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On Solutions to $\min(X, Y) \stackrel{d}{=} aX$ and $\min(X, Y) \stackrel{d}{=} aX \stackrel{d}{=} bY$

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Suppose that the minimum of a pair of independent non-negative random variables X and Y has the same distribution, up to a scale factor, as the first of the two random variables. The restricted class of possible distributions for X and Y is identified. If in addition it is required that X and Y have distributions only differing by a scale factor, it is shown under mild regularity conditions that X and Y have Weibull distributions.

1. Introduction

Suppose that X and Y are independent non-negative random variables with the property that $\min(X, Y)$ has the same distribution as aX, for some $a>0$, which we write as

$$\min(X, Y) \stackrel{d}{=} aX. \tag{1}$$

What can be said about the distributions of X and Y? If X and Y are required to be identically distributed with common distribution function $F(x)$, then following Arnold (1971), we may conclude that, subject to $\lim_{x \to 0^+} F(x)/x^\alpha = c$ for some $\alpha > 0$, the common distribution must be of the Weibull type. A convenient reference for this and related results is Galambos (1975) (see also Gupta (1973)). It is not difficult to verify that if X and Y are independent identically distributed Weibull random variables then for any positive d there exists a constant $a>0$ such that $\min(X, dY) \stackrel{d}{=} aX$. Consequently, Weibull pairs provide examples of non-identically distributed solutions to (1). Do other solutions exist? In Section 2, the family of solutions to (1) is identified. It may be noted that Weibull pairs, independent and with possibly different scale parameters, satisfy the more stringent requirement that, for some a, $b>0$, we have

$$\min(X, Y) \stackrel{d}{=} aX \stackrel{d}{=} bY. \tag{2}$$

In Section 3, it is shown that subject to $\lim_{x \to 0^+} F(x)/x^\alpha = c$, and $0 < a, b < 1$, the only non-degenerate solutions to (2) are Weibull random variables. Section 3 thus represents a generalization of one of the characterizations presented in Arnold (1971) and Gupta (1973).

2. The Family of Solutions to $\min(X, Y) \overset{d}{=} aX$

Suppose that X and Y are independent non-negative random variables with distribution functions F and G respectively and suppose that they satisfy (1). If we define $\bar{F}(x) = 1 - F(x)$ and $\bar{G}(x) = 1 - G(x)$, it follows that

$$\bar{F}(x)\,\bar{G}(x) = \bar{F}(x/a), \qquad \forall\, x. \tag{3}$$

Iterating this result we find, $\forall\, n$ and $\forall\, x$ that

$$\bar{F}(x) = \bar{F}(a^n x) \prod_{i=1}^{n} \bar{G}(a^i x). \tag{4}$$

If $a > 1$, then (4) implies $\bar{F}(x) = 0$ for all $x \geqq 0$ so that no non-degenerate solutions exist. If $a = 1$, one may conclude that the support of F lies to the left of the support of G and, except for this restriction, F and G are arbitrary. For $a \in (0, 1)$ we conclude from (4), letting $n \to \infty$, that

$$\bar{F}(x) = \bar{F}(0) \prod_{i=1}^{\infty} \bar{G}(a^i x), \qquad \text{for } x \geqq 0.$$

This will be satisfied in the degenerate case where $\bar{F}(x) \equiv 0$, $x \geqq 0$. Nontrivial solutions will arise if $\bar{F}(0) = 1$ and if, for some $x > 0$, the indicated infinite product does not diverge to 0. The standard criterion for convergence of an infinite product then permits the following description of the class of solutions to (1), with $a \in (0, 1)$. Let G be a distribution function such that for some $a \in (0, 1)$,

$$\sum_{i=1}^{\infty} G(a^i x) < \infty \qquad \text{for some } x > 0. \tag{5}$$

Define

$$F(x) = 1 - \prod_{i=1}^{\infty} [1 - G(a^i x)]. \tag{6}$$

Any pair of independent random variables X and Y with respective distribution functions F and G given by (5) and (6) will satisfy (1). Any distribution function G, satisfying (5) may be used to generate such a solution to (1). If we choose G to be a Weibull distribution, the accompanying distribution function F (obtained from (6)), will also correspond to a Weibull distribution. An example of a non-Weibull solution is provided by considering the distribution function

$$\begin{aligned} G(x) &= 0, \qquad x < 0 \\ &= 1 - e^{-x^2/2}, \qquad 0 \leqq x < 1 \\ &= 1 - e^{-x+1/2}, \qquad 1 \leqq x < \infty. \end{aligned} \tag{7}$$

The accompanying distribution function $F(x)$ has the form

$$\begin{aligned} F(x) &= 0, \quad x<0 \\ &= 1-\exp\{-a^2 x^2/2(1-a^2)\}, \quad 0 \leqq x < a^{-1} \\ &= 1-\exp\left\{-\frac{a-a^{j+1}}{1-a}x+\frac{j}{2}-\frac{a^{2j+2}x^2}{2(1-a^2)}\right\}, a^{-j} \leqq x < a^{-(j+1)}, \quad j=1,2,3,\ldots. \end{aligned} \tag{8}$$

Distributions of the form (7) arise in the study of limiting distributions for waiting times in birthday problems with finite memory (see Arnold (1972)).

An example of essentially different character is the following, supplied by a referee. Let X be uniform $(1, 2)$ and Y uniform $(1/2, 1)$, independent of X. It is evident that $\min(X, Y) \stackrel{d}{=} X/2$ so that (1) is satisfied. Actually this example also satisfies (2). It is believed that the cases of most interest will be those which can serve as models of failure distributions with time of failure not bounded away from zero. It thus remains of interest to find a distribution function G satisfying (5), which is non-Weibull and has support that is not bounded away from zero, but which leads to an accompanying distribution function F with a simple closed form.

3. Solutions to $\min(X, Y) \stackrel{d}{=} aX \stackrel{d}{=} bY$

To avoid cases of minimal interest restrict attention to non-negative independent random variables X and Y satisfying (2) for some pair (a, b) such that $0<a, b<1$. If F is the distribution function of X and $\bar{F}=1-F$, it follows from (2) that

$$\bar{F}\left(\frac{x}{a}\right)=\bar{F}\left(\frac{bx}{a}\right)\bar{F}(x)$$

which may be rewritten in the form

$$\bar{F}(x)=\bar{F}(ax)\,\bar{F}(bx), \quad \forall x. \tag{9}$$

Without loss of generality, in the rest of this section attention is restricted to $x \geqq 0$. If (9) is iterated, one finds

$$\bar{F}(x)=\left[\prod_{i=0}^{n-1}\bar{F}(ab^i x)\right]\bar{F}(b^n x).$$

Consequently, provided that $\sum_{i=0}^{\infty} F(ab^i x)<\infty$, $\bar{F}$ will satisfy

$$\bar{F}(x)=\prod_{i=0}^{\infty}\bar{F}(ab^i x)>0. \tag{10}$$

As an example, consider the Weibull distribution $\bar{F}(x)=e^{-\lambda x^\alpha}$. If (10) is to be satisfied we must have

$$e^{-\lambda x^\alpha}=e^{-\lambda a^\alpha x^\alpha/(1-b^\alpha)}$$

which implies

$$1 = a^\alpha/(1-b^\alpha).$$

Thus $\bar{F}(x)=e^{-\lambda x^\alpha}$ will provide a solution to (2) provided that

$$a^\alpha = 1 - b^\alpha. \tag{11}$$

It is of interest to determine conditions under which this family of Weibull solutions exhausts the lists of solutions to (2). Some conditions are necessary since, when $a=b=1/2$, non-Weibull solutions to (2) can be constructed (see Arnold (1971) and Huang (1974)). For the case $a=b$ (the identically distributed case), Arnold (1971) showed that the assumption

$$\lim_{x\to 0^+} F(x)/x^\alpha = c > 0 \tag{12}$$

guarantees a Weibull solution. In fact, assumption (12) is adequate to ensure that all solutions are Weibull for any pair a, b satisfying (11).

By considering X^α for suitable α, the problem reduces to that of finding solutions to

$$\min(X, Y) \stackrel{d}{=} \lambda X \stackrel{d}{=} (1-\lambda)\, Y$$

for $\lambda \in (0, 1)$ [i.e. choose α such that $a^\alpha + b^\alpha = 1$ and set $\lambda = a^\alpha$]. The characterization problem may then be stated in terms of exponential distributions as follows.

Theorem 3.1. *Let X and Y be non-degenerate non-negative independent random variables satisfying*

$$\min(X, Y) \stackrel{d}{=} \lambda X \stackrel{d}{=} (1-\lambda)\, Y \tag{13}$$

for some $\lambda \in (0, 1)$. Let F be the distribution function of X and assume F has a right derivative at zero denoted by $F'(0)$. Then F must be an exponential distribution.

Proof. Observe that (13) implies $F(0)=0$ or 1. Consequently since F is non-degenerate we must have $F(0)=0$. Let $\psi(x) = -\log \bar{F}(x)$. We see that ψ has a right derivative at zero which will be denoted by $\psi'(0)$.

From (13) it follows that

$$\bar{F}(x) = \bar{F}(\lambda x)\,\bar{F}((1-\lambda)\,x), \qquad \forall x$$

[cf. Eq. (9)]. Taking logarithms one obtains

$$\psi(x) = \psi(\lambda x) + \psi((1-\lambda)\,x). \tag{14}$$

Iteration of (14) yields, for every integer n,

$$\psi(x) = \sum_{k=0}^{n} \binom{n}{k}\, \psi[\lambda^k (1-\lambda)^{n-k} x]. \tag{15}$$

Let $\beta = \max(\lambda, 1-\lambda)$ and note that $\beta < 1$. Fix $x > 0$ and let $\varepsilon > 0$. There exists $\delta > 0$ such that whenever $x < \delta$ we have $\left|\frac{\psi(x)}{x} - \psi'(0)\right| < \varepsilon$. Choose $N = N(\varepsilon, \lambda, x)$ such that $\lambda^k(1-\lambda)^{n-k} x < \delta$ whenever $n \geqq N$. (This is possible since $\lambda^k(1-\lambda)^{n-k} \leqq \beta^n$). Hence for $n \geqq N$ we get

$$\begin{aligned}
|\psi(x) - x\psi'(0)| &= \left| \sum_{k=0}^{n} \binom{n}{k} \psi[\lambda^k(1-\lambda)^{n-k} x] - x\psi'(0) \right| \\
&= \left| \sum_{k=0}^{n} \binom{n}{k} \lambda^k(1-\lambda)^{n-k} x \left\{ \frac{\psi[\lambda^k(1-\lambda)^{n-k} x]}{\lambda^k(1-\lambda)^{n-k} x} - \psi'(0) \right\} \right| \\
&\leqq \varepsilon \left| \sum_{k=0}^{n} \binom{n}{k} \lambda^k(1-\lambda)^{n-k} x \right| = \varepsilon x.
\end{aligned}$$

It thus follows since ε is arbitrary, that, $\psi(x) = x\psi'(0)$. If $\psi'(0) = 0$ then $\psi(x) \equiv 0$ and $F(x) \equiv 0$, $x \geqq 0$, which is impossible, so $\psi'(0) > 0$. Consequently $F(x)$ is an exponential distribution function with mean $[\psi'(0)]^{-1}$.

By considering powers of exponential random variables the above permits the following conclusion. If X and Y are non-degenerate non-negative random variables with $\min(X, Y) \stackrel{d}{=} aX \stackrel{d}{=} bY$ where $0 < a, b < 1$ then X and Y have Weibull distributions provided that, for that α for which $a^\alpha + b^\alpha = 1$, we have

$$\lim_{x \to 0^+} F_X(x)/x^\alpha = c.$$

By using multinomial expansions instead of binomial expansions, Theorem 3.1 can be generalized and used to prove:

Theorem 3.2. *Let $X_1, X_2, \ldots, X_m$ be independent non-degenerate non-negative random variables. Suppose that*

$$\min(X_1, X_2, \ldots, X_m) \stackrel{d}{=} a_1 X_1 \stackrel{d}{=} a_2 X_2 \stackrel{d}{=} \cdots \stackrel{d}{=} a_m X_m,$$

where $0 < a_i < 1$, $i = 1, 2, \ldots, m$. If, for that α for which $\sum_{i=1}^{m} a_i^\alpha = 1$, $\lim_{x \to 0^+} F_X(x)/x^\alpha = c$ then the X_i's have Weibull distributions.

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