to an ordinary branching process with the exception that, when state 0 is reached, it is restarted with probability γ_j of being in state j on the next "run". This process is clearly transient if and only if $m=f'(1-)>1$ (cf. Harris, 1963), and one may apply a result of Kingman (1965) to show that P must have an invariant measure but that this may be non-unique. ||

The condition (2.1) is non-trivial (although in Example 1 it is always satisfied for invariant measures for P_1). The next example exhibits a matrix P such that the first diagonal submatrix admits an invariant measure, but P itself does not, because (2.1) fails.

Example 2. From Vere-Jones ((1967), Theorem 5.1), we have that if x satisfies
$$x(j)\leqq\sum_k x(k)\,p(k,j) \tag{2.6}$$
and y satisfies
$$y(j)\geqq\sum_k p(j,k)\,y(k), \tag{2.7}$$
and P is transient, then (provided neither x nor y is identically zero)
$$\sum x(k)\,y(k)=\infty.$$
If x_1 is invariant for P_1, then x_1 satisfies (2.6); the vector $y(k)=p(k,0)$ satisfies (2.7) if and only if
$$p(j,0)\geqq p^{(2)}(j,0) \qquad (j=0,1,2,\ldots).$$
Hence any transient P such that this holds and P_1 admits an invariant measure provides an example where (2.1) is false. Such a matrix is
$$P=\begin{pmatrix} \frac{1}{8} & \frac{7}{8} & 0 & 0 & \ldots & & & \\ \frac{1}{8} & \frac{5}{8} & \frac{1}{4} & 0 & \ldots & & & \\ \frac{1}{16} & \frac{1}{16} & \frac{5}{8} & \frac{1}{4} & 0 & \ldots & & \\ \frac{1}{32} & 0 & \frac{1}{32} & \frac{11}{16} & \frac{1}{4} & 0 & \ldots & \\ \cdots & \cdots & \cdots & \cdots & \ddots & & & \\ \frac{1}{2^{i+2}} & 0\;\ldots\;0 & \frac{1}{2^{i+2}} & & \frac{2^{i+2}-2^i-2}{2^{i+2}} & & \frac{1}{4} & 0 \\ \vdots & & \vdots & & & & \vdots & \vdots \end{pmatrix}. \quad ||$$

[1] This result has been derived independently, using special properties of branching processes, by Seneta (1974), who also shows that P possesses a unique invariant measure satisfying a certain regular variation criterion.

Example 3. Let P be the transition matrix of a renewal process,

$$P=\begin{pmatrix} f_0 & 1-f_0 & 0 & 0 & 0 & \dots \\ f_1 & 0 & 1-f_1 & 0 & 0 & \dots \\ f_2 & 0 & 0 & 1-f_2 & 0 & \dots \\ f_3 & 0 & 0 & 0 & 1-f_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

Derman (1955) showed by direct calculation that such a matrix has no invariant measure when transient; this result follows trivially from Theorem 1, since the only solution to

$$x_1 = x_1 P_1$$

is

$$x_1(j) \equiv 0. \quad \|$$

We can iterate Theorem 1 for the set of diagonal submatrices of P to find

Theorem 2. *If P is a transient transition matrix, then a necessary and sufficient condition for the existence of an invariant measure x for P is the existence of an invariant measure x_n for some one diagonal submatrix P_n satisfying*

$$N_\kappa(x_n) = \sum_k x_n(k)\, p(k,\kappa) < \infty \qquad (\kappa = 0, 1, \dots, n-1). \tag{2.8}$$

If such a measure x_n exists, then for every $m \geqq 0$ there is an invariant measure x_m for P_m satisfying the κ-th *assertion in* (2.8) *for each $\kappa < m$. To each x_n invariant for P_n and satisfying* (2.8), *there corresponds biuniquely an x_m invariant for P_m with $N_\kappa(x_m) < \infty$ for each $\kappa < m$.*

Proof. We define a sequence of measures $L_n = (L_n(j))$ (each $L_n(j)$ finite and non-negative) by

$$L_n(j) = \begin{cases} 0 & (j<n), \\ 1 & (j=n), \\ \sum_{r=0}^{\infty} {}_{[n]}l^{(r)}(n,j) & (j>n), \end{cases} \tag{2.9}$$

where ${}_{[n]}l^{(0)}(n,j) = 0$ and for $r \geqq 1$,

$${}_{[n]}l^{(r)}(n,j) = Pr\{X_r = j, X_s \notin \{0, 1, \dots, n\}\ (s=1, \dots, r-1) \mid X_0 = n\}.$$

Now suppose x is invariant for P. Then from Theorem 1, there exists x_1 invariant for P_1. Suppose that $\alpha \geqq 1$ is the first α such that $x_1(\alpha) > 0$. Then x_1 is also invariant for $P_1, P_2, \dots, P_\alpha$. For P_α, the measure L_α is the minimal subinvariant measure, from Lemma A; the measure $x_{\alpha+1}$ defined by

$$x_{\alpha+1}(j) = x_1(j) - L_\alpha(j)$$

must then, as in Theorem 1, be invariant for $P_{\alpha+1}$ when normalised as usual to be unity at its first non-zero component. One can in this manner construct a sequence of measures x_n invariant for P_n by putting inductively

$$x_{n+1} = x_n, \qquad \text{when} \quad x_n(n) = 0,$$
$$x_{n+1} = [x_n - L_n]/c_n(x_n) \quad \text{when} \quad x_n(n) > 0,$$

where

$$c_n(x_n) = x_n(\alpha(n)) - L_n(\alpha(n))$$

and

$$\alpha(n) = \min(j\colon x_n(j) - L_n(j) > 0).$$

Since $x_n(j) \leqq c x(j)$ for each j, where $c = \prod_{r=0}^{n-1} c_r$ and $c_r = 1$ or $1/c_r(x_r)$, it follows that $N_\kappa(x_n) \leqq c N_\kappa(x) = c x(\kappa) < \infty$ $(\kappa = 0, 1, \ldots, n-1)$. By this construction, distinct invariant measures for P lead to distinct invariant measures satisfying (2.8) for P_n, for each value of n.

Now suppose that x_{n+1} satisfies *the* κ-th *assertion in* (2.8) for $\kappa \leqq n$, and is invariant for P_{n+1}; as in Theorem 1, we show that there is an invariant measure x_n for P_n which satisfies *the* κ-th *assertion in* (2.8) for $\kappa \leqq n-1$ (i.e. (2.8) as it stands).

This is trivially true if x_{n+1} itself is invariant for P_n; and if x_{n+1} is not invariant for P_n, we must have

$$0 = x_{n+1}(n) \neq \sum_{\beta \geqq n+1} x_{n+1}(\beta)\, p(\beta, n) < \infty,$$

and so there must exist $\beta > n$ such that $x_{n+1}(\beta) > 0$ and $p(\beta, n) > 0$.

Define $\phi_{nn} = \sum_{r=0}^{\infty} {}_{[n]}l^{(r)}(n, n) \leqq Pr\{\text{the chain } \{X_k\} \text{ ever returns to } n | X_0 = n\} < 1$, and put

$$x_n = \frac{1 - \phi_{nn}}{N_n(x_{n+1})}\, x_{n+1} + L_n;$$

this is well defined and finite and non-negative since $\infty > N_n(x_{n+1}) > x_{n+1}(\beta)\, p(\beta, n)$. To check P_n-invariance it suffices to look at

$$\begin{aligned}
\sum_{j \geqq n} x_n(j)\, p(j, n) &= \frac{1 - \phi_{nn}}{N_n(x_{n+1})} \sum_{j \geqq n} x_{n+1}(j)\, p(j, n) + \sum_{j \geqq n} L_n(j)\, p(j, n) \\
&= \frac{1 - \phi_{nn}}{N_n(x_{n+1})} N_n(x_{n+1}) + \phi_{nn} \\
&= 1 = x_n(n).
\end{aligned}$$

Moreover, for $\kappa < n$, writing $\gamma = [1 - \phi_{nn}]/N_n(x_{n+1})$,

$$\begin{aligned}
\sum_{k \geqq n} x_n(k)\, p(k, \kappa) &= \sum_{k \geqq n} [L_n(k) + \gamma x_{n+1}(k)]\, p(k, \kappa) \\
&= p(n, \kappa) + \sum_{k > n} \Big(\sum_r {}_{[n]}l^{(r)}(n, k)\, p(k, \kappa)\Big) + \gamma \sum_{k \geqq n+1} x_{n+1}(k)\, p(k, \kappa)
\end{aligned}$$

and since the third of these terms is finite for all $\kappa < n+1$ by hypothesis, and the second is bounded by

$$\sum_r Pr\{X_r = \kappa, X_s \notin \{0, 1, \ldots, n\}\, (s = 1, \ldots, r-1) | X_0 = n\},$$

which is finite for all κ, x_n satisfies (2.8).

By iterating this construction, one obtains an invariant measure for P, and then on applying the first part of the theorem, one obtains invariant measures x_m

satisfying the appropriate finiteness conditions for each diagonal submatrix P_m; since distinct invariant measures for P_n lead to distinct invariant measures for P, and conversely, in the above constructions, the theorem is proved. ||

Example 2 shows that a chain whose diagonal submatrix admits an invariant measure may not have an invariant measure itself. However, from Theorem 2 it is simple to prove

Theorem 3. *Suppose P and P' are transient transition matrices which share a diagonal submatrix; that is $P_n = P'_m$ for some n, m. Then P has an invariant measure if and only if P' has, and there is a 1:1 correspondence between their invariant measures.*

Proof. If P has an invariant measure, then from the necessity part of Theorem 2, there exists x_n such that $x_n = x_n P_n$, and

$$\sum_{\kappa=0}^{n-1} \sum_k x_n(k)\, p(k,\kappa) < \infty.$$

Put $x'_m(j) = x_n(j+n-m)$, for $j \geqq m$, and elsewhere let $x'_m(j)=0$, so that x'_m is P'_m-invariant; we wish to show that for $\kappa < m$, $\sum_k x'_m(k)\, p'(k,\kappa) < \infty$. It is certainly enough to show that $\sum_{\kappa<m} \sum_k x'_m(k)\, p'(k,\kappa)$ is finite. Since $x'_m(k)=0$ for $k<m$, this sum equals

$$\sum_{\kappa<m} \sum_{k \geqq m} x'_m(k)\, p'(k,\kappa) = \sum_{k \geqq m} x'_m(k) \Big[\sum_{\kappa<m} p'(k,\kappa) \Big],$$

which in turn is equal to

$$\sum_{h \geqq n} x_n(h) \Big[\sum_{\kappa<n} p(h,\kappa) \Big] < \infty,$$

because for $k \geqq m$ we have

$$\begin{aligned} \sum_{\kappa=0}^{m-1} p'(k,\kappa) &= 1 - \sum_{\kappa} p'_m(k,\kappa) \\ &= 1 - \sum_{\kappa} p_n(k+n-m,\kappa) \\ &= \sum_{\kappa=0}^{n-1} p(k+n-m,\kappa). \end{aligned}$$

Hence x'_m satisfies the m-fold analogue of (2.8) for P'.

From the sufficiency part of Theorem 2, there is then an invariant measure for P'. The 1:1 correspondence of measures for P and P' comes from the correspondence between measures for $P(P')$ and $P_n(P'_m)$. ||

We have concentrated on proving results for the diagonal submatrices of P. However, by rearranging rows and columns, it is easy to see that, if $P(j_1, \ldots, j_n)$ is any *square submatrix* of P, obtained by removing the elements of the rows and columns corresponding to any n states $j_1, \ldots, j_n$, there exists an invariant vector for P if and only if there exists an invariant vector x for $P(j_1, j_2, \ldots, j_n)$ satisfying

$$\sum_k x(k)\, p(k, j_\kappa) < \infty$$

for each $\kappa = 1, \ldots, n$.

§3. The Representation of Invariant Measures

Theorem 4. *If P is transient, a necessary and sufficient condition for the existence of an invariant measure for P is the existence of a sequence of non-negative constants $\beta_0=1, \beta_1, \beta_2, \ldots$, not necessarily all positive, such that*

$$x^{\beta}=\sum_{0}^{\infty} \beta_k L_k \tag{3.1}$$

has finite components and is invariant for P; indeed every invariant measure can be written uniquely in the form (3.1) *for some sequence $\{\beta_k\}$. If x^{β} is invariant for P and has the canonical representation* (3.1), *then the corresponding invariant measure x_n^{β} for the n-th diagonal submatrix P_n, whose existence was established in Theorem 2, is*

$$x_n^{\beta}=c_n \sum_{k=n}^{\infty} \beta_k L_k,$$

where $c_n=[\beta_{\alpha(n)}]^{-1}$ and $\alpha(n)=\min(j \geqq n: \beta_j>0)$.

Proof. The sufficiency is trivial. Suppose x is P-invariant; from (2.2), there is a P_1-invariant measure x_1 and a non-negative constant η_1 such that

$$x(j)=L_0(j)+\eta_1 x_1(j). \tag{3.2}$$

If $\alpha(1)=\min(j \geqq 1: x_1(j)>0)$, we again have, as in the proof of Theorem 2, that for some $x_{\alpha(1)+1}$ which is invariant for $P_{\alpha(1)+1}$,

$$x_1(j)=L_{\alpha(1)}(j)+\eta_{\alpha(1)+1} x_{\alpha(1)+1}(j), \tag{3.3}$$

where $\eta_{\alpha(1)+1}$ is again a non-negative normalising constant. Putting (3.3) into (3.2) we can write

$$x(j)=L_0(j)+\sum_{1}^{\alpha(1)} \beta_k L_k(j)+\eta_1 \eta_{\alpha(1)+1} x_{\alpha(1)+1}(j) \quad (j=0,1,\ldots),$$

where $\beta_k=\eta_1$ if $k=\alpha(1)$ and 0 if $0<k<\alpha(1)$. Iterating this procedure m times will give us

$$x(j)=\sum_{0}^{\alpha(m)} \beta_k L_k(j)+\eta_1 \eta_{\alpha(1)+1} \cdots \eta_{\alpha(m)+1} x_{\alpha(m)+1}(j) \tag{3.4}$$

where $x_{\alpha(m)+1}$ is invariant for $P_{\alpha(m)+1}$ and

$$\beta_k=\eta_1 \eta_{\alpha(1)+1} \cdots \eta_{\alpha(r)+1} \quad \text{if } k=\alpha(r) \quad \text{and} \quad \beta_k=0 \text{ if } \alpha(r)<k<\alpha(r+1),$$

$r=1, \ldots, \alpha(m)$. Of course the analogue of (2.8) will hold for $x_{\alpha(m)+1}$.

But for each k, by definition $L_k(j)=0$ when $j<k$; applying this to (3.4) for fixed j, we must have a terminating sequence of no more than $j+1$ terms, and we can write (3.4) as

$$x(j)=\sum_{k \leqq j} \beta_k L_k(j) \tag{3.5}$$

where the β_k are given (uniquely, because of the unique correspondence between P_n-invariant and P_m-invariant measures of Theorem 2) as above. Thus x has the form (3.1), as claimed. Comparing this construction with that in Theorem 2 proves the statement regarding the form of the P_n-invariant measure corresponding to x. ∥

We now investigate conditions on a sequence β of non-negative numbers which will make (3.1) a P-invariant measure. Extending the definition of ϕ_{nn}, we write

$$\phi(n,i)=\begin{cases}\sum_{m=0}^{\infty} {}_{[n]}l^{(m)}(n,i) & (i\leqq n),\\ 0 & \text{otherwise.}\end{cases}$$

Now suppose that x^{β} given by (3.1) is to be invariant for P; this is equivalent to

$$\sum_{k=0}^{\infty}\beta_k L_k(j)=\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\beta_k L_k(i)\,p(i,j), \tag{3.6}$$

for every $j=0,1,\ldots$. Using the definition of $L_k(i)$ and $\phi(k,i)$, we have

$$\begin{aligned}\sum_{i=0}^{\infty}L_k(i)\,p(i,j)&=p(k,j)+\sum_{i=k+1}^{\infty}\left(\sum_{r=0}^{\infty}{}_{[k]}l^{(r)}(k,i)\right)p(i,j)\\ &=p(k,j)+\sum_{r=1}^{\infty}{}_{[k]}l^{(r+1)}(k,j)\\ &=\sum_{r=0}^{\infty}{}_{[k]}l^{(r)}(k,j),\end{aligned}\tag{3.7}$$

and the right hand side of (3.7) is $L_k(j)$ if $j>k$, and $\phi(k,j)$ if $j\leqq k$. Hence the right hand side of (3.6) is

$$\sum_k\beta_k\left(\sum_0^{\infty}{}_{[k]}l^{(r)}(k,j)\right)=\sum_{k=0}^{j-1}\beta_k L_k(j)+\sum_{k=j}^{\infty}\beta_k\,\phi(k,j)$$

so that $\{\beta_k\}$ satisfies (3.6) if and only if

$$\sum_{k=j}^{\infty}\beta_k L_k(j)=\sum_{k=j}^{\infty}\beta_k\,\phi(k,j)\qquad (j\in\mathscr{S}). \tag{3.8}$$

But as in (3.4), $L_k(j)=0$, $k>j$; and $L_j(j)=1$. Thus (3.8) is in fact

$$\beta_j=\sum_{k=j}^{\infty}\beta_k\,\phi(k,j)\qquad (j\in\mathscr{S}).$$

Denote by Φ the triangular matrix whose elements are $\phi(k,j)$. We have proved

Theorem 5. *If P is transient, there is a non-negative finite solution x to $x=xP$, $x(0)=1$, if and only if there is a non-negative finite solution $\beta=(\beta_k)$ with $\beta_0=1$ to*

$$\beta=\beta\Phi;$$

that is, if and only if there exists a sequence of non-negative finite numbers $\beta_0=1,\beta_1,\beta_2,\ldots$ satisfying

$$\beta_j=\sum_{k\geqq j}\beta_k\,\phi(k,j)\qquad (j\in\mathscr{S}). \tag{3.9}$$

There is a 1:1 convex-linear correspondence between the allowed solutions of (3.9) *and the allowed solutions of* (1.1), *given by* (3.1) (*or equivalently* (3.5)). ‖

Let us call a set of states $\{k_0, k_1, \ldots\}$ a *simple path from infinity* if all the states k_i are distinct and if, for each i, $p(k_{i+1}, k_i) > 0$. Harris (1957) showed that a necessary condition for the existence of a P-invariant measure for transient P is the existence of a simple path from infinity. Harris' result can be derived from Theorem 5 as follows. The matrix Φ need not be irreducible, and if it is not, in general solutions to (3.9) will have some of the coefficients $\beta_j = 0$, as has already been mentioned. However, since L_k is not invariant for P_k for any k, x^β defined by (3.1) can be P-invariant only if $\beta = (\beta_k)$ *is an infinite vector*. Suppose x^β is P-invariant, and let $K = \{k: \beta_k > 0\}$; we can write (3.9) as

$$\beta_k = \sum_{j \in K} \beta_j \phi(j, k) \qquad (k = 0, 1, 2, \ldots), \tag{3.10}$$

and since $\beta_k = 0$ for $k \notin K$, this shows that K is a closed class under Φ. Since $\phi(j, k) = 0$ for $j < k$, and $\phi(j, j) < 1$, (3.10) also shows that for any $k \in K$ there exists in K a $j > k$ such that $\phi(j, k) > 0$, which in turn implies that there is a state $k_1 \geqq j > k$ such that $p(k_1, k) > 0$. Since K is closed, k_1 is again in K, and by repeating this procedure, we see that, because K is infinite, we can construct a simple path from infinity.

Example 4. Given a probability distribution $\{c_i, i = \ldots -2, -1, 0, 1, 2, \ldots\}$ on the integers, define the transition matrix of *random walk with boundaries near zero* by

$$P = \begin{pmatrix} p(0,0) & p(0,1) \ldots p(0,\alpha) & p(0,\alpha+1) & p(0,\alpha+2) & \\ p(1,0) & p(1,1) \ldots p(1,\alpha) & p(1,\alpha+1) & p(1,\alpha+2) & \ldots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ p(\alpha,0) & p(\alpha,1) \ldots p(\alpha,\alpha) & p(\alpha,\alpha+1) & p(\alpha,\alpha+2) & \ldots \\ p(\alpha+1,0) & p(\alpha+1,1) \ldots p(\alpha+1,\alpha) & c_0 & c_1 & c_2 \ \ldots \\ p(\alpha+2,0) & p(\alpha+2,1) \ldots p(\alpha+2,\alpha) & c_{-1} & c_0 & c_1 \ \ldots \\ \vdots & \vdots \qquad\qquad \vdots & c_{-2} & c_{-1} & c_0 \ \ldots \end{pmatrix} \tag{3.11}$$

so that $p(j, k) = c_{k-j}$ for both j and $k > \alpha$, whilst the first α rows and columns are arbitrary, subject only to the conditions

$$\sum_k p(j, k) = 1 \qquad (j = 0, 1, \ldots, \alpha),$$

$$\sum_0^\alpha p(j, k) = \sum_{-\infty}^{\alpha - j} c_i \qquad (j > \alpha).$$

This transition matrix occurs naturally in the context of embedded Markov chains for queueing processes (cf. Miller, 1965).

If P is transient, it follows from the above theorems that there will be a solution to $x = xP$ if and only if there is a solution to $x_\alpha = x_\alpha P_\alpha$, where $P_\alpha(i, j) = c_{j-i}$ $(i, j = 1, 2, \ldots)$, such that

$$\sum_j x_\alpha(j) \left(\sum_{-\infty}^{-j} c_i \right) < \infty. \tag{3.12}$$

(This finiteness condition corresponds to (2.8) summed over κ.) It is proved in Tweedie (1971), that such a solution x_α exists if and only if there is a positive real number $\beta<1$ such that

$$\sum_{-\infty}^{\infty} c_j \beta^j = 1. \tag{3.13}$$

Hence the existence of such a root is also necessary and sufficient for the existence of an invariant measure for P given by (3.11), since it is shown in the above paper that when such a root exists, the unique solution x_α of $x_\alpha = x_\alpha P_\alpha$ satisfies (3.12).

The method used in Tweedie (1971) is that of finding the form (3.1) for invariant measures in the special case of random walk on a half line, and then solving (3.9) for this case. This provides an example where the equation $\beta = \beta\Phi$ is rather easier to solve than the original invariant equations; the unique solution when (3.13) holds is given by $\beta_j = \beta^{-j}, j \geqq \alpha$. ||

Example 5. It is possible to construct an irreducible transition matrix P with any desired number of invariant measures using the results of this section. One may utilise the result of the previous example, say, and construct d random walks on the half lattices $r+dj$ $(j=1,2,\ldots; r=1,\ldots,d)$ each of which has

$$p(r+dj, r+dk) = c_{k-j},$$
$$p(r+dj, 0) = \sum_{-\infty}^{-j} c_k,$$

for some probability distribution satisfying (3.13); one thus generates d different invariant measures for P, each with a corresponding vector β concentrated on a different lattice. ||

§4. Non-Negative Matrices

Many of the properties of transition matrices carry over to arbitrary non-negative matrices, and the above work is not exceptional. We sketch some of these results; the proofs are identical with those for transition matrices.

Let $T = \big(t(i,j)\big)$ be a matrix with non-negative terms, and with finite iterates $T^n = \big(t^{(n)}(i,j)\big)$. The more usual problem here is to consider the equation

$$x = r x T \tag{4.1}$$

for real non-negative r; results on these equations can be found in Vere-Jones (1967). We assume T is irreducible and *r-transient*; that is, for all (i,j),

$$\sum_n t^{(n)}(i,j)\, r^n < \infty.$$

We define the vectors L_n^r, $n=0,1,\ldots$ by

$$L_n^r(j) = \begin{cases} 0 & (j<n), \\ 1 & (j=n), \\ \sum_{k=0}^{\infty} {}_{[n]}l^{(k)}(n,j)\, r^k & (j>n), \end{cases}$$

where ${}_{[n]}l^{(k)}(n,j)$ is zero for $k=0$ and is defined iteratively for $k \geqq 1$ by

$$\begin{aligned} {}_{[n]}l^{(1)}(n,j) &= t(n,j) \\ {}_{[n]}l^{(k+1)}(n,j) &= \sum_{i>n} {}_{[n]}l^{(k)}(n,i)\, t(i,j). \end{aligned}$$

The vectors L_n^r play the same role in the theory of r-invariant measures (solutions to (4.1)) as do the vectors L_n, similarly defined, in the theory of invariant measures: this is because (Vere-Jones, 1967) they are the minimal r-subinvariant vectors for the diagonal submatrices of T.

Defining the matrix $\Phi^{(r)} = (\phi^{(r)}(i,j))$ by

$$\phi^{(r)}(i,j) = \begin{cases} 0 & (j>i) \\ \sum_{k=0}^{\infty} {}_{[i]}l^{(k)}(i,j)\, r^k & (j \leqq i), \end{cases} \tag{4.2}$$

we have

Theorem 6. (i) *There is a non-negative solution $x^{(r)}$ to the r-invariant equations* (4.1) *if and only if there are non-negative solutions $x_n^{(r)}$ to the r-invariant equations*

$$x_n^{(r)} = r\, x_n^{(r)}\, T_n$$

for each diagonal submatrix T_n of T, such that

$$\sum_j x_n^{(r)}(j)\, t(j,\kappa) < \infty \qquad (\kappa = 0, 1, \ldots, n-1).$$

(ii) *There is a solution $x^{(r)}$ to* (4.1) *if and only if there is a solution $\beta^{(r)} = (\beta_j^{(r)})$ to*

$$\beta^{(r)} = r\, \beta^{(r)}\, \Phi^{(r)} \tag{4.3}$$

where $\Phi^{(r)}$ is defined by (4.2). *Solutions of* (4.1) *and* (4.3) *are in* 1:1 *correspondence, and are related by*

$$x^{(r)}(j) = \sum_{k=0}^{j} \beta_k^{(r)}\, L_k^{(r)}(j). \quad \|$$

The only one of Theorems 1–5 whose analogue does not hold is Theorem 3: for non-negative matrices, the existence of r-invariant measures is not governed entirely by the behaviour in the tail: the behaviour of individual columns can also affect this existence. Our final example demonstrates this.

Example 6. Let P be as in Example 2, and let x_1 be the invariant measure for P_1. Define T by

$$T = \left(\begin{array}{c:cccc} \frac{1}{8} & \frac{7}{8} & 0 & 0 & \ldots \\ \hdashline T' & & P_1 & & \end{array} \right)$$

where T' is a column vector with $T'(j) = [2^j x_1(j)]^{-1}$.

Then P does not admit an invariant measure, since the elements of the first column do not satisfy

$$\sum_j x_1(j)\, p(j,0) < \infty;$$

T does admit an invariant measure, having been constructed so that

$$\sum_j x_1(j)\, t(j, 0) = 1.$$

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