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Moments of the Maximum of Normed Partial Sums of Random Variables with Multidimensional Indices

Allan Gut

Uppsala University, Department of Mathematics, Thunbergsvägen 3, S-75238 Uppsala, Sweden

Summary. For a set of i.i.d. random variables indexed by the positive integer d-dimensional lattice points we give conditions for the existence of moments of the supremum of normed partial sums, thereby obtaining results related to the Kolmogorov-Marcinkiewicz strong law of large numbers and the law of the iterated logarithm.

1. Introduction

Let $\{X_n; n \geqq 1\}$ be a sequence of independent, identically distributed (i.i.d.) random variables and let S_n, $n \geqq 1$, denote their partial sums. Further, let X be a random variable which has the same distribution as X_1 and which is independent of all other random variables. Let the common distribution function be $F(x) = P(X \leqq x)$.

Marcinkiewicz and Zygmund [13] proved that

$$E \sup_n |S_n/n|^p < \infty, \qquad p \geqq 1, \tag{1.1}$$

provided

$$E|X| \log^+ |X| < \infty \quad \text{if } p=1 \quad \text{and} \quad E|X|^p < \infty \quad \text{if } p>1. \tag{1.2}$$

A related but more general result was proved by Wiener [18] in the context of ergodic theory.

Since $\sup_n |S_n/n| \geqq |S_1/1| = |X_1|$, (1.1) implies that $E|X|^p < \infty$ and therefore (1.2) is trivially necessary for (1.1) to hold if $p>1$. Burkholder [1] proved the necessity for $p=1$ and further that (1.1), (1.2) and

$$E \sup_n |X_n/n|^p < \infty, \qquad p \geqq 1, \tag{1.3}$$

are equivalent.

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New proofs of the equivalences of (1.1)–(1.3) were given in [12] and (for $P(X \geqq 0)=1$) in [5]. A variation of this result has been established in [2]. Gabriel has generalized the result to the case of i.i.d. random variables with multidimensional indices for the case $p=1$, (see [3] and [4], Theorem 5).

Teicher [16] studied the problem for more general normalizing sequences and, finally, Siegmund [14] and Teicher [17] proved that, if $EX=0$, then

$$EX^2 \cdot (l_2(|X|))^{-1} \cdot l(|X|) < \infty \quad \text{if } p=2 \quad \text{and} \quad E|X|^p \quad \text{if } p>2 \tag{1.4}$$

$$E \sup_n |X_n/\sqrt{n\, l_2(n)}|^p < \infty \tag{1.5}$$

$$E \sup_n |S_n/\sqrt{n\, l_2(n)}|^p < \infty \tag{1.6}$$

are all equivalent, Siegmund proving it for the integers 2, (3, 4, ...) and Teicher for $p>2$.

Here and in the following $l(x)=\max\{1, \log x\}$ and $l_2(x)=l(l(x)) =\max\{1, \log\log x\}$.

The main purpose of this paper is to generalize the last result to random variables with multidimensional indices. The proofs in [14] and [17] use sharp tail estimates of the partial sums (exponential bounds), whereas our approach is different, making use of an extension of a result by Hoffmann-Jørgensen, [8], Corollary 3.4, which will provide a general result for proving implications of the type (1.3)⇒(1.1) and (1.5)⇒(1.6). This also gives a new proof for the case $d=1$. With this, more general, method we also prove a result corresponding to the normalizing sequence $\{n^{1/r}\}_{n=1}^{\infty}$, $0<r<2$, which, for $r \neq 1$ relates to the Marcinkiewicz strong law (see e.g. Loève [11], pp. 242–243) in the same way as the case $r=1$ relates to the classical Kolmogorov strong law. Note also, that while martingale methods work if $r=1$, this is not the case if $r \neq 1$.

After some preliminaries in Sect. 2, the results are formulated in Sect. 3 and proved in Sect. 4 and 5. Sect. 6 finally, contains some remarks on how the results relate to the law of large numbers and the law of the iterated logarithm and on the proofs.

2. Preliminaries

We follow the notation of [15, 6], and [7]. Let Z_+^d, $d \geqq 1$, be the positive integer d-dimensional lattice points with coordinate-wise partial ordering, $\prec$. Points in Z_+^d are denoted by $\mathbf{m}$, $\mathbf{n}$ etc. (We use, however, m, n etc. if $d=1$.) For $\mathbf{n} \in Z_+^d$ we define $|\mathbf{n}|$ to be the product of the coordinates n_i, $1 \leqq i \leqq d$, and $\mathbf{n} \to \infty$ is to be interpreted as $\min_{1 \leqq i \leqq d} n_i \to \infty$.

Also, $[x]$ denotes the integral part of x and $I\{\cdot\}$ denotes the indicator function of the set in braces.

Following Smythe [15], let $d(x)=\operatorname{card}\{\mathbf{n} \in Z_+^d; |\mathbf{n}|=[x]\}$ and $M(x) =\operatorname{card}\{\mathbf{n} \in Z_+^d; |\mathbf{n}| \leqq [x]\} = \sum_{k=1}^{[x]} d(k)$ for $x \geqq 1$ and $M(x)=1$ for $0 \leqq x \leqq 1$. Then

$$M(x)=O(x(\log x)^{d-1}) \quad \text{and} \quad d(x)=o(x^\delta)\ \forall\, \delta>0 \quad \text{as } x \to \infty.$$

Lemma 2.1. *For any random variable X, the following are equivalent:*

$$E|X| \cdot (l(|X|))^d < \infty \tag{2.1}$$

$$\sum_{m=1}^{\infty} \sum_{\mathbf{k}} P(|X| \geqq m \cdot |\mathbf{k}|) < \infty. \tag{2.2}$$

Proof. By considering Z_+^{d+1} with elements $\mathbf{k}^* = (k_1, \ldots, k_d, m)$ we note that (2.2) can be written

$$\sum_{\mathbf{k}^*} P(|X| \geqq |\mathbf{k}^*|) < \infty. \tag{2.3}$$

The conclusion now follows from [6], Lemma 2.1 (cf. also [15], Lemma 2.1).

The following result is an extension of Corollary 3.4 of [8] to the d-dimensional case.

Lemma 2.2. *Let E be a Banach space with norm $\|\cdot\|$ and let $\{Y_{\mathbf{n}}; \mathbf{n} \in Z_+^d\}$ be independent E-valued random variables. Further, let $\{a_{\mathbf{n}}; \mathbf{n} \in Z_+^d\}$ be a set of positive real numbers which are functions of $|\mathbf{n}|$ only and such that $a_{\mathbf{n}} \leqq a_{\mathbf{m}}$ if $|\mathbf{n}| \leqq |\mathbf{m}|$. Set*

$$U_{\mathbf{n}} = a_{\mathbf{n}}^{-1} \cdot \sum_{\mathbf{k} \prec \mathbf{n}} Y_{\mathbf{k}}, \qquad V = \sup_{\mathbf{n}} \|U_{\mathbf{n}}\|, \qquad W = \sup_{\mathbf{n}} \|a_{\mathbf{n}}^{-1} \cdot Y_{\mathbf{n}}\|.$$

and suppose that $V < \infty$ a.s. Then $W < \infty$ a.s. and if $EW^p < \infty$ for some p such that $0 < p < \infty$, then $EV^p < \infty$.

Remarks. 1. For $d=1$ and $a_n \equiv 1$, this is Theorem 3.1 of Hoffmann-Jørgensen [8], and for $d=1$ and general nondecreasing sequences $\{a_n\}_{n=1}^{\infty}$ it is Corollary 3.4 of [8].

2. Since $V \leqq \sup_{|\mathbf{n}| \leqq n_0} \|U_{\mathbf{n}}\| + \sup_{|\mathbf{n}| > n_0} \|U_{\mathbf{n}}\|$ it is obvious that the lemma remains true if the monotonicity only holds for $|\mathbf{n}| >$ some n_0.

The proof is an appropriate modification of those given in [8] for Theorem 3.1 and Corollary 3.4.

Proof. Since $W \leqq 2^d V$ we always have $W < \infty$ a.s.

First assume that $a_{\mathbf{n}} \equiv 1$ and that $Y_{\mathbf{n}}$ has a symmetric distribution for all $\mathbf{n} \in Z_+^d$. The Lévy inequality for symmetric Banach space valued random variables indexed by Z_+^d now becomes

$$P(\sup_{\mathbf{n}} \|U_{\mathbf{n}}\| \geqq \lambda) \leqq 2^d \sup_{\mathbf{n}} P(\|U_{\mathbf{n}}\| \geqq \lambda), \qquad \lambda > 0. \tag{2.4}$$

This is seen by combining the proof in Kahane [9], p. 12 for the case $d=1$ with the method of Gabriel [3], where a related problem is treated for general d in the real valued case.

Therefore, the inequality on the bottom of p. 164 of [8] is modified into

$$P(V \geqq 2t + s) \leqq 2^d \cdot P(W \geqq s) + 2^{d+2} \cdot (P(V \geqq t))^2. \tag{2.5}$$

The conclusion now follows exactly as in [8].

For the general case we proceed roughly as in [8], p. 167. For each $\mathbf{j}$, let us define

$$\tilde{Y}_{\mathbf{jn}} = \begin{cases} a_{\mathbf{n}}^{-1} \cdot Y_{\mathbf{j}} & \text{if } \mathbf{j} \prec \mathbf{n} \\ 0 & \text{otherwise} \end{cases} \tag{2.6}$$

and $\tilde{Y}_{\mathbf{j}} = \{\tilde{Y}_{\mathbf{jn}}; \mathbf{n} \in Z_+^d\}$.

By defining $\tilde{S}_{\mathbf{j}} = \sum_{\mathbf{k} \prec \mathbf{j}} \tilde{Y}_{\mathbf{k}}$ it follows that $\tilde{S}_{\mathbf{j}} = \{\tilde{S}_{\mathbf{jn}}; \mathbf{n} \in Z_+^d\}$, where

$$\tilde{S}_{\mathbf{jn}} = \begin{cases} a_{\mathbf{n}}^{-1} \cdot \sum_{\mathbf{k} \prec \mathbf{j}} Y_{\mathbf{k}} & \text{if } \mathbf{j} \prec \mathbf{n} \\ U_{\mathbf{n} \wedge \mathbf{j}} & \text{otherwise,} \end{cases} \tag{2.7}$$

and where $\mathbf{n} \wedge \mathbf{j}$ denotes coordinate-wise minimum. The conclusion now follows from the first part of the lemma just as in [8], p. 167. We omit the details.

Remark. For our purposes we need Lemma 2.2 in the real valued case only. The proof for general $\{a_{\mathbf{n}}\}$, however, is based on the validity of the case $a_{\mathbf{n}} \equiv 1$ for Banach space valued random variables (cf. [8]) and this is why the lemma has been given in the more abstract setting.

3. Results

In this section we state two theorems, the proofs of which are given in Sections 4 and 5 respectively.

Theorem 3.1. *Let X and $\{X_{\mathbf{n}}; \mathbf{n} \in Z_+^d\}$ be i.i.d. random variables with $EX=0$. Let $p \geqq 2$. The following statements are equivalent:*

$$EX^2 \cdot (l_2(|X|))^{-1} \cdot (l(|X|))^d < \infty \quad \textit{if } p=2 \quad \textit{and} \quad E|X|^p < \infty \quad \textit{if } p>2. \tag{3.1}$$

$$E \sup_{\mathbf{n}} \left| \frac{X_{\mathbf{n}}}{\sqrt{|\mathbf{n}| \cdot l_2(|\mathbf{n}|)}} \right|^p < \infty. \tag{3.2}$$

$$E \sup_{\mathbf{n}} \left| \frac{S_{\mathbf{n}}}{\sqrt{|\mathbf{n}| \cdot l_2(|\mathbf{n}|)}} \right|^p < \infty. \tag{3.3}$$

For $d=1$, the above are also equivalent to

$$\sup_N E \left| \frac{X_N}{\sqrt{N \cdot l_2(N)}} \right|^p < \infty \tag{3.4}$$

$$\sup_N E \left| \frac{S_N}{\sqrt{N \cdot l_2(N)}} \right|^p < \infty, \tag{3.5}$$

where $\sup_N$ means that the supremum is taken over all stopping rules (cf. [12, 2]).

For $d=1$ and $p=2$, (3, 4, ...) the equivalence of (3.1)–(3.3) has been proved by Siegmund [14] and for $d=1$ and $p>2$ by Teicher [17], in both cases with a method which is different from the one presented below. The fact that (3.1)–(3.3) are equivalent to (3.4) and (3.5) for $d=1$ is new.

As mentioned in the introduction, several authors have treated the problem of relating moments of X to moments of $\sup_n |n^{-1} \cdot X_n|$ and $\sup_n |n^{-1} \cdot S_n|$. Lemma 2.2 makes it possible to derive the following generalization.

Theorem 3.2. *Let X and $\{X_{\mathbf{n}}; \mathbf{n} \in Z_+^d\}$ be i.i.d. random variables and suppose that $EX=0$ whenever it is finite. Let $0<r<2$ and $p \geqq r$. The following statements are equivalent:*

$$E|X|^p \cdot (l(|X|))^d < \infty \quad \text{if } p=r \quad \text{and} \quad E|X|^p < \infty \quad \text{if } p>r. \tag{3.6}$$

$$E \sup_{\mathbf{n}} \left| \frac{X_{\mathbf{n}}}{|\mathbf{n}|^{1/r}} \right|^p < \infty. \tag{3.7}$$

$$E \sup_{\mathbf{n}} \left| \frac{S_{\mathbf{n}}}{|\mathbf{n}|^{1/r}} \right|^p < \infty. \tag{3.8}$$

For $d=1$ the above are also equivalent to

$$\sup_N E \left| \frac{X_N}{N^{1/r}} \right|^p < \infty \tag{3.9}$$

$$\sup_N E \left| \frac{S_N}{N^{1/r}} \right|^p < \infty, \tag{3.10}$$

where, as before, the supremum is taken over all stopping rules.

Remark. It is in fact no loss of generality to assume that $EX=0$ if $r \leqq 1 \leqq p$. This is seen as follows:

Consider e.g. (3.8) and suppose that $EX=\mu \neq 0$. Then

$$\begin{aligned}\sup_{\mathbf{n}} \big| |\mathbf{n}|^{-1/r} \cdot S_{\mathbf{n}} \big| &\leqq \sup_{\mathbf{n}} \big| |\mathbf{n}|^{-1/r} \cdot (S_{\mathbf{n}} - |\mathbf{n}|\, \mu) \big| + \mu \sup_{\mathbf{n}} |\mathbf{n}|^{1-1/r} \\ &\leqq \sup_{\mathbf{n}} \Big| |\mathbf{n}|^{-1/r} \cdot \sum_{\mathbf{k} \prec \mathbf{n}} (X_{\mathbf{k}} - \mu) \Big| + \mu,\end{aligned}$$

which shows that if (3.8) holds for $EX=0$ it also holds for $EX \neq 0$.

The case $d=1$, $r=1$, has been studied by several authors. Marcinkiewicz and Zygmund [13] proved that (3.6)$\Rightarrow$(3.8). Burkholder [1] proved that (3.6)–(3.8) are equivalent if $p=1$. Further, when $p=1$, McCabe and Shepp [12] proved the equivalence of (3.6)–(3.10) partly with a different method, Davis [2] proved the equivalence of (3.6), (3.9), and (3.10) and, for $P(X \geqq 0)=1$, Gundy [5], proved the equivalence of (3.6) and (3.8).

For $d \geqq 2$, $r=p=1$, Gabriel [3] and [4], Theorem 5 has demonstrated the equivalence of (3.6)–(3.8).

4. Proof of Theorem 3.1.

(3.1)$\Rightarrow$(3.2). Set $a_{\mathbf{n}}=\sqrt{|\mathbf{n}|\cdot l_2(|\mathbf{n}|)}$ and $b_{\mathbf{n}}=\sqrt{|\mathbf{n}|\cdot(l_2(|\mathbf{n}|))^{-1}}$, $\mathbf{n}\in Z_+^d$ and define

$$X'_{\mathbf{n}}=X_{\mathbf{n}}\cdot I\{|X_{\mathbf{n}}|\leqq b_{\mathbf{n}}\}\quad\text{and}\quad X''_n=X_{\mathbf{n}}-X'_{\mathbf{n}}. \tag{4.1}$$

Lemma 4.1. *Let X and $\{X_{\mathbf{n}};\mathbf{n}\in Z_+^d\}$ be i.i.d. random variables. If $X'_{\mathbf{n}}$ and $X''_{\mathbf{n}}$ are defined as in* (4.1), *then*

$$\sum_{\mathbf{n}} a_{\mathbf{n}}^{-p}\cdot E|X''_{\mathbf{n}}|^p\leqq\begin{cases} c\cdot EX^2\cdot(l_2(|X|))^{1-p}\cdot(l(|X|))^{d-1} & \text{if } 0<p<2\\ c\cdot EX^2\cdot(l_2(|X|))^{-1}\cdot(l(|X|))^{d} & \text{if } p=2\\ c\cdot E|X|^p & \text{if } p>2,\end{cases}$$

where c is a constant.

Proof. For convenience we write $a_j(b_j)$ instead of $a_{(j,1,\dots,1)}(b_{(j,1,\dots,1)})$. We first show that, for large j,

$$\sum_{k=1}^{j}(k\cdot l_2(k))^{-p/2}\cdot d(k)\leqq\begin{cases} c\cdot b_j^{2-p}\cdot(l_2(b_j))^{1-p}\cdot(l(b_j))^{d-1} & \text{if } 0<p<2\\ c\cdot(l_2(b_j))^{-1}\cdot(l(b_j))^{d} & \text{if } p=2\\ c & \text{if } p>2.\end{cases} \tag{4.2}$$

Let $p>2$. Since $d(x)=o(x^\delta)\ \forall\,\delta>0$ as $x\to\infty$ we have $d(j)<j^\delta$ if $j>j_0$. Choose δ such that $0<\delta<p/2-1$ to obtain

$$\sum_{k=1}^{j}(k\cdot l_2(k))^{-p/2}\cdot d(k)\leqq\sum_{k=1}^{j_0}k^{-p/2}\cdot d(k)+\sum_{k=j_0+1}^{j}k^{-(p/2)+\delta}$$
$$\leqq M(j_0)+\sum_{k=1}^{\infty}k^{-(p/2)+\delta}<\infty,$$

since $p/2-\delta>1$.

This proves (4.2) for $p>2$.

Now, let $p=2$. Summation by parts together with the facts that $M(x)=O(x(\log x)^{d-1})$ as $x\to\infty$ and

$$\sum_{k=1}^{j}(k\,l_2(k))^{-1}\cdot(l(k))^{d-1}=O((\operatorname{loglog} j)^{-1}\cdot(\log j)^d)\quad\text{as } j\to\infty \tag{4.3}$$

(see [14], p. 528 for the case $d=1$) yield

$$\sum_{k=1}^{j}(k\,l_2(k))^{-p/2}\cdot d(k)=\sum_{k=1}^{j}(k\,l_2(k))^{-1}\cdot d(k)$$
$$=(j\cdot l_2(j))^{-1}\cdot M(j+1)-1$$
$$+\sum_{k=1}^{j}((k\,l_2(k))^{-1}-((k+1)\,l_2(k+1))^{-1})\cdot M(k)$$
$$\leqq(j\cdot\operatorname{loglog} j)^{-1}\cdot M(j+1)$$
$$+\sum_{k=1}^{j}k^{-2}\cdot(l_2(k))^{-1}\cdot M(k)\sim(j\cdot\operatorname{loglog} j)^{-1}\cdot M(j+1)$$
$$+\sum_{k=1}^{j}(k\,l_2(k))^{-1}\cdot(l(k))^{d-1}$$

$$=O((\log\log j)^{-1}\cdot(\log j)^{d-1})+O((\log\log j)^{-1}\cdot(\log j)^{d})$$
$$=O((\log\log j)^{-1}\cdot(\log j)^{d})$$
$$=O((\log\log b_j)^{-1}\cdot(\log b_j)^{d}),\quad \text{as } j\to\infty.$$

Thus (4.2) also holds if $p=2$.

Finally, if $0<p<2$, (4.2) follows by the same procedure as for the case $p=2$ together with the fact that

$$\sum_{k=1}^{j}(k\,l_2(k))^{-p/2}\cdot(l(k))^{d-1}=O((\log\log j)^{-p/2}\cdot j^{1-p/2}\cdot(\log j)^{d-1})$$
$$=O(b_j^{2-p}\cdot(\log\log b_j)^{1-p}\cdot(\log b_j)^{d-1})\quad \text{as } j\to\infty.$$

This concludes the proof of (4.2) and we return to the proof of the lemma. We have

$$\sum_{\mathbf{n}} a_{\mathbf{n}}^{-p}\cdot E|X_{\mathbf{n}}''|^p=\sum_{k=1}^{\infty}d(k)\cdot a_k^{-p}\sum_{j=k}^{\infty}\int_{b_j<|x|\leqq b_{j+1}}|x|^p\,dF$$
$$=\sum_{j=1}^{\infty}\left(\sum_{k=1}^{j}(k\cdot l_2(k))^{-p/2}\cdot d(k)\right)\cdot\int_{b_j<|x|\leqq b_{j+1}}|x|^p\,dF.$$

The conclusion follows by inserting the estimates obtained in (4.2) in the last expression. (Cf. also [14], p. 528.)

The implication (3.1)$\Rightarrow$(3.2) now is immediate, because of

$$\begin{aligned}&E\sup_{\mathbf{n}}|a_{\mathbf{n}}^{-1}\cdot X_{\mathbf{n}}'|^p\leqq\sup_{\mathbf{n}}(a_{\mathbf{n}}^{-1}\cdot b_{\mathbf{n}})^p\leqq 1\\&E\sup_{\mathbf{n}}|a_{\mathbf{n}}^{-1}\cdot X_{\mathbf{n}}''|^p\leqq\sum_{\mathbf{n}}a_{\mathbf{n}}^{-p}\cdot E|X_{\mathbf{n}}''|^p.\end{aligned}\tag{4.4}$$

This terminates the first step of the proof.

(3.2)$\Rightarrow$(3.1). Since $\sup_{\mathbf{n}}|X_{\mathbf{n}}/\sqrt{|\mathbf{n}|\,l_2(|\mathbf{n}|)}|\geqq|X_1|$ it follows that (3.2)$\Rightarrow E|X|^p<\infty$. If $p>2$ there is nothing more to prove, so suppose that $p=2$.

For $d=1$ the implication has been proved by Siegmund [14]. Since the proof below is based on induction on the dimension we present a proof for the case $d=1$ which is related to the proof given in Gabriel [3, 4] for the implication (3.7)$\Rightarrow$(3.6) with $p=r=1$.

Since the conclusion is trivial for uniformly bounded random variables we may assume without loss of generality that X is unbounded. We may also assume that $P(|X|<1)>0$. Recall that we already have $EX^2<\infty$.

Set $A=\prod_{j=1}^{\infty}P(X^2\leqq j\cdot l_2(j))$. Then, from the well-known fact that for a sequence of nonnegative real numbers $\{\alpha_n\}_{n=1}^{\infty}$, $\prod_{n=1}^{\infty}(1-\alpha_n)$ converges if and only if $\sum_{n=1}^{\infty}\alpha_n$ converges, it follows that $A>0$ if and only if $\sum_{j=1}^{\infty}P(X^2>j\cdot l_2(j))<\infty$. Since this sum is majorized by $\sum_{j=1}^{\infty}P(X^2>j)\leqq EX^2<\infty$, we conclude that

$$A>0.\tag{4.5}$$

By using the argument of Gabriel [3] and [4], pp. 892–893, we obtain, for $d=1$, and $m=1, 2, \ldots$

$$P\left(\sup_n \frac{X_n^2}{n \cdot l_2(n)} > m\right) \geqq \sum_{n=1}^{\infty} P(X^2 \geqq mn \cdot l_2(n)) \prod_{j=1}^{n-1} P(|X_j| \leqq \sqrt{j \cdot l_2(j)})$$
$$\geqq A \cdot \sum_{n=1}^{\infty} P(X^2 \geqq mn \cdot l_2(n))$$

and thus

$$\infty > E \sup_n \frac{X_n^2}{n \cdot l_2(n)} \geqq \sum_{m=1}^{\infty} P\left(\sup_n \frac{X_n^2}{n \cdot l_2(n)} > m\right)$$
$$\geqq A \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} P(X^2 > mn \cdot l_2(n))$$
$$\geqq A \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} P(X^2 \cdot (l_2(|X|))^{-1} > mn).$$

It now follows from Lemma 2.1 that $EX^2 \cdot (l_2(|X|))^{-1} \cdot l(|X|) < \infty$, which proves the conclusion for $d=1$.

Next, let $d \geqq 2$ and suppose that the conclusion is known to hold for $d-1$ dimensions. Since

$$\infty > E \sup_{\mathbf{n}} \left(\frac{X_{\mathbf{n}}^2}{|\mathbf{n}| \cdot l_2(|\mathbf{n}|)}\right) \geqq E \sup_{\substack{n_1, n_2, \ldots, n_{d-1} \\ n_d = 1}} \left(\frac{X_{\mathbf{n}}^2}{|\mathbf{n}| \cdot l_2(|\mathbf{n}|)}\right)$$

it follows from the induction hypothesis that $EX^2 \cdot (l_2(|X|))^{-1} \cdot (l(|X|))^{d-1} < \infty$.

According to [6], Lemma 2.1 (with $\alpha = r = \varepsilon = 1$ and $m=0$) the moment condition is equivalent to

$$\sum_{j=1}^{\infty} d(j) \cdot P(X^2 \cdot (l_2(|X|))^{-1} > j) < \infty,$$

which in turn is equivalent to

$$\sum_{j=1}^{\infty} d(j) \cdot P(X^2 > j \cdot l_2(j)) < \infty, \tag{4.6}$$

because $P(X^2 \cdot (l_2(|X|))^{-1} > j) \leqq P(X^2 > j \cdot l_2(j)) \leqq P(X^2 \cdot (l_2(|X|))^{-1} > j/2)$.

Define $Y_j = \sup_{|\mathbf{k}|=j} X_{\mathbf{k}}^2$, $j=1, 2, \ldots$. The Y_j:s are independent (but not identically distributed) random variables. We wish to show that

$$B = \prod_{j=1}^{\infty} P(Y_j \leqq j \cdot l_2(j)) > 0, \tag{4.7}$$

or, equivalently, (cf. above) that

$$\sum_{j=1}^{\infty} P(Y_j > j \cdot l_2(j)) < \infty. \tag{4.8}$$

To prove this we first note that

$$P(Y_j > j \cdot l_2(j)) = 1 - (P(X^2 \leqq j \cdot l_2(j)))^{d(j)} = 1 - (1 - P(X^2 > j \cdot l_2(j)))^{d(j)}.$$

Furthermore, since (4.6) in particular implies that

$$d(j) \cdot P(X^2 > j \cdot l_2(j)) \to 0 \quad \text{as } j \to \infty$$

we obtain

$$\tfrac{1}{2} \cdot d(j) \cdot P(X^2 > j \cdot l_2(j)) \leqq P(Y_j > j \cdot l_2(j)) \leqq d(j) \cdot P(X^2 > j \cdot l_2(j)), \tag{4.9}$$

the first inequality only being true for $j \geqq$ some j_0, where j_0 is large.

Consequently, $B > 0$ if and only if

$$\sum_{j=1}^{\infty} d(j) \cdot P(X^2 > j \cdot l_2(j)) < \infty.$$

In view of the computations which lead to (4.6) this sum is finite if and only if $EX^2 \cdot (l_2(|X|))^{-1} \cdot l(|X|)^{d-1} < \infty$. However, this moment condition was already known to hold because of the induction hypothesis.

Therefore,

$$\begin{aligned} P\left(\sup_{\mathbf{k}} \frac{X_{\mathbf{k}}^2}{|\mathbf{k}| \cdot l_2(|\mathbf{k}|)} > m\right) &= P\left(\sup_{j} \sup_{|\mathbf{k}|=j} \frac{X_{\mathbf{k}}^2}{|\mathbf{k}| \cdot l_2(|\mathbf{k}|)} > m\right) \\ &= P\left(\sup_{j} \frac{Y_j}{j \cdot l_2(j)} > m\right) \geqq B \cdot \sum_{j=1}^{\infty} P(Y_j > m j\, l_2(j)) \\ &\geqq \frac{B}{2} \cdot \sum_{j=j_0}^{\infty} d(j) \cdot P(X^2 > m j\, l_2(j)), \end{aligned}$$

where the last inequality follows from (4.9). Finally,

$$\begin{aligned} \infty > E \sup_{\mathbf{k}} \frac{X_{\mathbf{k}}^2}{|\mathbf{k}| \cdot l_2(|\mathbf{k}|)} &\geqq \frac{B}{2} \sum_{m=1}^{\infty} \sum_{j=j_0}^{\infty} d(j) \cdot P(X^2 > m j\, l_2(j)) \\ &\geqq \frac{B}{2} \sum_{m=1}^{\infty} \sum_{j=j_0}^{\infty} d(j) \cdot P(X^2 \cdot (l_2(|X|))^{-1} > m j) \\ &= \frac{B}{2} \sum_{m=1}^{\infty} \sum_{\{k;\, |\mathbf{k}| \geqq j_0\}} P(X^2 \cdot (l_2(|X|))^{-1} > m \cdot |\mathbf{k}|) \\ &\geqq \frac{B}{2} \sum_{m=1}^{\infty} \sum_{\mathbf{k}} P(X^2 \cdot (l_2(|X|))^{-1} > m|\mathbf{k}|) \\ &\quad - \frac{B}{2} \cdot \sum_{m=1}^{\infty} M(j_0)\, P(X^2 \cdot (l_2(|X|))^{-1} > m) \\ &\geqq \frac{B}{2} \sum_{m=1}^{\infty} \sum_{\mathbf{k}} P(X^2 \cdot (l_2(|X|))^{-1} > m|\mathbf{k}|) - \frac{B}{2} \cdot M(j_0) \cdot EX^2 \cdot (l_2(|X|))^{-1}. \end{aligned}$$

An application of Lemma 2.1 now shows that (3.1) holds and so the proof of this part is complete.

(3.2)⇒(3.3). We already know that (3.2)⇔(3.1). In order to apply Lemma 2.2 we set $Y_{\mathbf{n}}=X_{\mathbf{n}}$ and $a_{\mathbf{n}}=\sqrt{|\mathbf{n}|\cdot l_2(|\mathbf{n}|)}$. Then, $U_{\mathbf{n}}=(|\mathbf{n}|\cdot l_2(|\mathbf{n}|))^{-1/2}\cdot S_{\mathbf{n}}$,

$$V=\sup_{\mathbf{n}}|(|\mathbf{n}|\cdot l_2(|\mathbf{n}|))^{-1/2}\cdot S_{\mathbf{n}}| \quad \text{and} \quad W=\sup_{\mathbf{n}}|(|\mathbf{n}|\cdot l_2(|\mathbf{n}|))^{-1/2}\cdot X_{\mathbf{n}}|.$$

The law of the iterated logarithm (see [19], p. 280) guarantees the a.s. finiteness of V, since (3.1) holds. Thus, assuming (3.2) i.e. that $EW^p<\infty$ it follows that $EV^p<\infty$, i.e. it follows that (3.3) holds.

(3.3)⇒(3.2). Immediate, since $W\leqq 2^d V$.

This concludes the proof of the theorem if $d\geqq 2$. We therefore assume that $d=1$ in order to complete the proof also for this case. This is accomplished by suitable modifications of the proof given in [12] for the case $a_n=n$.

(3.2)⇒(3.4). Immediate, since for every rule N,

$$|(N\cdot l_2(N))^{-1/2}\cdot X_N|\leqq \sup_n |(n\cdot l_2(n))^{-1/2}\cdot X_n|.$$

(3.3)⇒(3.5). Similarly, since

$$|(N\cdot l_2(N))^{-1/2}\cdot S_N|\leqq \sup_n |(n\cdot l_2(n))^{-1/2}\cdot S_n|.$$

(3.4)⇒(3.1). By choosing $N\equiv 1$ we find that (3.4)⇒$E|X|^p<\infty$ and the proof is complete unless $p=2$. Furthermore, like in the proof of (3.2)⇒(3.1) we note that it is no loss of generality to assume that X is unbounded and that $P(|X|<1)>0$. Define

$$N=\inf\{n;\,|X_n|>\sqrt{n\cdot l_2(n)}\},\qquad N=\infty \qquad \text{if no such } n \text{ exists.} \tag{4.11}$$

Since we already know that $EX^2<\infty$ it follows that (4.5) holds. Furthermore,

$$P(N=n+1)=P(N=n)\cdot\frac{P(|X_n|\leqq a_n)\cdot P(|X_{n+1}|>a_{n+1})}{P(|X_n|>a_n)}, \tag{4.12}$$

which, after iteration, implies that

$$\frac{P(N=n)}{P(|X_n|>a_n)}\geqq A\cdot\frac{P(N=1)}{P(|X|>1)}=A>0, \tag{4.13}$$

by (4.5) and the fact that X is unbounded.

An application of (4.3) shows that

$$\sum_{\{n;\,|x|>\sqrt{n\,l_2(n)}\}}(n\cdot l_2(n))^{-1}\geqq \sum_{\{n;\,n\leqq x^2/2\cdot l_2(|x|)\}}(n\cdot l_2(n))^{-1}$$
$$=O((\log\log|x|)^{-1}\cdot\log|x|)\quad \text{as } |x|\to\infty. \tag{4.14}$$

Finally, independence, (4.13) and (4.14) yield

$$\infty > E((N \cdot l_2(N))^{-1} \cdot X_N^2 \cdot I\{N<\infty\}) = \sum_{n=1}^{\infty} a_n^{-2} \cdot E(X_n^2 | N=n) \cdot P(N=n)$$
$$= \sum_{n=1}^{\infty} a_n^{-2} \cdot E(X_n^2 \,|\, |X_n| > a_n) \cdot P(N=n)$$
$$= \sum_{n=1}^{\infty} a_n^{-2} \cdot (P(|X_n| > a_n))^{-1} \cdot \int_{|x|>a_n} x^2\, dF(x) \cdot P(N=n)$$
$$\geqq A \cdot \sum_{n=1}^{\infty} a_n^{-2} \cdot \int_{|x|>a_n} x^2\, dF(x) = A \cdot \int_{-\infty}^{\infty} x^2 \Big(\sum_{\{n: |x|>a_n\}} a_n^{-2}\Big) \cdot dF(x)$$
$$= A \cdot \int_{-\infty}^{\infty} x^2 \Big(\sum_{\{n, |x| > \sqrt{n\, l_2(n)}\}} (n \cdot l_2(n))^{-1}\Big) \cdot dF(x)$$
$$\geqq \text{const.} \int_{-\infty}^{\infty} x^2 (l_2(|x|))^{-1} \cdot l(|x|)\, dF(x)$$

which proves the assertion.

(3.5)⇒(3.1). By choosing $N \equiv 1$ we first obtain $(3.5) \Rightarrow E|X|^p < \infty$, which completes the proof unless $p=2$. For this case we define N as in (4.11) and impose the usual conditions on X. We have

$$E((N\, l_2(N))^{-1} \cdot X_N^2 \cdot I\{N<\infty\})$$
$$\leqq 2\, E((N\, l_2(N))^{-1} \cdot S_N^2 \cdot I\{N<\infty\}) + 2\, E((N\, l_2(N))^{-1} \cdot S_{N-1}^2 \cdot I\{N<\infty\}). \quad (4.15)$$

The first term on the RHS of (4.15) is finite by assumption and we now want to show that the second term is finite too.

Towards this end we note that

$$E(X_k^2 \,|\, |X_k| \leqq \sqrt{k\, l_2(k)}) = \frac{EX_k^2 \cdot I\{|X_k| \leqq \sqrt{k\, l_2(k)}\}}{P(|X_k| \leqq \sqrt{k\, l_2(k)})} \leqq \frac{EX^2}{P(|X| \leqq 1)} \quad (4.16)$$

and, since $EX=0$, that

$$|E(X_k \,|\, |X_k| \leqq \sqrt{k\, l_2(k)})| = \frac{|E(X_k \cdot I\{|X_k| > \sqrt{k\, l_2(k)}\})|}{P(|X_k| \leqq \sqrt{k\, l_2(k)})}$$
$$\leqq \frac{EX^2}{\sqrt{k\, l_2(k)} \cdot P(|X| \leqq 1)} \leqq \frac{EX^2}{\sqrt{k} \cdot P(|X| \leqq 1)} \quad (4.17)$$

Further, if $i \neq j$, then

$$E(X_i X_j \,|\, |X_i| \leqq \sqrt{i\, l_2(i)}, |X_j| \leqq \sqrt{j\, l_2(j)})$$
$$= \frac{E(X_i X_j \cdot I\{|X_i| \leqq \sqrt{i\, l_2(i)}, |X_j| \leqq \sqrt{j\, l_2(j)}\})}{P(\{|X_i| \leqq \sqrt{i\, l_2(i)}\} \cap \{|X_j| \leqq \sqrt{j\, l_2(j)}\})}$$
$$= \frac{E(X_i \cdot I\{|X_i| \leqq \sqrt{i\, l_2(i)}\} \cdot X_j \cdot I\{|X_j| \leqq \sqrt{j\, l_2(j)}\})}{P(|X_i| \leqq \sqrt{i\, l_2(i)}) \cdot P(|X_j| \leqq \sqrt{j\, l_2(j)})}$$
$$= \frac{E(X_i \cdot I\{|X_i| \leqq \sqrt{i\, l_2(i)}\})}{P(|X_i| \leqq \sqrt{i\, l_2(i)})} \cdot \frac{E(X_j \cdot I\{|X_j| \leqq \sqrt{j\, l_2(j)}\})}{P(|X_j| \leqq \sqrt{j\, l_2(j)})}.$$

It follows from (4.17) that

$$|E(X_i X_j \,|\, |X_i| \leqq \sqrt{i\, l_2(i)},\, |X_j| \leqq \sqrt{j\, l_2(j)})| \leqq \frac{(EX^2)^2}{\sqrt{ij}\,(P(|X| \leqq 1))^2} \qquad \text{if } i \neq j. \tag{4.18}$$

Finally, independence (4.16) and (4.18) yield

$$\begin{aligned}
&E((N \cdot l_2(N))^{-1} \cdot S_{N-1}^2 \cdot I\{N<\infty\}) \\
&= \sum_{n=1}^{\infty} a_n^{-2} \cdot E(S_{n-1}^2 \,|\, N=n) \cdot P(N=n) \\
&= \sum_{n=1}^{\infty} a_n^{-2} \cdot \left(\sum_{k=1}^{n-1} E(X_k^2 \,|\, |X_k| \leqq a_k) \right. \\
&\quad \left. + \sum\sum_{i \neq j} E(X_i X_j \,|\, |X_i| \leqq a_i,\, |X_j| \leqq a_j) \right) \cdot P(N=n) \\
&\leqq \sum_{n=1}^{\infty} a_n^{-2} \cdot \left((n-1) \frac{EX^2}{P(|X| \leqq 1)} + \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \left(\frac{EX^2}{P(|X| \leqq 1)} \right)^2 \cdot \frac{1}{\sqrt{ij}} \right) \cdot P(N=n) \\
&\leqq \frac{EX^2}{P(|X| \leqq 1)} \cdot \sum_{n=1}^{\infty} (l_2(n))^{-1} \cdot P(N=n) \\
&\quad + \left(\frac{EX^2}{P(|X| \leqq 1)} \right)^2 \cdot \sum_{n=1}^{\infty} (n\, l_2(n))^{-1} \cdot (2\sqrt{n})^2 \cdot P(N=n) \\
&\leqq \frac{EX^2}{P(|X| \leqq 1)} + \frac{4(EX^2)^2}{(P(|X| \leqq 1))^2} < \infty.
\end{aligned}$$

We now know that the LHS of (4.15) is finite, i.e. we know that $E(N \cdot l_2(N))^{-1} \cdot X_N^2 \cdot I\{N<\infty\} < \infty$, where N is defined as in (4.11). From the proof of the previous step we know that this fact implies that (3.1) holds. The proof of this step and thus of the whole theorem is therefore complete.

Remark 1. The proof of the theorem also yields a new proof of (3.2)⇒(3.1) for the case $d=1$ since we have shown that (3.2)⇒(3.4)⇒(3.1).

5. Proof of Theorem 3.2.

The proof follows the pattern of Section 4 and is a little easier.

(3.6)⇒(3.7). Set $X'_{\mathbf{n}} = X_{\mathbf{n}} \cdot I\{|X_{\mathbf{n}}| \leqq |\mathbf{n}|^{1/r}\}$ and $X''_{\mathbf{n}} = X_{\mathbf{n}} - X'_{\mathbf{n}}$. Computations as in the proof of Theorem 3.1 show that

$$E \sup_{\mathbf{n}} \left| |\mathbf{n}|^{-1/r} \cdot X''_{\mathbf{n}} \right|^p \leqq \begin{cases} c \cdot E|X|^p \cdot (l(|X|))^d & \text{if } p=r \\ c \cdot E|X|^p & \text{if } p>r \end{cases} \tag{5.1}$$

and

$$E \sup_{\mathbf{n}} \left| |\mathbf{n}|^{-1/r} \cdot X'_{\mathbf{n}} \right|^p \leqq 1. \tag{5.2}$$

(3.7)⇒(3.6). Trivially, (3.7)$\Rightarrow E|X|^p<\infty$. If $p>r$ we are done, so assume that $p=r$, in which case (3.7) can be re-written into

$$E\sup_{\mathbf{n}}(|\mathbf{n}|^{-1}\cdot|X_{\mathbf{n}}|^p)<\infty. \tag{5.3}$$

Since $\{|X_{\mathbf{n}}|^p\}$ is a sequence of i.i.d. integrable random variables, the problem is reduced to the case $r=1$ and the conclusion thus follows from [3] and [4], Theorem 5, according to which (5.3) is equivalent to $E|X|^p\cdot(l(|X|^p))^d<\infty$, which in turn is equivalent to $E|X|^p\cdot(l(|X|))^d<\infty$. Thus (3.6) follows.

(3.7)⇒(3.8). By the preceding step we also known that (3.6) holds. From the Kolomogorov-Marcinkiewicz law (see [11], pp. 242–243 for the case $d=1$ and [6], Theorem 3.1 for the case $d\geqq 2$) it follows that $V=\sup_{\mathbf{n}}\big||\mathbf{n}|^{-1/r}\cdot S_{\mathbf{n}}\big|<\infty$ a.s. Since (3.7) holds, i.e. $EW^p<\infty$, where $W=\sup_{\mathbf{n}}\big||\mathbf{n}|^{-1/r}\cdot X_{\mathbf{n}}\big|$, an application of Lemma 2.2 yields $EV^p<\infty$, i.e. (3.8).

(3.8)⇒(3.7). Immediate, since $W\leqq 2^d\cdot V$.

This concludes the proof if $d\geqq 2$. We therefore assume that $d=1$ in the rest of the proof.

The implications (3.7)⇒(3.9) and (3.8)⇒(3.10) follow just as the corresponding implications of Theorem 3.1. By choosing $N\equiv 1$ it follows that (3.9) and (3.10) both imply that $E|X|^p<\infty$, which completes the proof for the case $p>r$.

If $p=r$ we define

$$N=\inf\{n;\,|X_n|>n^{1/r}\},\qquad N=\infty\qquad\text{if no such } n \text{ exists.} \tag{5.4}$$

By performing the same kind of computations as in [12] and Sect. 4 it follows that (3.9) and (3.10) both imply that (3.6) holds also for this case. The details are omitted.

6. Complements

1. There are various ways of obtaining additional information about the law of the iterated logarithm.

Let $V=\sup_{\mathbf{n}}|a_{\mathbf{n}}^{-1}\cdot S_{\mathbf{n}}|$, $L_d=\sup\{|\mathbf{n}|;\ |S_{\mathbf{n}}|\geqq\varepsilon\cdot a_{\mathbf{n}}\}$, $(\sup\emptyset=0)$, and $N_d=\sum_{\mathbf{n}} I\{|S_{\mathbf{n}}|\geqq\varepsilon\cdot a_{\mathbf{n}}\}$, with $a_{\mathbf{n}}=\sqrt{|\mathbf{n}|\cdot l_2(|\mathbf{n}|)}$. Thus, V is the quantity studied in Theorem 3.1, L_d is a last exit time and N_d equals the number of partial sums exceeding $\varepsilon\cdot a_{\mathbf{n}}$ in absolute value.

The law of the iterated logarithm, (see Wichura [19], p. 280) implies e.g. that, if $\sigma^2=EX^2<\infty$ for the case $d=1$ and $EX^2\cdot(l_2(|X|))^{-1}\cdot(l(|X|))^{d-1}<\infty$ for the case $d\geqq 2$, then N_d and L_d are a.s. finite if $\varepsilon>\sigma\sqrt{2d}$ and then $V<\infty$ a.s. Also, in [7], Theorem 6.2, it is shown that the same assumptions imply that $\sum_{\mathbf{n}}|\mathbf{n}|^{-1}\cdot P(|S_{\mathbf{n}}|\geqq\varepsilon\cdot a_{\mathbf{n}})<\infty$ for $\varepsilon>\sigma\sqrt{2d}$.

Theorem 3.1 states e.g. that, if in addition $EX^2 \cdot (l_2(|X|))^{-1} \cdot (l(|X|))^d < \infty$, then $EV^2 < \infty$. In [7] it is shown that, for $\varepsilon > \sigma\sqrt{2(d+1)}$, the same conditions entail $E\,l(L_d) < \infty$, (Theorem 8.3) and $E\,l(N_d) < \infty$, (Corollary 8.5), and also that $\sum_{j=1}^{\infty} j^{-1} \cdot P(\sup_{j \leq |\mathbf{k}|} |a_{\mathbf{k}}^{-1} \cdot S_{\mathbf{k}}| \geq \varepsilon) < \infty$, (Theorem 6.1).

2. In the same way one can obtain additional information about the Kolomogorov-Marcinkiewicz strong law of large numbers, which states that, if $EX = 0$ whenever it is finite, then $|\mathbf{n}|^{-1/r} \cdot S_{\mathbf{n}} \to 0$ a.s. as $\mathbf{n} \to \infty$ if and only if $E|X|^r \cdot l(|X|)^{d-1} < \infty$, where $0 < r < 2$. See e.g. Loève [11], pp. 242–243 for the case $d = 1$ and [6], Theorem 3.1 for the case $d \geq 2$.

Let V, L_d and N_d be defined as in the previous remark with $a_{\mathbf{n}} = |\mathbf{n}|^{1/r}$. The strong law thus implies that, under the above moment conditions on X, N_d, L_d and V are all a.s. finite.

Theorem 3.2 shows e.g. that, if in addition $E|X|^r \cdot (l(|X|))^d < \infty$, then $EV^r < \infty$ for $0 < r < 2$. In [6], Theorem 4.2, it is shown that the same conditions are equivalent to $\sum_{j=1}^{\infty} j^{-1} \cdot P(\sup_{j \leq |\mathbf{k}|} |a_{\mathbf{k}}^{-1} \cdot S_{\mathbf{k}}| \geq \varepsilon) < \infty$ for all $\varepsilon > 0$ and, by the method of [7], Section 8, this can be used to show that $E\,l(L_d) < \infty$. Furthermore, if $0 < r < 2$, $p > r$, the statements

$$E|X|^p \cdot (l(|X|))^{d-1} < \infty \qquad \text{and, if } p \geq 1, \quad EX = 0,$$

$$EL_d^{\frac{p}{r}-1} < \infty \qquad \text{for all } \varepsilon > 0,$$

$$\sum_{j=1}^{\infty} j^{\frac{p}{r}-2} \cdot P(\sup_{j \leq |\mathbf{k}|} |a_{\mathbf{k}}^{-1} \cdot S_{\mathbf{k}}| \geq \varepsilon) < \infty \qquad \text{for all } \varepsilon > 0$$

are all equivalent (see [7], Theorem 8.1 and [6], Theorem 4.1). Further, these statements all imply (see [7], Corollary 8.3) that

$$EN^{\frac{p}{r}-1} \cdot l(N_d)^{-(d-1)(\frac{p}{r}-1)} < \infty \qquad \text{for all } \varepsilon > 0.$$

3. The cases $p < 2$ (Theorem 3.1) and $p < r$ (Theorem 3.2) do not yield much more than cases $p = 2$ and $p = r$ respectively. This is because a necessary condition for $EV^p < \infty$, (where $V = \sup_{\mathbf{n}} |a_{\mathbf{n}}^{-1} \cdot S_{\mathbf{n}}|$), is that $V < \infty$ a.s. and the necessary conditions for this to hold are given by the law of the iterated logarithm and the strong law of large numbers respectively, conditions which are very close to the boundary conditions given in Theorems 3.1 and 3.2. For the sufficiency of these conditions when $d = 1$, $p = 1$, see Siegmund [14] for $a_n = \sqrt{n \cdot l_2(n)}$ and Klass [10], p. 904, for $a_n = n^{1/r}$, $1 < r < 2$.

4. By modifying the above proofs it is possible to obtain necessary and sufficient conditions for finiteness of $E \sup_{\mathbf{n}} |(|\mathbf{n}|\, l(|\mathbf{n}|))^{-1/2} \cdot X_{\mathbf{k}}|^p$ and $E \sup_{\mathbf{n}} |(|\mathbf{n}|\, l(|\mathbf{n}|))^{-1/2} \cdot S_{\mathbf{n}}|^p$. For the case $d = 1$, $p = 2$ see [14], p. 530, where it is

mentioned that, if $EX=0$, then $E\sup_n((n\,l(n))^{-1}\cdot S_n^2)<\infty$ if and only if $EX^2\cdot l_2(|X|)<\infty$.

5. It is also possible to show that (3.1)⇒(3.3) by an extension of the method of Siegmund and, since the crucial estimates are available from [7], we present a brief sketch of how this can be achieved in the symmetric case.

Again $X'_{\mathbf{n}}$ and $X''_{\mathbf{n}}$ are as in Sect. 4, $S'_{\mathbf{n}}=\sum_{\mathbf{k}\prec\mathbf{n}} X'_{\mathbf{k}}$ and $S''_{\mathbf{n}}=S_{\mathbf{n}}-S'_{\mathbf{n}}$. Also, $a_{\mathbf{n}}$, $b_{\mathbf{n}}$, a_j, b_j are as before.

We first wish to show that

$$\int_{x_0}^{\infty} x^{p-1}\cdot P\left(\sup_{\mathbf{n}}\left|\frac{S'_{\mathbf{n}}}{\sqrt{|\mathbf{n}|\,l_2(|\mathbf{n}|)}}\right|>x\right)dx<\infty \qquad \text{for some } x_0>0. \tag{6.1}$$

With minor modifications of the arguments leading to formula (4.3) of [7] we obtain

$$P(|S'_{\mathbf{n}}|\geqq x\cdot a_{\mathbf{n}})\leqq 2(\log|\mathbf{n}|)^{-\frac{x}{4}} \qquad \text{for } |\mathbf{n}| \text{ and } x \text{ large.} \tag{6.2}$$

Next, by performing computations like those made in the derivation of formula (5.2) of [6] together with Lévy's inequality, and (6.2) we obtain

$$\begin{aligned}
P(\sup_{\mathbf{n}}|a_{\mathbf{n}}^{-1}\cdot S'_{\mathbf{n}}|>x) &\leqq \sum_{j=1}^{\infty} P(\sup_{|\mathbf{n}|\leqq 2^j}|S'_{\mathbf{n}}|\geqq x\cdot a_{2^{j-1}})\\
&\leqq \sum_{j=1}^{\infty}\sum_{|\mathbf{n}|=2^{j+d}} P(\sup_{\mathbf{k}\prec\mathbf{n}}|S'_{\mathbf{k}}|\geqq x\cdot a_{2^{j-1}})\\
&\leqq 2^d\cdot\sum_{j=1}^{\infty}\sum_{|\mathbf{n}|=2^{j+d}} P(|S'_{\mathbf{n}}|\geqq x\cdot a_{2^{j-1}})\\
&\leqq c\cdot\sum_{j=j_0}^{\infty} d(2^{j+d})\cdot(\log(2^{j+d}))^{-xc}\\
&\leqq c\cdot\sum_{j=j_0}^{\infty} j^{d-1}\cdot((j+d)\log 2)^{-xc}\leqq c\cdot e^{-xc},
\end{aligned}$$

where c is a constant.

Thus, for x_0 sufficiently large, it follows that

$$\int_{x_0}^{\infty} x^{p-1}\cdot P(\sup_{\mathbf{n}}|a_{\mathbf{n}}^{-1}\cdot S'_{\mathbf{n}}|>x)\,dx<c\int_{x_0}^{\infty} x^{p-1}e^{-xc}\,dx<\infty,$$

which proves (6.1).

To see that $E\sup_{\mathbf{n}}|a_{\mathbf{n}}^{-1}\cdot S''_{\mathbf{n}}|^p<\infty$ we use the fact that $E\sup_{\mathbf{n}}|a_{\mathbf{n}}^{-1}\cdot S''_{\mathbf{n}}|^p \leqq E(\sum_{\mathbf{n}} a_{\mathbf{n}}^{-1}\cdot|X''_{\mathbf{n}}|)^p$ together with the lemma on p. 2157 of [17] and Lemma 4.1. We omit the details.

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