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Convergence and Regularity of Multiparameter Strong Martingales

John B. Walsh

Department of Mathematics, University of British Columbia Vancouver, B.C., Canada

Give $\mathbb{R}^2$ the usual order: $(s_1, s_2) \prec (t_1, t_2)$ if $s_1 \leqq t_1$ and $s_2 \leqq t_2$, and $(s_1, s_2) \ll (t_1, t_2)$ if $s_1 < t_1$ and $s_2 < t_2$. The question that prompted this study is this: if M is a martingale indexed by $\mathbb{R}^2$, does it necessarily have a version which is right-continuous and has left limits? (A function f on $\mathbb{R}^2$ is right-continuous at t if $\lim_{\substack{s \to t \\ t \prec s}} f(s) = f(t)$, and f has a left limit at t if $\lim_{\substack{s \to t \\ s \ll t}} f(s)$ exists.)

This question has been answered when M is a functional of the Brownian sheet. In this case, M will even have a continuous version as long as it is bounded in $L \log L$, but if it is only L^1-bounded, it may have no continuous version whatsoever [2]. This is, however, a very special case, and one knows that in general, martingales may have at least jump discontinuities. The real question is whether they can have oscillatory discontinuities.

We will give a partial answer to these questions. If M is a *strong* martingale, it does indeed have a right continuous version.

In order to prove this result, we first prove a maximal inequality for strong martingales which sharpens Cairoli's inequality [1], and then use it to prove a new convergence theorem for strong martingales: a strong martingale which is bounded in L^1 converges a.s. This is in contrast to ordinary two-parameter martingales, which converge a.e. if they are bounded in $L \log L$ [1] but which may not converge a.e. if they are only bounded in L_1 [2].

However, our methods give us no information on the important question of left limits, and they do not extend to cover ordinary martingales. We think that strong martingales should have left limits a.s., and that ordinary martingales which are bounded in $L \log L$ should have versions which are right continuous and have left limits, but these questions are open at the moment.

§ 1. Strong Martingales

Let I_1 and I_2 be subsets of R and put $I = I_1 \times I_2$. Let $(\Omega, \mathscr{F}, P)$ be a complete probability space and let $\{\mathscr{F}_t, t \in I\}$ be a family of sub-σ-fields of $\mathscr{F}$ such that each $\mathscr{F}_t$ contains all null-sets of $\mathscr{F}$ and such that, if $s \prec t \in I$, then $\mathscr{F}_s \subset \mathscr{F}_t$. If $t = (t_1, t_2)$, put

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$$\mathscr{F}_t^1 = \mathscr{F}_{t_1,\infty} \overset{\text{def}}{=} \bigvee_{\tau \in I_2} \mathscr{F}_{t_1,\tau};$$

$$\mathscr{F}_t^2 = \mathscr{F}_{\infty,t_2} \overset{\text{def}}{=} \bigvee_{\tau \in I_1} \mathscr{F}_{\tau,t_2},$$

and put

$$\mathscr{F}_t^* = \mathscr{F}_t^1 \vee \mathscr{F}_t^2.$$

If we think of the parameter as being two-dimensional time, then $\mathscr{F}_t$ represents the past before t. But this "time" is only partially ordered, so we could also think of the past as being "everything not in the future". $\mathscr{F}_t^*$ represents the past in this wider sense.

A process $M = \{M_t, \mathscr{F}_t, t \in I\}$ is a *martingale* if $M_t \in \mathscr{F}_t$ for all $t \in I$, and if $E\{M_t | \mathscr{F}_s\} = M_s$ whenever $s \prec t \in I$. It is a *1-martingale* if $M_t \in \mathscr{F}_t^1$ for all $t \in I$ and if for each $\sigma \leqq \sigma'$ in I_1 and $\tau \in I_2$

$$E\{M_{\sigma',\tau} | \mathscr{F}_{\sigma\tau}^1\} = M_{\sigma\tau},$$

and it is a *2-martingale* if $M_t \in \mathscr{F}_t^2$ for all $t \in I$ and if, for each $\sigma \in I_1$ and $\tau < \tau' \in I_2$

$$E\{M_{\sigma\tau'} | \mathscr{F}_{\sigma\tau}^2\} = M_{\sigma\tau}.$$

If $s \ll t$, let $(s,t]$ denote the rectangle $\{u \in \mathbb{R}^2 : s \ll u \prec t\}$. Given a process $\{X_t, t \in I\}$, we define a finitely-additive signed measure on the rectangles with corners in I by

$$X(s,t] = X_{t_1 t_2} - X_{t_1 s_2} - x_{t_2 s_1} + x_{s_1 s_2}, \tag{1.1}$$

where $s = (s_1, s_2) \ll t = (t_1, t_2)$. The points (t_1, s_2) and (s_2, t_1) are in I since I is a product set.

We say that $M = \{M_t, \mathscr{F}_t, t \in I\}$ is a *strong martingale* if it is a martingale and if, in addition, for each $s \ll t \in I$

$$E\{M(s,t] | \mathscr{F}_s^*\} = 0. \tag{1.2}$$

Remark. If $M = 0$ on the axes, then M is a strong martingale if it is adapted and if (1.2) holds.

The easiest non-trivial example of a strong martingale is this. Let $\{X_{ij}, i,j = 1,2,\ldots\}$ be independent mean zero random variables, and let $M_{mn} = \sum_{i=1}^{m} \sum_{j=1}^{n} X_{ij}$. Then $\{M_{m,n}, m,n = 1,2,\ldots\}$ is a strong martingale. More generally, let μ be a random set function defined on the rectangles of $\mathbb{R}_+^2$ with the properties that $E\{\mu(A)\} = 0$ for each rectangle and that, if $A_1, \ldots, A_n$ are disjoint, then $\mu(A_1), \ldots, \mu(A_n)$ are independent. Set $M_t = \mu(0,t]$. Then $\{M_t, \mathscr{F}_t, t \in \mathbb{R}_+^2\}$ is a strong martingale, where $\mathscr{F}_t = \sigma\{M_s, s \prec t\}$. Moreover, stochastic integrals relative to M are also strong martingales [2]. The most-studied case of this is that in which μ is Gaussian, and $\mu(A)$ has variance equal to the area of A. M is then the Brownian sheet.

In studying two-parameter martingales, one often assumes that the fields $\mathscr{F}_t$ satisfy a conditional independence hypothesis: for each $t \in I$, $\mathscr{F}_t^1$ and $\mathscr{F}_t^2$ are conditionally independent given $\mathscr{F}_t$. But, as we deal exlusively with strong martingales, this hypothesis will not be assumed – the strong martingale property takes its place. Here, for instance, is a result which is true for ordinary martingales only under the conditional independence hypothesis.

Proposition 1.1. *A strong martingale is both a* 1- *and a* 2-*martingale.*

Proof. Fix $\sigma \leqq \sigma'$ in I_1 and $\tau \leqq \rho$ in I_2. Let Δ be the rectangle $((\sigma, \tau), (\sigma', \rho)]$, so that

$$M(\Delta) = M_{\sigma' \rho} - M_{\sigma \rho} - (M_{\sigma' \tau} - M_{\sigma \tau}).$$

Then

$$E\{M(\Delta) | \mathscr{F}_{\sigma \rho}\} = E\{E\{M(\Delta) | \mathscr{F}^*_{\sigma \tau}\} | \mathscr{F}_{\sigma \rho}\} = 0.$$

But, as M is a martingale,

$$E\{M_{\sigma' \rho} - M_{\sigma \rho} | \mathscr{F}_{\sigma \rho}\} = 0$$

so that

$$E\{M_{\sigma' \tau} - M_{\sigma \tau} | \mathscr{F}_{\sigma \rho}\} = 0.$$

Now let $\rho \to \infty$ and notice that $\mathscr{F}_{\sigma \rho} \uparrow \mathscr{F}^1_{\sigma \tau}$, so that in the limit

$$E\{M_{\sigma' \tau} - M_{\sigma \tau} | \mathscr{F}^1_{\sigma \tau}\} = 0,$$

hence M is a 1-martingale, and, by symmetry, a 2-martingale as well.

§ 2. Stopping Domains and Lines: The Discrete Case

Let $I = \{(i,j): i = 0, 1, \ldots, m, j = 0, 1, \ldots, n\}$ where m and n are fixed (possibly infinite) integers. If $t = (i,j)$, we denote $t^{++} = (i+1, j+1)$, $t^{+0} = (i+1, j)$ and $t^{0+} = (i, j+1)$. For any set $A \subset \mathbb{R}^2_+$, $A^0 = \{s \in A: \exists\, t \in A \ni s \ll t\}$. If $t \in \mathbb{R}^2_+$, R_t will denote the closed rectangle $[0, t]$.

Definition. A *stopping domain* D is a random subset of $\mathbb{R}^2_+$ such that

(i) for each ω, there is a set $D' \subset I$ such that

$$D(\omega) = \bigcup_{t \in D'} R_t;$$

(ii) if $t \in I$, $\{\omega: t^{++} \in D(\omega)\} \in \mathscr{F}_t$; in addition, if t is on the axes, the sets $\{t \in D\}$, $\{t^{0+} \in D\}$ and $\{t^{+0} \in D\}$ are also in $\mathscr{F}_t$.

The *stopping line associated with* D is $D - D^0$.

Remarks. 1°. Stopping domains have been defined by Wong and Zakai in the continuous case [4], under the name of stopping times. The definition appears more complex in the discrete case principally because there is no discrete analogue of right-continuous σ-fields.

2°. If D is a stopping domain, it is easily seen that $\{t \in D\}$, $\{t^{0+} \in D\}$, and $\{t^{+0} \in D\}$ are in $\mathscr{F}_t$ for all t, not just those t on the axes. It was necessary to state this explicitly in the definition since if t is on one of the axes, there is no s such that $t = s^{++}$.

Condition (ii) could be stated more succinctly as follows: if $t \in I$, and if s is a nearest neighbor of t, then $\{s \in D\} \in \mathscr{F}_t$.

3°. To see why it is $\{t^{++} \in D\}$, rather than $\{t \in D\}$, which is $\mathscr{F}_t$-measurable, observe that a domain in the one-parameter case would be an interval $[0, T]$, where T is a stopping time. Then $\{T \leqq i\} \in \mathscr{F}_i$. But this is the complement of $\{i \in [0, T-1]\} = \{i+1 \in [0, T]\}$, so the latter must be $\mathscr{F}_i$-measurable.

4°. A stopping domain D is a closed, simply-connected subset of $\mathbb{R}^2_+$, and a stopping line L is a continuous curve with the property that if $s, t \in L$, then $s \not\ll t$.[1] It is of course $D \cap I$ and $L \cap I$ that are of interest, but it seems convenient to extend D and L to be continuous, rather than discrete.

Most elementary properties of stopping domains are proved exactly as are the analogous properties of stopping times. One just needs to keep Remark 3° in mind. For instance

Proposition 2.1. (i) *If $A \subset I$, $\bigcup_{t \in A} R_t$ is a stopping domain.*

(ii) *If D_1 and D_2 are stopping domains, so are $D_1 \cup D_2$ and $D_1 \cap D_2$. In particular, if $t \in I$, $D_1 \cap R_t$ is a stopping domain.*

(iii) *If D is a stopping domain and L its associated stopping line, then for each $t \in I$, the sets $\{t \in D^0\}$, $\{t \in D\}$ and $\{t \in L\}$ are all in $\mathscr{F}_t$.*

Proof. Both (i) and (ii) are immediate from the definition. As for (iii), if $t \in I$, $\{t \in D^0\} = \{t^{++} \in D\} \in \mathscr{F}_t$ and $\{t \in D\} = \{s \in D^0, \forall s \ll t, s \in I\}$ which is in $\bigvee_{s \ll t} \mathscr{F}_s \subset \mathscr{F}_t$. Finally, $\{t \in L\} = \{t \in D\} - \{t \in D^0\} \in \mathscr{F}_t$. qed.

Let $K \subset \mathbb{R}$ be a Borel set and let $\{X_t, t \in I\}$ be a process adapted to the $\mathscr{F}_t$. Define

$$D_K = \bigcup_{t \in I} \{R_{t^{++}} \colon X_s \in K^c, \forall s \in R_t \cap I\}.$$

Then D_K is a stopping domain and its associated stopping line L_K is called the *first hitting line of* K. Notice that if $X_s \in K$ for some $s \in I$, then $X_s \in K$ for some – but not necessarily all – $s \in L_K$. To verify that D_K is a stopping domain, let $t \in I$ and write

$$\{t^{++} \in D_K\} = \{X_s \in K^c, \forall s \in R_t \cap I\} \in \mathscr{F}_t.$$

If t is on one of the axes,

$$\{t^{0+} \in D_K\} = \{t^{+0} \in D_K\} = \{t^{++} \in D_K\} \in \mathscr{F}_t.$$

Definition. Let D be a stopping domain. Then

$$\mathscr{F}_D = \{\Lambda \in \mathscr{F} : \Lambda \cap \{t \notin D^0\} \in \mathscr{F}_t^*, \forall t \in I\}.$$

Remarks. 5°. $\mathscr{F}_D$ is a σ-field.

6°. If $D \subset D'$ are stopping domains, $\mathscr{F}_D \subset \mathscr{F}_{D'}$.

7°. $\{t \in D\} \in \mathscr{F}_D$. More generally, if Y is $\mathscr{F}_s$-measurable, then $Y I_{\{s \in D\}}$ is $\mathscr{F}_D$-measurable.

8°. If $D \equiv R_t$, $\mathscr{F}_t \subset \mathscr{F}_D$.

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[1] $\not\ll$ means "not $\ll$".

Remarks 5° and 6° are proved as in the one-parameter case. To verify 7°, suppose $Y>0$ is $\mathscr{F}_s$-measurable. Then

$$\{YI_{\{s\in D\}}\geqq\lambda\}\cap\{t\notin D^0\}=\{Y\geqq\lambda\}\cap\{s\in D\}\cap\{t\notin D^0\}.$$

If $t\ll s$ this is empty, hence in $\mathscr{F}_t^*$. If $t\not\ll s$, it is in $\mathscr{F}_s\vee\mathscr{F}_t\subset\mathscr{F}_t^*$. Remark 8° follows from Remark 7° on taking Y to be the indicator function of an arbitrary event in $\mathscr{F}_s$.

If the fields satisfy the conditional independence hypothesis, and if $\mathscr{F}_0^*$ is trivial, then there is equality in 8°, but in general there is only inclusion.

If R is a rectangle with corners in I and if $\{M_t, t\in I\}$ is any process, $M(R)$ is defined by (1.1). Stopping domains are unions of this type of ractangle, so we can define $M(D)$ for any stopping domain. We then have the following analogue of Doob's martingale stopping theorem, due to Wong and Zakai in the continuous case.

Proposition 2.2. *Let $D_1\subset D_2$ be stopping domains and let $\{M_t,\mathscr{F}_t,t\in I\}$ be a strong martingale. Then*

$$E\{M(D_2)|\mathscr{F}_{D_1}\}=M(D_1).$$

Proof. Set $\Delta_t=(\mathrm{t},\mathrm{t}^{++}]$ for $t\in I$. D_1 and D_2 are both disjoint unions of the Δ_t so that, for $i=1,2$:

$$\begin{aligned}M(D_i)&=\sum_{t\in I}M(\Delta_t)\,I_{\{\Delta_t\subset D_i\}}\\&=\sum_{t\in I}M(\Delta_t)\,I_{\{t^{++}\in D_i\}},\end{aligned}$$

this last because $\Delta_t\subset D_i$ iff $t^{++}\in D_i$. It follows by Remark 7 that $M(D_i)$ is $\mathscr{F}_{D_i}$-measurable. Now note that

$$\begin{aligned}M(D_2)-M(D_1)&=M(D_2-D_1)\\&=\sum_{t\in I}M(\Delta_t)\,I_{\{t^{++}\notin D_1\}}\,I_{\{t^{++}\in D_2\}}.\end{aligned}$$

If $\Lambda\in\mathscr{F}_{D_1}$,

$$\int_\Lambda M(\Delta_t)\,I_{\{t^{++}\in D_2-D_1\}}=\int_{\Lambda\cap\{t^{++}\notin D_1\}\cap\{t^{++}\in D_2\}}M(\Delta_t)=0,$$

because both $\{t^{++}\in D_2\}$ and $\Lambda\cap\{t^{++}\notin D_1\}$ are in $\mathscr{F}_t^*$, and, as M is a strong martingale, $E\{M(\Delta_t)|\mathscr{F}_t^*\}=0$. It follows that $E\{M(D_2)-M(D_1)|\mathscr{F}_{D_1}\}=0$, and we are done.

§ 3. The Decomposition of a Strong Martingale along a Stopping Line

Let $\{M_t,\mathscr{F}_t,t\in I\}$ be a strong martingale, where $I=\{(i,j),\ i=0,1,\dots,m,\ j=0,1,\dots,n\}$. Let L be a stopping line and consider M along L: $\{M_t, t\in L\cap I\}$. This is essentially a one-parameter process, but, as the points of L are not ordered, we can't expect it to be a martingale. Nevertheless, we shall see that it is a sum of one-parameter martingales. These martingales are with repsect to different fields, which is why their sum is not a martingale.

If D is the stopping domain associated with L, $D \subset R_{mn}$, and, on the set $\{(i,j) \in L\}$ we have

$$D = (D \cap R_{in}) \cup (D \cap R_{mj})$$

so that

$$M(D) = M(D \cap R_{in}) + M(D \cap R_{mj}) - M(R_{ij}).$$

Since

$$M_{ij} = M(R_{ij}) + M_{i0} + M_{0j} - M_{00},$$

we have on the set $\{(i,j) \in L\}$:

$$M_{ij} = M(D \cap R_{in}) + M(D \cap R_{mj}) - M(D) + M_{i0} + M_{0j} - M_{00}. \tag{3.1}$$

This equation simplifies it M vanishes on the axes, and it turns out that, modulo a trivial re-parametrization, one can always assume that this happens, as the following lemma shows.

Lemma 3.1. *Let $\{M_{ij}, \mathscr{F}_{ij}, 0 \leqq i \leqq m, 0 \leqq j \leqq n\}$ be a strong martingale. Then there exists a strong martingale $\{\hat{M}_{ij}, \hat{\mathscr{F}}_{ij}, 0 \leqq i \leqq m+1, 0 \leqq j \leqq n+1\}$ such that for each $0 \leqq i \leqq m$, $0 \leqq j \leqq n$*

$$M_{ij} = \hat{M}_{i+1, j+1} + E\{M_{00}\},$$
$$\mathscr{F}_{ij} = \hat{\mathscr{F}}_{i+1, j+1},$$

and

$$\hat{M}_{0j} = \hat{M}_{i0} = 0.$$

Proof. By subtracting $E\{M_{00}\}$ if necessary, we may suppose $E\{M_{ij}\} = 0$. If $i \geqq 1$, $j \geqq 1$, set $\hat{M}_{ij} = M_{i-1, j-1}$, and $\hat{\mathscr{F}}_{ij} = \mathscr{F}_{i-1, j-1}$, while if either i or j is zero, set $\hat{M}_{ij} = 0$ and $\hat{\mathscr{F}}_{ij} = \{\phi, \Omega\}$. $\hat{M}$ is clearly a martingale, and we need only verify the strong martingale property. Let $s = (i,j) \ll t = (k,h)$. Suppose first that $i \geqq 1$ and $j \geqq 1$, and let $s^{--} = (i-1, j-1)$ and $t^{--} = (k-1, h-1)$. Then $\hat{\mathscr{F}}_s^* = \mathscr{F}_{s^{--}}^*$, so

$$E\{\hat{M}(s,t] \mid \hat{\mathscr{F}}_s^*\} = E\{M(s^{--}, t^{--}] \mid \mathscr{F}_{s^{--}}^*\} = 0.$$

If $i \geqq 1$ and $j = 0$, then $\hat{\mathscr{F}}_s^* = \mathscr{F}_{i-1, 0}^1$ and, as $\hat{M}_{i0} = \hat{M}_{i'0} = 0$, $\hat{M}(s,t] = M_{k-1, h-1} - M_{i-1, h-1}$, so that

$$E\{\hat{M}(s,t] \mid \hat{\mathscr{F}}_s^*\} = E\{M_{k-1, h-1} - M_{i-1, h-1} \mid \mathscr{F}_{i-1}^1\} = 0$$

by Proposition 1.1. Finally, if $i = j = 0$, $\hat{\mathscr{F}}_s^*$ is trivial, so

$$E\{\hat{M}(s,t] \mid \hat{\mathscr{F}}_s^*\} = E\{M_{t^{--}}\} = 0. \quad \text{qed.}$$

Theorem 3.2. *Let $\{M_t, \mathscr{F}_t, t \in I\}$ be a strong martingale which vanishes on the axes, and let D be a stopping domain with associated stopping line L. Then there are two* *processes* $\{X_i, 0 \leqq i \leqq m\}$ *and* $\{Y_j, 0 \leqq j \leqq n\}$ *such that for each* $(i,j) \in I$,

$$M_{ij} = X_i + Y_j - M(D) \quad \text{on } \{(i,j) \in L\}. \tag{3.2}$$

Moreover, $\{X_0, \ldots, X_m, M_{mn}\}$, $\{Y_0, \ldots, Y_n, M_{mn}\}$, *and* $\{M(D), M_{mn}\}$ *are all martingales.*

Proof. Set $X_i = M(D \cap R_{in})$ and $Y_j = M(D \cap R_{mj})$. Then (3.2) follows from (3.1). To verify that the three processes are martingales, set $D_i = D \cap R_{in}$. Then $D_i \subset D_{i+1} \subset R_{mn}$ and the D_i are stopping regions. Thus $\mathscr{F}_{D_i} \subset \mathscr{F}_{D_{i+1}}$, $X_i = M(D_i)$ is $\mathscr{F}_{D_i}$-measurable, and, by Proposition 2.2

$$X_i = E\{M(R_{mn}) | \mathscr{F}_{D_i}\}.$$

Thus $\{X_i, \mathscr{F}_{D_i}, 0 \leqq i \leqq m\}$ is a martingale and we can add $M(R_{mn}) = M_{mn}$ as a final element. Since $D = D_n$, this also shows that $\{M(D), M_{mn}\}$ is a martingale. By symmetry, $Y_0, \ldots, Y_n, M_{mn}$ is also a martingale. qed.

This brings us to the maximal inequality for strong martingales.

Theorem 3.3. *Let* $\{M_t, \mathscr{F}_t, t \in I\}$ *be a strong martingale. Then for* $\lambda > 0$

$$\lambda P\{\sup_t |M_t| \geqq \lambda\} \leqq 13 E\{|M_{mn}|\}. \tag{3.3}$$

Proof. Write $M_t = N_{t++} + E\{M_0\}$, where $\{N_t\}$ is a strong martingale which vanishes on the axes (Lemma 3.1), and let $a = |E\{M_0\}|$.

$$P\{\sup_t |M_t| \geqq \lambda\} \leqq P\{\sup_t |N_t| \geqq \lambda - a\}.$$

Let L be the first hitting line of the set $\{|N_t| \geqq \lambda - a\}$ (see § 2). If $|N_t|$ exceeds $\lambda - a$ in R_{mn}, it must do so somewhere along L. Now write

$$N_{ij} = X_i + Y_j - N(D) \quad \text{on } \{(i,j) \in L\},$$

where X and Y are the martingales of Theorem 3.2. Note that N_t can exceed $\lambda - a$ only if at least one of $\sup |X_i|$, $\sup |Y_j|$ or $|N(D)|$ exceeds $\frac{1}{3}(\lambda - a)$. Since $N(D) = X_m$, we see that

$$P\{\sup |N_t| \geqq \lambda - a\} \leqq P\left\{\sup_i |X_i| \geqq \frac{\lambda - a}{3}\right\} + P\left\{\sup_j |Y_j| \geqq \frac{\lambda - a}{3}\right\}.$$

Apply the maximal inequality to both X and Y, remembering that N_{mn} is the final element of each:

$$\leqq \frac{6}{\lambda - a} E\{|N_{mn}|\}.$$

Now

$$E\{|N_{mn}|\} \leqq a + E\{|M_{mn}|\} \leqq 2E\{|M_{mn}|\},$$

so the above is

$$\leqq \frac{12}{\lambda - a} E\{|M_{mn}|\}.$$

This inequality is only useful if the right-hand side is smaller than one. Since $E\{|M_{mn}|\} \geqq a$, this can only happen if $a \leqq \lambda/13$. Setting $a = \lambda/13$ above gives (3.3).

Remark. If $E\{M_{00}\} = 0$, the same argument shows that one can replace 13 by 6 in (3.3), but we have made no effort to find the best constant.

Corollary 3.4. *Let I_1 and I_2 be arbitrary subsets of* $\mathbb{R}$, *and let $I = I_1 \times I_2$. If $\{M_t, \mathscr{F}_t, t \in I\}$ is a separable strong martingale, then*

$$\lambda P\{\sup_{t \in I} |M_t| \geqq \lambda\} \leqq 13 \sup_{t \in I} E\{|M_t|\}. \tag{3.4}$$

This follows from Theorem 3.3 as usual by choosing a sequence of finite subsets of I whose union is a separability set, applying Theorem 3.3 to each, and taking the limit.

The above maximal inequality allows us to sharpen Cairoli's convergence theorem in the case of strong martingales.

Theorem 3.5. *Let I_1 and I_2 be subsets of* $\mathbb{R}$ *which are unbounded both above and below, and put $I = I_1 \times I_2$. Suppose $\{M_t, \mathscr{F}_t, t \in I\}$ is a separable strong martingale. Then*

(i) $\lim_{\substack{t \to (-\infty, -\infty) \\ t \in I}} M_t = M_{-\infty}$ *exists a.s. and in* L^1;

(ii) *if* $\sup_t E\{|M_t|\} < \infty$, *then* $\lim_{\substack{t \to (\infty, \infty) \\ t \in I}} M_t = M_\infty$ *exists a.s.*

Before proving this, we need a lemma concerning ordinary martingales.

Lemma 3.6. *Let $\{X_n, \mathscr{F}_n, n \geqq 1\}$ be an L^1-bounded martingale. Then there exists a sequence (n_k) and sets (Λ_k) such that $\Lambda_k \in \mathscr{F}_{n_k}$, and such that*

$$P\{\Lambda_k\} \geqq 1 - 2^{-k} \quad \textit{and} \quad \int_{\Lambda_k} |X_p - X_{n_k}| < 4^{-k}, \quad \textit{all } p > n_k.$$

Proof. Using Krickeberg's decomposition, it is easy to show that we can write

$$X_n = Y_n^+ - Y_n^- + Z_n,$$

where $Y_n^\pm$ are positive martingales with limit zero, and Z_n is a uniformly integrable martingale. Choose n_k large enough so that

$$P\left\{Y_{n_k}^+ < \frac{1}{6.4^k}, Y_{n_k}^- < \frac{1}{6.4^k}\right\} > 1 - 2^{-k}$$

and

$$E\{|Z_\infty - Z_p|\} < \tfrac{1}{6} 4^{-k}, \quad \text{for all } p \geqq n_k.$$

Let $\Lambda_k=\{Y^+_{n_k}<\frac{1}{6}4^{-k}, Y^-_{n_k}<\frac{1}{6}4^{-k}\}$. Then, if $p>n_k$

$$\int_{\Lambda_k}|X_p-X_{n_k}|\leqq\int_{\Lambda_k}(Y^+_p+Y^-_p+Y^+_{n_k}+Y^-_{n_k})+E\{|Z_p-Z_\infty|\}+E\{|Z_{n_k}-Z_\infty|\}.$$

As $Y^\pm$ is a martingale, $\int_{\Lambda_k} Y^\pm_p=\int_{\Lambda_k} Y^\pm_{n_k}\leqq\frac{1}{6}4^{-k}$, and both expectations are bounded by $\frac{1}{6}4^{-k}$, so the above expression is bounded by 4^{-k}. qed.

We can now prove Theorem 3.5. By a theorem of Helms [3], M converges in L^1 as $t\to(-\infty,-\infty)$. To show that there is almost-everywhere convergence, let $M_{-\infty}$ be the limit and choose a sequence $(t_n)\subset I$ tending to $(-\infty,-\infty)$ such that

$$E\{|M_{t_n}-M_{-\infty}|\}\leqq 4^{-n}.$$

Now $\{M_t-M_{-\infty}, t\in I\}$ is a strong martingale, so by Corollary 3.4,

$$P\{\sup_{\substack{t\prec t_n\\ t\in I}}|M_t-M_{-\infty}|\geqq 2^{-n}\}\leqq 13\cdot 2^{-n}.$$

It follows by the Borel-Cantelli lemma that M_t converges a.s. as $t\to(-\infty,-\infty)$.

The proof of (ii) is similar. If $(t_n)\subset I$ is an increasing sequence – i.e. $t_n\ll t_{n+1}$ – tending to (∞,∞), $\{M_{t_n}, n\geqq 1\}$ is an L^1-bounded one-parameter martingale. By Lemma 3.6, there is a subsequence (n_k) and sets $\Lambda_k\in\mathscr{F}_{t_{n_k}}$ with $P\{\Lambda_k\}\geqq 1-2^{-k}$ and

$$\int_{\Lambda_k}|M_{t_p}-M_{t_{n_k}}|\leqq 4^{-k}\qquad\text{for all } p>n_k.$$

Now if $t_{n_k}\prec t$, there is an i for which $t\prec t_{n_i}$, and so, since $|M_t-M_{t_{n_k}}|$ is a sub-martingale

$$\int_{\Lambda_k}|M_t-M_{t_{n_k}}|\leqq\int_{\Lambda_k}|M_{t_{n_i}}-M_{t_{n_k}}|\leqq 4^{-k}.$$

We can now apply Corollary 3.4 to the strong martingale

$$\{(M_t-M_{t_{n_k}})I_{\Lambda_k}, t_{n_k}\prec t, t\in I\}$$

to see that

$$P\{\Lambda_k;\ \sup_{\substack{t>t_{n_k}\\ t\in I}}|M_t-M_{t_{n_k}}|\geqq 2^{-k}\}\leqq 13\cdot 2^{-k},$$

or, since $P\{\Lambda_k\}\geqq 1-2^{-k}$

$$P\{\sup_{\substack{t_{n_k}\prec t\\ t\in I}}|M_t-M_{t_{n_k}}|\geqq 2^{-k}\}\leqq 14\cdot 2^{-k}.$$

If $M_\infty=\lim_{k\to\infty}M_{t_{n_k}}$ (which exists a.s.), an application of the Borel-Cantelli lemma shows that $M_t\to M_\infty$ a.s. as $t\to(\infty,\infty)$, and we are done.

§ 4. The Right Continuity of the Paths

Let $\{\mathscr{F}_t, t\in\mathbb{R}^2_+\}$ be a family of sub-σ-fields of $\mathscr{F}$ and define $\mathscr{F}_t^i$, $i=1,2$, and $\mathscr{F}_t^*$ as in § 1. We assume that the fields are complete, increasing, and right-continuous, that is:

1) if $s \prec t$, then $\mathscr{F}_s \subset \mathscr{F}_t$;
2) $\mathscr{F}_s$ contains all null-sets of $\mathscr{F}$;
3) for each s, $\mathscr{F}_s = \bigcap_{s \ll t} \mathscr{F}_t$ and $\mathscr{F}_s^* = \bigcap_{s \ll t} \mathscr{F}_t^*$.

Then we have:

Theorem 4.1. *Let $\{M_t, \mathscr{F}_t, t\in\mathbb{R}^2_+\}$ be a strong martingale. Then it has a version which is a.s. right-continuous.*

Before embarking on a proof of this theorem, we will introduce and establish some properties of a second two-parameter analogue of stopping times which, for lack of a better name, we call *weak stopping points.* A weak stopping point T is a random variable with values in $\mathbb{R}_+ \cup \{\infty\}$ such that for each $t\in\mathbb{R}^2_+$, $\{T \prec t\} \in \mathscr{F}_t^*$. We define

$$\mathscr{F}_T^* = \{\Lambda\in\mathscr{F}: \Lambda \cap \{T\prec t\} \in \mathscr{F}_t^*, \text{ all } t\in\mathbb{R}^2_+\}.$$

The elementary properties of $\mathscr{F}_T^*$ are derived exactly as in the classical case, so we will leave the proofs of the following to the reader.

a) $\mathscr{F}_T^*$ is a σ-field;
b) if $T\equiv t$, then $\mathscr{F}_T^* = \mathscr{F}_t^*$;
c) if $S \prec T$ are weak stopping points, $\mathscr{F}_S^* \subset \mathscr{F}_T$;
d) T is $\mathscr{F}_T^*$-measurable;
e) if $T \prec T_n$ are weak stopping points and $T_n \to T$, then $\mathscr{F}_T^* = \bigcap_n \mathscr{F}_{T_n}^*$.

Here is one example of a weak stopping point. For $i=1,2$, let τ_i be an ordinary stopping time relative to the fields $\{\mathscr{F}_{a0}^1, a\geqq 0\}$ and $\{\mathscr{F}_{0b}^2, b\geqq 0\}$ respectively. We call τ_i an *$\mathscr{F}^i$-stopping time.* Then $T=(\tau_1,\tau_2)$ is a weak stopping point. Indeed, if $t=(t_1,t_2)$,

$$\{T\prec t\} = \{\tau_1 \leqq t_1\} \cap \{\tau_2 \leqq t_2\} \subset \mathscr{F}_t^1 \vee \mathscr{F}_t^2 = \mathscr{F}_t^*.$$

In the simplest non-trivial case (which is the only case we shall use in this article[2] one of the τ_i is constant. Then we have

f) if τ_1 is an $\mathscr{F}^1$-stopping time, then

$$\mathscr{F}_{\tau_1}^1 \subset \mathscr{F}_{\tau_1 0}^*,$$

where

$$\mathscr{F}_{\tau_1}^1 = \{\Lambda\in\mathscr{F}: \Lambda\cap\{\tau_1\leqq a\} \in \mathscr{F}_{a,0}^1, \text{ all } a\geqq 0\}.$$

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[2] Our original proof of this theorem used more complex stopping points. We wish to thank R. Cairoli for pointing out that they weren't necessary and that our proof could be shortened and simplified by omitting its middle third

Lemma 4.2. *Let $\{M_t, \mathscr{F}_t, t \in \mathbb{R}_+^2\}$ be a strong martingale and let $S \prec T$ be bounded weak stopping points such that T is $\mathscr{F}_S^*$-measurable. Suppose both S and T take on only countably many values. Then*

$$E\{M(S,T] \mid \mathscr{F}_S^*\} = 0. \tag{4.1}$$

Proof. If $\Lambda \in \mathscr{F}_S$, and if both S and T take their values in the set $\{t_n\}$,

$$E\{M(S,T];\Lambda\} = \sum_{t_m \prec t_n} E\{M(t_m, t_n]; \Lambda \cap \{T=t_n\} \cap \{S=t_m\}\}.$$

But $\Lambda \cap \{T=t_n\} \in \mathscr{F}_S^*$, so $\Lambda \cap \{T=t_n\} \cap \{S=t_m\} \in \mathscr{F}_{t_m}^*$, and each term in the sum vanishes by the strong martingale property. qed.

We can now prove Theorem 4.1. Notice first that we can replace $M_{t_1 t_2}$ by $M_{t_1 t_2} - M_{t_1 0} - M_{0 t_2} + M_{00}$ to get a strong martingale which vanishes on the axes. Since M is an ordinary martingale along each of the coordinate axes, it has a version which is right-continuous and has left limits there. Thus we can replace M by a martingale which is zero along the axes without affecting the continuity of M in the interior. We will accordingly assume that M vanishes on the coordinate axes.

If $t=(t_1,t_2)$, then $t_i \to M_t$ is right-continuous in probability and a.s. right-continuous along the rationals, $i=1,2$, since M is an i-martingale and the fields are right-continuous. Thus, define a standard modification, also denoted by M, by

$$M_{t_1 t_2} = \lim_{\substack{r \downarrow t_1 \\ r \in \mathbb{Q}}} M_{r t_2}.$$

For each fixed t_2, this version will be right continuous in t_1. We can extend Lemma 4.2 to a rather special case of weak stopping times. Let $\sigma \leqq \tau$ be bounded $\mathscr{F}^1$-stopping times, such that τ is $\mathscr{F}_\sigma^1$-measurable, let $0 \leqq s_2 \leqq t_2$ be real, and set $S = (\sigma, s_2)$ and $T=(\tau, t_2)$. We claim that (4.1) holds for this S and T. Let $\sigma_j = k2^{-j}$ if $(k-1)2^{-j} \leqq \sigma \prec k2^{-j}$ and $\tau_j = k2^{-j}$ if $(k-1)2^{-j} \leqq \tau < k2^{-j}$. We can apply Lemma 4.2 to the times $S_j=(\sigma_j, s_2)$ and $T_j=(\tau_j, t_2)$:

$$E\{M(S_j, T_j] \mid \mathscr{F}_S^*\} = E\{E\{M(S_j, T_j] \mid \mathscr{F}_{S_j}^*\} \mid \mathscr{F}_S^*\} = 0. \tag{4.2}$$

Now $\sigma_j \downarrow \sigma$ and $\tau_j \downarrow \tau$, and M is a.s. right continuous in its first parameter for both values s_2 and t_2 of the second, so $M(S_j, T_j] \to M(S,T]$. Moreover, the family $\{M(S_j, T_j]\}$ is uniformly integrable, since if N is a bound for the σ_j and τ_j, then

$$M(S_j, T_j] = M_{\tau_j, t_2} - M_{\tau_j, s_2} - M_{\sigma_j, t_2} + M_{\sigma_j, s_2},$$

and each term of the difference is a conditional expectation of either $M_{N s_2}$ or $M_{N t_2}$, e.g. $M_{\tau_j, t_2} = E\{M_{N t_2} \mid \mathscr{F}_{\tau_j}^1\}$. We can thus go to the limit in (4.2) to get (4.1).

Now let σ be a bounded $\mathscr{F}^1$-stopping time, let $S=(\sigma, 0)$, $t=(t_1,t_2)$, and consider the process

$$\begin{aligned} N_t &= M_{\sigma+t_1, t_2} - M_{\sigma t_2} \\ &= M(S, S+t], \end{aligned}$$

where this last equality holds because M vanishes on the axes. We claim that N is a strong martingale relative to its natural fields $\mathscr{G}_t=\sigma\{N_s, s\leqq t\}$. Indeed, if $s\prec t$, $N(s,t]=M(S+s,S+t]$ and $\mathscr{G}_s^*\subset\mathscr{F}_{S+s}^*$, so by the above extension of Lemma 4.2

$$E\{N(s,t]\,|\,\mathscr{G}_s^*\}=E\{E\{M(S+s,S+t]\,|\,\mathscr{F}_{S+s}^*\}\,|\,\mathscr{G}_s^*\}=0. \tag{4.3}$$

But now N vanishes on the axes, so (4.3) implies that N is a strong martingale.

Next, let K be a bounded subset of the rationals and notice that $N_{u,v}$ is a.s. right continuous in u simultaneously for all $v\in K$, so by Corollary 3.4, if $\rho=\sup K$ and $\lambda>0$, $\alpha>0$

$$\lambda P\{\sup_{\substack{u\leqq\alpha\\ r\in K}}|N_{ur}|>\lambda\}\leqq 6E\{|N_{\alpha\rho}|\}.$$

But $E\{|N_{\alpha\rho}|\}\to 0$ as $\alpha\downarrow 0$, so we can conclude that with probability one, $N_{ur}\to 0$ as $u\to 0$, *uniformly* over all $r\in K$. In terms of M, we have proved that for any $\mathscr{F}^1$-stopping time σ_1,

$$\lim_{u\downarrow\sigma_1} M_{ur}=M_{\sigma_1 r},$$

a.s. uniformly for $r\in K$.

Now let's apply Doob's transfinite induction argument. Let $\sigma_0=0$, $\sigma_1=\inf\{u\geqq 0\colon \sup_{r\in K}|M_{ur}-M_{0r}|\geqq\varepsilon\}$, and, by induction,

$$\sigma_{n+1}=\inf\{u\geqq\sigma_n\colon \sup_{r\in K}|M_{ur}-M_{\sigma_n r}|\geqq\varepsilon\}.$$

Then $P\{\sigma_{n+1}>\sigma_n\,|\,\sigma_n<\infty\}=1$. If $\lim\sigma_n<\infty$, define $\sigma_\omega=\lim\sigma_n$, and define $\sigma_{\omega+1},\sigma_{\omega+2},\ldots$ and so on. Thus we define σ_α for all countable ordinals α. Since $\sigma_{\alpha+1}>\sigma_\alpha$ a.s. on $\{\sigma_\alpha<\infty\}$, it is easily seen that there is a countable ordinal β such that $\sigma_\beta(\omega)=\infty$ for a.e. ω. Now for such an ω and for any $u\geqq 0$, there is some ordinal α such that $\sigma_\alpha(\omega)\leqq u<\sigma_{\alpha+1}(\omega)$, so that

$$\lim_{u'\downarrow u}[\sup_{r\in K}|\mathrm{M}_{u'r}-\mathrm{M}_{ur}|]<2\varepsilon.$$

This being true simultaneously for a sequence of $\varepsilon\to 0$, we conclude that $u\to M_{ur}$ is a.s. right-continuous, uniformly for $r\in K$.

The above was derived using a standard modification of M, but we can conclude for any version of M that $u\to M_{ur}$ is right-continuous along the rationals, uniformly for $r\in K$. By symmetry, $v\to M_{rv}$ is also right continuous along the rationals, uniformly for $r\in K$. This uniformity implies that for a.e. ω and any $s\in\mathbb{R}_+^2$

$$\lim_{t,t'\to s}\{|M_t-M_{t'}|\colon s\prec t, s\prec t', t,t'\in K^2\}=0;$$

as K was arbitrary we have that for a.e. ω,

$$\lim_{t\to s}\{M_t(\omega), s\prec t, t\in\mathbb{Q}_+^2\} \qquad \text{exists.}$$

Thus, define for each $s\in\mathbb{R}_+^2$:

$$\overline{M}_s(\omega)=\lim_{t\to s}\{M_t(\omega)\colon s\prec t, t\in\mathbb{Q}_+^2\}$$

if the limit exists, and set $\bar{M}_s=0$ if not. Then $\bar{M}$ is right continuous, and it is a standard modification of M since M is already right-continuous in probability. This finishes the proof.

We can squeeze a little more out of the proof. For each rational $v, u\to\bar{M}_{uv}$ has left limits. But, in view of the fact that $v\to\bar{M}_{uv}$ is right continuous, and uniformly so for u in bounded sets, we see that $u\to\bar{M}_{uv}$ has left limits for *every* v, and, moreover, that

Corollary 4.3. *For a.e. ω, the following sectorial limits exist for all $t=(t_1,t_2)$:*

$$\lim_{(u,v)\to t}\{\bar{M}_{uv}(\omega):\ u\geqq t_1, v<t_2\}$$

and

$$\lim_{(u,v)\to t}\{\bar{M}_{uv}(\omega):\ u<t_1, v\geqq t_2\}.$$

§ 5. Extension to n Parameters

The foregoing work extends easily from 2 to n parameters. The partial order in $\mathbb{R}^n$ becomes $(s_1,\dots,s_n)\prec(t_1,\dots,t_n)$ if $s_i\leqq t_i$, $i=1,\dots,n$. Let $I_1,\dots,I_n$ be subsets of $\mathbb{R}$ and put $I=I_1\times\dots\times I_n$. Let $\{\mathscr{F}_t, t\in I\}$ be a family of sub-σ-fields of $\mathscr{F}$ such that $s\prec t\Rightarrow\mathscr{F}_s\subset\mathscr{F}_t$. If $t=(t_1,\dots,t_n)$, define

$$\mathscr{F}_t^i=\mathscr{F}_{(\infty,\dots,t_i,\dots,\infty)} \quad\text{and}\quad \mathscr{F}_t^*=\mathscr{F}_t^1\vee\dots\vee\mathscr{F}_t^n.$$

If $\{M_t, t\in\mathbb{R}_+^n\}$ is a process which is zero on the axes, define an additive set function $M(A)$ on rectangles in $\mathbb{R}^n$ by $M(R_t)=M_t$, and then extend it by additivity to all rectangles. The notions of martingale, strong martingale, and i-martingale, $i=1,\dots,n$, are then the same as before. A stopping domain is a subset of $\mathbb{R}_+^n$. In the discrete case, where the I_j above are of the form $I_j=\{0,1,\dots,m_j\}$, then D is a *stopping domain* if it is a union of R_t, for $t\in I$, and if, whenever $t\in I$ and s is a nearest neighbor of t, $\{s\in D\}\in\mathscr{F}_t$. The set $L=D-D^0$ is a *stopping surface*, rather than a stopping line. The elementary properties of martingales and stopping domains are the same in n parameters as in two. The first result to change is the decomposition of a strong martingale along a stopping surface.

Suppose $\{M_t,\mathscr{F}_t, t\in I\}$ is a strong martingale which vanishes on the axes. To see how the decomposition changes, if $t=(t_1,\dots,t_n)$, put $R_t^i=R_{(\infty,\dots,t_i,\dots,\infty)}=\{s=(s_1,\dots,s_n):\ s_i\leqq t_i\}$, and note that if D is a stopping domain and if $t\in L$, that $D=(D\cap R_t^1)\cup\dots\cup(D\cap R_t^n)$. By the inclusion-exclusion principle, on $\{t\in L\}$ we have

$$\begin{aligned}M(D)=&\sum_{j=1}^{n}M(D\cap R_t^j)-\sum_{i\neq j}M(D\cap R_t^i\cap R_t^j)\\&+\sum_{i\neq j\neq k}M(D\cap R_t^i\cap R_t^j\cap R_t^k)-\dots+(-1)^{n+1}M(D\cap R_t^1\cap\dots\cap R_t^n).\end{aligned}\tag{5.1}$$

But, if $t\in L\subset D$, $D\cap R_t^1\cap\cdots\cap R_t^n=R_t$ and, since M vanishes on the axes, the last term is $(-1)^{n+1}M_t$. If $t=(t_1,\ldots,t_n)$ we define for each of the indices $1\leqq i<j<\cdots<k\leqq n$

$$M_t^{ij\ldots k}=M(D\cap R_t^i\cap R_t^j\cap\cdots\cap R_t^k),\qquad M_t^0\equiv M(D). \tag{5.2}$$

If we solve (5.1) for M_t, we get

$$(-1)^n M_t=\sum_{i=1}^{n} M_t^i-\sum_{i\neq j} M_t^{ij}+\cdots-M_t^0\qquad \text{on } \{t\in L\}. \tag{5.3}$$

Notice that M_t^i depends only on the i^{th} coordinate t_i of t, so it is a one-parameter process. Similarly, M_t^{ij} is two-parameter process, M_t^{ijk} is a three-parameter process, and so on. Suppose the stopping domain is bounded, so that $M(D)$ is integrable. Then one verifies as before that $\{M_t^i, t\in I\}$ is a martingale for each i, that $\{M_t^{ij}, t\in I\}$ and $\{M_t^{ijk}, t\in I\}$ are two-parameter and three-parameter strong martingales, respectively, relative to the fields $\mathscr{G}_t^i=\mathscr{F}_{D\cap R_t^i}$, $\mathscr{G}_t^{ij}=\mathscr{F}_{D\cap R_t^i\cap R_t^j}$, and so on. This follows from the n-parameter analogue of Proposition 2.2. Indeed, consider the case of M^{ij}. If A is a rectangle in $\mathbb{R}_+^2$ with lower-left-hand and upper-right-hand corners (u_1,u_2) and (v_1,v_2) respectively, put $\hat{A}=\{t\in\mathbb{R}_+^n$: $(t_i,t_j)\in A)$ and note that $M^{ij}(A)=M(D\cap\hat{A})$. Further, $D\cap\hat{A}$ can be written as the difference D_2-D_1 of the two stopping regions D_1 and D_2, where $D_1=D\cap\{t\in\mathbb{R}_+^n\colon t_i\leqq u_1$ or $t_j\leqq u_2\}$ and $D_2=D\cap\{t\in\mathbb{R}_+^n\colon t_i\leqq v_i$ and $t_j\leqq v_2\}$. Thus, from Proposition 2.2:

$$E\{M^{ij}(A)|(\mathscr{G}_{u_1u_2}^{ij})^*\}=E\{M(D_2)-M(D_1)|\mathscr{F}_{D_1}\}=0.$$

To summarize, we have

Theorem 5.1. *Let $\{M_t, t\in I\}$ be an n-parameter strong martingale which vanishes on the axes and let f be the final element of I. Let D be a stopping domain and let L be its associated stopping surface. Then the decomposition* (5.3) *is valid, where the $M_t^0, M_t^i, M_t^{ij},\ldots$ are all strong martingales of* $0,1,2,\ldots,n-1$ *parameters respectively, and, moreover, one can add M_f as a final element of each of these strong martingales.*

The maximal inequality follows nearly as before except that we must use an induction on the number of parameters.

Theorem 5.2. *There exists a constant c_n such that if $\{M_t,\mathscr{F}_t, t\in\mathbb{R}_t^2\}$ is an n-parameter separable strong martingale and $\lambda>0$, then*

$$\lambda P\{\max_{t\in I}|M_t|\geqq\lambda\}\leqq c_n\sup_{t\in I}E\{|M_t|\}. \tag{5.4}$$

Proof. First assume I is finite and discrete and that M vanishes on the axes. Let L be the first hitting line of the set $\{|M_t|\geqq\lambda\}$. If M_t exceeds λ, it must do so somewhere on L, and if it exceeds λ on L, one of the 2^n-1 processes on the right-hand side of (5.3) must exceed $\lambda/2^n-1$. Note that each value of $M_t^{ij\ldots k}$ is a value of one of the $(n-1)$-parameter martingales $M^{\hat{\imath}}$, where $\hat{\imath}=1,2,\ldots,i-1,i$

$+1, \ldots, n$. Then

$$\lambda P\{\sup_t |M_t| \geqq \lambda\} \leqq \sum_1^n \lambda P\left\{\sup_t |M_t^{\hat{i}}| \geqq \frac{\lambda}{2^n - 1}\right\}.$$

Since M^i has final element M_f, we can write, if d_{n-1} is the constant occurring in (5.4) for $(n-1)$-parameter strong martingales which vanish on the axes:

$$\leqq (2^n - 1)\, n d_{n-1} E\{|M_f|\}.$$

If M does not vanish on the axes, we may reparameterize and extend it to be constant on the axes by an extension of Lemma 3.1. If $a = |E\{M_0\}|$ and $N_t = M_t - E\{M_0\}$

$$P\{\sup |M_t| \geqq \lambda\} \leqq P\{\sup |N_t| \geqq \lambda - a\} \leqq d_n \frac{E\{|N_f|\}}{\lambda - a}$$

$$\leqq \frac{2 d_n E\{|M_f|\}}{\lambda - a}.$$

This inequality is only useful if the right hand side is smaller than one, which only happens if $\frac{2 d_n a}{\lambda - a} \leqq 1$ or $a \leqq \frac{\lambda}{2 d_n + 1}$. Setting a equal to this, we get

$$\lambda P\{\sup |M_t| \geqq \lambda\} \leqq (2 d_n + 1) E\{|M_f|\},$$

proving (5.3) in this case with $c_n = 2 d_n + 1$. One then passes to the case where $I = \mathbb{R}_+^n$ by taking I_n to be a finite discrete set, and letting I_n increase to a separability set for $\{M_t\}$. qed.

Remark. $d_1 = 1$; since $d_n = n(2^n - 1) d_{n-1}$, the c_n in (5.4) grow roughly as e^{n^2}.

The convergence theorem for n-parameter strong martingales follows exactly as before:

Theorem 5.3. *Let $I_1, \ldots, I_n$ be subsets of $\mathbb{R}$ which are unbounded both above and below, and put $I = I_1 \times \cdots \times I_n$. If $\{M_t, \mathscr{F}_t, t \in I\}$ is a separable strong martingale, then*

i) $\lim_{\substack{t \to -\infty \\ t \in I}} M_t$ *exists a.s. and in L^1;*

ii) *If* $\sup_t E\{|M_t|\} < \infty$, *then* $\lim_{\substack{t \to \infty \\ t \in I}} M_t$ *exists a.s.*

The regularity properties of n-parameter martingales now follow from this as before: fix $n-1$ of the parameters, and show that in the i^{th} parameter, $t_i \to M_t$ is right continuous along the rationals, uniformly for all values of the other $n-1$ parameters which fall in any bounded countable set. We conclude from this as before that

Theorem 5.4. *Let $\{M_t, \mathscr{F}_t, t \in \mathbb{R}_+^n\}$ be a strong martingale. Then it has a version which is a.s. right-continuous.*

The analogue of the sectorial convergence (Corollary 4.3) looks slightly different. With $n>2$ parameters, the uniform right continuity only allows us to conclude that:

Corollary 5.5. *Let* $\{M_t, \mathscr{F}_t, t\in\mathbb{R}^n_+\}$ *be a right-continuous strong martingale. Then for a.e.* ω *and each* $i=1,\dots,n$,

$$\lim_{t\to s}\{M_t\colon t_j\geqq s_j \text{ if } j\neq i, \textit{ and } t_i<s_i\} \qquad \textit{exists a.s.}$$

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