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A Note on Uniform Strong Convergence of Bivariate Density Estimates

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Abstract. In this paper we consider a class of estimates of a bivariate density function f based on an independent sample of size n. Under the assumption that f is uniformly continuous, the uniform strong consistency of such estimates was first proved by Nadaraya (1970) for a large class of kernel functions. In this note we show that the assumption of the uniform continuity of f is necessary for this type of convergence.

1. Introduction

Let (X_1, Y_1), $(X_2, Y_2), \ldots$ be a sequence of independent two dimensional random variables with a common distribution function F and let

$$f_n(x, y) = \frac{1}{n h_n^2} \sum_{j=1}^{n} k_1\left(\frac{x - X_j}{h_n}\right) k_2\left(\frac{y - Y_j}{h_n}\right)$$

where $\{h_n\}$ is a sequence of positive numbers converging to zero and k_1 and k_2 are probability density functions.

We assume the following conditions:

(i) the series $\sum_{n=1}^{\infty} \exp\{-\gamma n h_n^4\}$ is convergent for all $\gamma > 0$,

(ii) k_1 and k_2 are functions of bounded variation.

In this note we prove the following

Theorem. *A necessary and sufficient condition for*

$$\lim_{n\to\infty} \sup_{(x,y)\in R_2} |f_n(x, y) - g(x, y)| = 0$$

with probability one for some function g is that g be the uniformly continuous Lebesgue density of F.

2. Proof of the Theorem

The sufficiency of the condition is due to Nadaraya (1970). To establish the necessity of the condition we need a series of lemmas.

Lemma 1.

$$\lim_{n\to\infty} \sup_{(x,y)\in R_2} |f_n(x, y) - E f_n(x, y)| = 0$$

with probability one.

Proof. Denote by $G(\cdot|u)$ the conditional distribution function of Y given $X=u$ and by $F_1(\cdot)$ the marginal distribution function of X. Hence,

$$Ef_n(x,y)=\int_{-\infty}^{\infty} h_n^{-2}\,k_1\left(\frac{x-u}{h_n}\right)\left\{\int_{-\infty}^{\infty} k_2\left(\frac{y-v}{h_n}\right)dG(v|u)\right\}dF_1(u).$$

Integrating by parts the integral in the bracket and interchanging the order of integration we get

$$\begin{aligned}Ef_n(x,y)&=\int_{-\infty}^{\infty} h_n^{-2}\,k_1\left(\frac{x-u}{h_n}\right)\left\{-\int_{-\infty}^{\infty} G(v|u)\,dk_2\left(\frac{y-v}{h_n}\right)\right\}dF_1(u)\\&=\int_{-\infty}^{\infty} h_n^{-2}\left\{-\int_{-\infty}^{\infty} k_1\left(\frac{x-u}{h_n}\right)G(v|u)\,dF_1(u)\right\}dk_2\left(\frac{y-v}{h_n}\right)\\&=\int_{-\infty}^{\infty} h_n^{-2}\left\{-\int_{-\infty}^{\infty} k_1\left(\frac{x-u}{h_n}\right)dF_v^*(u)\right\}dk_2\left(\frac{y-v}{h_n}\right)\end{aligned}$$

where for fixed v_0 the function $F_{v_0}^*(u)$ is defined by

$$F_{v_0}^*(u)=\int_{-\infty}^{u} G(v_0|t)\,dF_1(t)=F(u,v_0).$$

Again, integrating by parts the integral inside the bracket we obtain

$$Ef_n(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_n^{-2}\,F(u,v)\,dk_1\left(\frac{x-u}{h_n}\right)dk_2\left(\frac{y-v}{h_n}\right). \tag{1}$$

Similarly,

$$\begin{aligned}f_n(x,y)&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_n^{-2}\,k_1\left(\frac{x-u}{h_n}\right)k_2\left(\frac{y-v}{h_n}\right)dS_n(u,v)\\&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_n^{-2}\,S_n(u,v)\,dk_1\left(\frac{x-u}{h_n}\right)dk_2\left(\frac{y-v}{h_n}\right),\end{aligned} \tag{2}$$

where $S_n(u,v)$ is the two dimensional empirical distribution function defined by

$$S_n(u,v)=\frac{1}{n}\sum_{j=1}^{n}\Phi(u-X_j)\,\Phi(v-Y_j)$$

and $\Phi(x-y)=1$ for $y\leqq x$ and vanishes for $y>x$. Hence,

$$\sup_{(x,y)\in R_2}|f_n(x,y)-Ef_n(x,y)|\leqq\sup_{(u,v)\in R_2}|S_n(u,v)-F(u,v)|\cdot\mu_1\mu_2 h_n^{-2}$$

where

$$\mu_1=\int_{-\infty}^{\infty}|dk_1(u)|\quad\text{and}\quad\mu_2=\int_{-\infty}^{\infty}|dk_2(u)|.$$

By a result due to Kiefer and Wolfowitz (1958) we now get for every $\varepsilon>0$

$$\begin{aligned}&P\{\sup_{(x,y)\in R_2}|f_n(x,y)-Ef_n(x,y)|\geqq\varepsilon\}\\&\quad\leqq P\left\{\sup_{(u,v)\in R_2}|S_n(u,v)-F(u,v)|\geqq\frac{\varepsilon h_n^2}{\mu_1\mu_2}\right\}<\lambda_0\exp\left\{-\frac{\lambda_1\varepsilon^2 n h_n^4}{\mu_1^2\mu_2^2}\right\}\end{aligned}$$

where λ_0 and λ_1 are positive constants. By condition (i) it now follows that

$$\sum_{n=1}^{\infty} P\{\sup_{(x,y)\in R_2} |f_n(x,y)-Ef_n(x,y)| \geqq \varepsilon\}$$

is finite. An application of the Borel-Cantelli lemma completes the proof.

Lemma 2. *If*

$$\lim_{n\to\infty} \sup_{(x,y)\in R_2} |f_n(x,y)-g(x,y)|=0$$

with probability one for some function g, then

$$\lim_{n\to\infty} \sup_{(x,y)\in R_2} |Ef_n(x,y)-g(x,y)|=0.$$

Proof. The proof follows from Lemma 1 and the following inequality

$$\sup_{(x,y)\in R_2} |Ef_n(x,y)-g(x,y)| \leqq \sup_{(x,y)\in R_2} |f_n(x,y)-Ef_n(x,y)| + \sup_{(x,y)\in R_2} |f_n(x,y)-g(x,y)|.$$

Lemma 3. *If*

$$\lim_{n\to\infty} \sup_{(x,y)\in R_2} |f_n(x,y)-g(x,y)|=0$$

with probability one for some function g, then F is continuous.

Proof. Suppose F is discontinuous at some point (x_0, y_0).

This implies that $P(X=x_0, Y=y_0)>0$. Then,

$$\begin{aligned}\sup_{(x,y)\in R_2} Ef_n(x,y) &= \sup_{(x,y)\in R_2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_n^{-2} k_1\left(\frac{x-u}{h_n}\right) k_2\left(\frac{y-v}{h_n}\right) dF(u,v) \\ &\geqq h_n^{-2}\cdot P(X=x_0, Y=y_0)\left[\sup_{(x,y)\in R_2}\left\{k_1\left(\frac{x-x_0}{h_n}\right) k_2\left(\frac{y-y_0}{h_n}\right)\right\}\right].\end{aligned} \tag{3}$$

Again, we get from (1)

$$\begin{aligned}\sup_{(x,y)\in R_2} Ef_n(x,y) &= \sup_{(x,y)\in R_2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_n^{-2} F(x-h_n u, y-h_n v)\, dk_1(u)\, dk_2(v) \\ &\leqq h_n^{-2} \mu_1 \mu_2.\end{aligned} \tag{4}$$

It is now easy to see that inequalities (3) and (4) contradict Lemma 2.

Lemma 4. *If*

$$\lim_{n\to\infty} \sup_{(x,y)\in R_2} |f_n(x,y)-g(x,y)|=0$$

with probability one for some function g, then g is uniformly continuous.

Proof. By Lemma 3, F is uniformly continuous. This implies that $Ef_n(x,y)$ is uniformly continuous. As g is by Lemma 2 the uniform limit of uniformly continuous functions it is itself uniformly continuous.

We now complete the proof of the theorem. By Lemma 2 it follows that for any point (x,y)

$$\lim_{n\to\infty} \int_{a_1}^{x}\int_{b_1}^{y} Ef_n(u,v)\, du\, dv = \int_{a_1}^{x}\int_{b_1}^{y} g(u,v)\, du\, dv.$$

Using Fubini's theorem and Lebesgue dominated convergence theorem we get

$$\lim_{n\to\infty} \int_{a_1}^{x} \int_{b_1}^{y} E f_n(u, v)\, du\, dv = F(x, y) - F(a_1, y) - F(x, b_1) + F(a_1, b_1)$$
$$= \int_{a_1}^{x} \int_{b_1}^{y} g(u, v)\, du\, dv.$$

By the fundamental theorem of calculus we conclude that

$$\frac{\partial^2}{\partial x\, \partial y} F(x, y) = g(x, y).$$

Since the point (x, y) is chosen arbitrarily, the desired conclusion now follows.

It may be mentioned that under similar conditions the theorem remains true for the estimates of a multivariate density function.

References

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