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Excursions of Brownian Motion and Bessel Processes

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1. Introduction

Let $B=(B_t)_{t\geqq 0}$ be the standard one dimensional Brownian motion process, and let P^x be the law of B under the condition $B_0=x$, $x\in\mathbb{R}$. It will be convenient to write $P=P^0$. Let $R=\inf\{t: B_t=0\}$ be the hitting time of 0. If $t>0$, define

$$(1.1)\quad G_t=\sup\{s<t: B_s=0\};\qquad D_t=\inf\{s>t: B_s=0\}.$$

Then $D_t=t+R\circ\theta_t$ and it is well known that $P[0<G_t<t<D_t<\infty]=1$. The interval (G_t, D_t) is called the *excursion* interval straddling t, and the process $|B_s|$, $G_t<s<D_t$ is called the *excursion* process. Note that we follow Chung [3] in letting $|B|$ be the excursion process rather than B as in [4]. This is more natural here.

If $a\geqq 0$ and $\varepsilon>0$ define

$$(1.2)\quad S(t; a, \varepsilon)=\int_{G_t}^{D_t} 1_{[a, a+\varepsilon]}(|B_s|)\, ds.$$

Thus $S(t; a, \varepsilon)$ is the amount of time the excursion process spends in the interval $[a, a+\varepsilon]$. When $a=0$ we write $S(t; \varepsilon)=S(t; 0, \varepsilon)$. In [3], Chung showed that $S(t; \varepsilon)/\varepsilon^2$ has a limiting distribution under P as $\varepsilon\downarrow 0$, and he proposed the problem of finding this limiting distribution. The solution of this problem was the starting point of the present investigation. In Section 2 we prove that

$$(1.3)\quad \lim_{\varepsilon\downarrow 0} E\{e^{-\beta S(t;\, \varepsilon)/\varepsilon^2}\}=(\cosh\sqrt{2\beta})^{-2}.$$

It is a standard fact (see p. 29 of [6], for example) that if $R^*=\inf\{s: |B_s|=1\}$, then

$$(1.4)\quad E^0(e^{-\beta R^*})=(\cosh\sqrt{2\beta})^{-1}.$$

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Thus (1.3) is the convolution of this first passage distribution with itself, and represents "independent" contributions from the two ends of the excursion. Our original proof of (1.3) was extremely computational in nature. However, after seeing how simple the answer was we were able to find a non-computational proof which clearly exhibits the "reason" behind (1.3). This proof is presented in Section 2. This method permits us to calculate the limiting distribution of

$$(1.5)\quad \varepsilon^{-(\gamma+2)}\int_{G_t}^{D_t}|B_s|^\gamma\, 1_{[0,\varepsilon]}(|B_s|)\,ds$$

as $\varepsilon\downarrow 0$ for $\gamma>-2$, and these results are presented in Section 6.

If $l^a(s)$ is the local time at a for $|B|$ appropriately normalized, then

$$(1.6)\quad S(t;a,\varepsilon)=\int_a^{a+\varepsilon}[l^x(D_t)-l^x(G_t)]\,dx,$$

and so the continuity of l^x implies

$$(1.7)\quad \varepsilon^{-1}S(t;a,\varepsilon)\to l^a(D_t)-l^a(G_t)\overset{\text{def}}{=}Z(a,t)$$

almost surely as $\varepsilon\downarrow 0$. Of course, $Z(0,t)=0$. It is not surprising that $a^{-1}Z(a,t)$ has a limiting distribution as $a\downarrow 0$, and, in Section 3, we show that this limiting distribution is the convolution of two exponential distributions each having parameter 1/2 (i.e. mean 2). The actual distribution of $Z(a,t)$ seems more difficult. Formula (3.17) contains the distribution of $Z(a,t)$ via a double Laplace transform so dear to the hearts of applied probabilists. Although we have been unable to invert (3.17) we are able to use it to find an explicit expression for $P[Z(a,t)=0]$. See (3.19). If $M_t=\sup\{|B_s|: G_t<s<D_t\}$ is the maximum of the excursion straddling t, then it is clear that almost surely $Z(a,t)=0$ if and only if $M_t<a$. Thus (3.19) gives the distribution of M_t and should be compared with Theorem 7 in Chung [3].

Our method for obtaining the limiting distributions in (1.5) involves certain first passage distributions for Bessel diffusions on $\mathbb{R}^+=[0,\infty)$, just as the case $\gamma=0$ involves a first passage distribution for $|B|$. In Section 5 we present the distributions that we need as well as some related facts. Many of these results are known, and the techniques for obtaining them are more or less standard. We defer the actual calculations to Section 8. However, a number of curious facts and relationships emerge from these calculations. We shall describe some of the more striking ones here in an informal manner. Let X^ν be the Bessel process of index ν as described in Section 5. If $\nu=(d-2)/2$, then X^ν is equivalent to the modulus of a d-dimensional Brownian motion, $d\geqq 1$. Note $d=1$ corresponds to $\nu=-1/2$. If $a\geqq 0$ let $T_a^\nu=\inf\{t: X_t^\nu=a\}$ be the first passage time to a for X^ν and $L_a^\nu=\sup\{t: X_t^\nu=a\}$ be the last exit time from a. We write P^x for the law of X^ν starting at x, but suppress the dependence on ν. The following remarks are expanded upon in Section 5. See (5.9) and (5.11) in particular.

(1.8) If $\nu>0$, the P^0 distribution of the total time X^ν spends in $(0,a)$ is the same as the P^0 distribution of $T_a^{\nu-1}$. If $\nu=(d-2)/2$ and $d\geqq 3$, this phenomenon was already observed by Ciesielski and Taylor [2].

(1.9) If $\nu>0$, the P^0 distribution of L_a^ν is the same as the P^a distribution of $T_0^{-\nu}$.

(1.10) Consider X^ν on $(0, a]$ with a reflecting barrier at a, and let $\bar{X}^\nu$ be this process (see Section 5 for a formal definition). If $\nu<0$ the P^a distribution of $\bar{T}_0^\nu$ is the same as the P^0 distribution of $T_a^{-\nu-1}$.

In the course of proving (1.10) we need an interesting fact about diffusions which is proved in Section 4. Let X be a regular diffusion on an interval I of $\mathbb{R}$ for which killing can occur only at the endpoints of I. Let $A=(A_t)$ be a continuous additive functional of X whose support is a compact interval $J=[a, b]$ contained in the interior of I. Let $\tilde{X}$ be the process obtained from X by a time change based on A. Then it is well known that $\tilde{X}$ is a diffusion on J with the same scale as X and speed measure the measure μ_A naturally associated with A. (See Section 4.) Thus to completely describe $\tilde{X}$ as a diffusion one needs to determine the "boundary conditions" that the generator $\tilde{\Gamma}$ of $\tilde{X}$ satisfies at the endpoints of J. These boundary conditions are given in (4.9) and (4.11). The interesting point is that if μ_A is continuous these boundary conditions depend only on the interval J and the original diffusion X and not on the particular form of A.

Sections 2 and 3 require only basic facts about Brownian motion. In the remaining sections some knowledge of diffusion theory and Bessel functions is needed. We actually obtain the limiting distribution of functionals of the form (1.5) with $|B|$ replaced by a Bessel process X^ν with $-1<\nu<0$ in Section 6.

2. Proof of (1.3)

The notation is that of Section 1. In particular $R=\inf\{t: B_t=0\}$ is the hitting time of 0. We begin the proof of (1.3) with the following proposition.

(2.1) **Proposition.** *Let* $x\geqq\varepsilon>0$. *Let* $A^\varepsilon(t)=\int_0^t 1_{[0,\varepsilon]}(B_s)\,ds$. *Then*

$$E^x\{\exp[-\beta A^\varepsilon(R)]\}=(\cosh \varepsilon\sqrt{2\beta})^{-1}.$$

Proof. Observe that $A^\varepsilon(R)$ is the amount of time the Brownian motion spends below the level ε before hitting zero. Clearly if $x>\varepsilon$

$$(2.2)\quad E^x\{\exp[-\beta A^\varepsilon(R)]\}=E^\varepsilon\{\exp[-\beta A^\varepsilon(R)]\},$$

and so we need only consider $x=\varepsilon$.

Let

$$(2.3)\quad H(t)=\int_0^t 1_{[0,\infty)}(B_s)\,ds,$$

and let $T=T(\varepsilon)=\inf\{t: B_t=\varepsilon\}$. Then the symmetry properties of the Brownian motion imply that the distribution of $A^\varepsilon(R)$ under P^ε is the same as the distribution of $H(T)$ under $P=P^0$. Note that $H(T)$ is the amount of time B spends *above* the level 0 before hitting ε. Next let

$$\tau(t)=\inf\{s: H(s)>t\}$$

be the right continuous inverse of H. Then it is well known that $X_t = B_{\tau(t)}$ under P^0 is the reflecting Brownian motion; that is, X and $|B|$ are equivalent processes under P^0. See for example Section 5.3 of [6]. This is also a special case of (4.8) in the present paper. But $H(T)$ is just the hitting time of ε by the process X. It is then immediate from (1.4) and the scaling property of Brownian motion [see also (5.7) (iv)] that

$$E(e^{-\beta H(T)}) = (\cosh \varepsilon \sqrt{2\beta})^{-1}.$$

Combining these observations completes the proof of (2.1).

Replacing β by β/ε^2, (2.1) may be written in the form

$$(2.4)\quad E^x\{\exp[-\beta A^\varepsilon(R)/\varepsilon^2]\} = (\cosh \sqrt{2\beta})^{-1}; \qquad x \geqq \varepsilon.$$

(2.5) **Corollary.** *Fix $x>0$ and $t>0$. Let h be a positive increasing continuous function defined on $[0, t)$ such that $E^x\{h(R); R<t\}<\infty$. Then*

$$E^x\{h(R)\, e^{-\beta A^\varepsilon(R)/\varepsilon^2}; R<t\} \to \frac{E^x[h(R); R<t]}{\cosh \sqrt{2\beta}}$$

as $\varepsilon \downarrow 0$.

Proof. Let $T(\varepsilon) = \inf\{s: B_s = \varepsilon\}$. If $x>0$, as $\varepsilon \downarrow 0$, $T(\varepsilon) \uparrow R$ almost surely P^x. For notational simplicity let $A(s) = \varepsilon^{-2} A^\varepsilon(s)$. For $0<\varepsilon<x$, a.s. P^x $A(R) = A(T(\varepsilon)) + A(R) \circ \theta_{T(\varepsilon)} = A(R) \circ \theta_{T(\varepsilon)}$ because A is an additive functional and $A(T(\varepsilon)) = 0$. In what follows $\theta(\varepsilon)$ will denote a quantity that tends to 0 as $\varepsilon \downarrow 0$, but which may differ from one occurrence to the next. Now using the properties of h

$$(2.6)\quad E^x\{h(R)\, e^{-\beta A(R)}; R<t\} = E^x\{h(T(\varepsilon))\, e^{-\beta A(R)}: T(\varepsilon)<t\} + \theta(\varepsilon).$$

But $B(T(\varepsilon)) = \varepsilon$, and so the strong Markov property and (2.4) give

$$\begin{aligned} &E^x\{h(T(\varepsilon))\, e^{-\beta A(R)}; T(\varepsilon)<t\} \\ &\quad = E^x\{h(T(\varepsilon))\, E^\varepsilon\{e^{-\beta A(R)}\}; T(\varepsilon)<t\} \\ &\quad = (\cosh \sqrt{2\beta})^{-1}\, E^x\{h(T(\varepsilon)): T(\varepsilon)<t\} \\ &\quad \to (\cosh \sqrt{2\beta})^{-1}\, E^x\{h(R); R<t\} \end{aligned}$$

as $\varepsilon \downarrow 0$, since $P^x(R=t) = 0$. Combining this with (2.6) yields (2.5).

We are now ready to prove (1.3). Fix $t>0$ and recall that $S(t;\varepsilon) = S(t; 0, \varepsilon)$ where $S(t; a, \varepsilon)$ is defined in (1.2). Let $f_\varepsilon(x) = \varepsilon^{-2}\, 1_{[0,\varepsilon]}(|x|)$. Then

$$(2.7)\quad \Gamma(\varepsilon) = E\{e^{-\beta S(t;\varepsilon)/\varepsilon^2}\} = E\left\{\exp\left[-\beta \int_{G_t}^{D_t} f_\varepsilon(B_s)\, ds\right]\right\}.$$

Writing $A(s) = \varepsilon^{-2} \int_0^s 1_{[0,\varepsilon]}(B_u)\, du$ as before, and $\Lambda(t) = \exp\left[-\beta \int_{G_t}^t f_\varepsilon(B_s)\, ds\right]$ we have

$$\begin{aligned} \Gamma(\varepsilon) &= E\left\{\Lambda(t) \exp\left[-\beta \int_t^{D_t} f_\varepsilon(B_s)\, ds\right]\right\} \\ &= E\{\Lambda(t)\, E^{|B_t|}(e^{-\beta A(R)})\} \\ &= E\{\Lambda(t)\, E^{|B_t|}(e^{-\beta A(R)}): |B_t| \geqq \varepsilon\} + \theta(\varepsilon), \end{aligned}$$

because $A(t)$ is $\mathscr{F}_t$ measurable. Using (2.4) and introducing one more $\theta(\varepsilon)$ term this becomes

(2.8) $$\Gamma(\varepsilon)=(\cosh\sqrt{2\beta})^{-1}\,E\left\{\exp\left[-\beta\int_{G_t}^{t} f_\varepsilon(B_s)\,ds\right]\right\}+\theta(\varepsilon).$$

We shall now use a "time reversal" argument to evaluate the limit of the expectation in (2.8) as $\varepsilon\downarrow 0$. Let $P^{x;\,t,\,y}$ be the usual "tied down by $B_0=x$ and $B_t=y$" Brownian probability on $\mathscr{F}_t^0$; that is, $P^{x;\,t,\,y}$ is the unique probability on $\mathscr{F}_t^0$ with the following property: If $0<t_1<\ldots<t_n<t$ and $E_1,\ldots,E_n$ are Borel subsets of $\mathbb{R}$, then

$$P^{x;\,t,\,y}(B_{t_j}\in E_j\colon 1\leqq j\leqq n)$$
$$=\int_{E_1}\ldots\int_{E_n}\frac{p(t_1,x,x_1)\,p(t_2-t_1,x_1,x_2)\ldots p(t-t_n,x_n,y)\,dx_n\ldots dx_1}{p(t,x,y)}$$

where $p(t,x,y)=(2\pi t)^{-1/2}e^{-(x-y)^2/2t}$ is the Brownian transition density. It is immediate that if $Z\in b\mathscr{F}_t^0$, then $E^{x;\,t,\,y}(Z)$ is a version $E^x\{Z\,|\,B_t=y\}$. Next let r_t be the reversal from t operator; that is, $B_s\circ r_t=B_{t-s}$, $0\leqq s\leqq t$. The following equation expresses the familiar and easily checked time reversal property of Brownian motion. If $Z\in b\mathscr{F}_t^0$, then

(2.9) $$E^{x;\,t,\,y}[Z]=E^{y;\,t,\,x}[Z\circ r_t].$$

We leave it to the reader to check the following lemma.

(2.10) **Lemma.** *If T is an $(\mathscr{F}_{t+}^0)$ stopping time and $Z\in b\mathscr{F}_{T+}^0$, then $Z1_{\{T<t\}}\in\mathscr{F}_{t+}^0$ and*

$$E^{x;\,t,\,y}[Z;\,T<t]=\frac{E^x\{Z\,p(t-T,B_T,y);\,T<t\}}{p(t,x,y)}.$$

Next observe that $R=\inf\{t\colon B_t=0\}$ is an $(\mathscr{F}_{t+}^0)$ stopping time, and recall that (see Sect. 1.7 of [6], for example)

(2.11) $$E^x\{e^{-\beta R}\}=e^{-|x|\sqrt{2\beta}}.$$

From (2.10), we have for $x\neq 0$

(2.12) $$P^{x;\,t,\,0}[R<t]=\frac{E^x\{p(t-R,0,0);\,R<t\}}{p(t,x,0)}$$
$$=[p(t,x,0)]^{-1}\int_0^t[2\pi(t-s)]^{-1/2}\,P^x(R\in ds)=1,$$

where the last equality follows from (2.11) by direct computation. It also appears as formula (2.8) in [3]. We prove the analogous result for a general diffusion in Lemma 6.2.

One easily checks that

(2.13) $$\int_{G_t}^{t} f(B_s)\,ds\circ r_t=\int_0^{t\wedge R} f(B_s)\,ds$$

whenever f is a positive Borel function. Let

$$C(t)=\int_{G_t}^{t} f_\varepsilon(B_s)\,ds=\varepsilon^{-2}\int_{G_t}^{t} 1_{[0,\varepsilon]}(|B_s|)\,ds.$$

Then using (2.9), (2.10), (2.11) and (2.13) we find

$$\begin{aligned}(2.14)\quad E^0\{e^{-\beta C(t)}\}&=2\int_0^\infty p(t,0,x)\,E^{0;\,t,x}\{e^{-\beta C(t)}\}\,dx\\ &=2\int_0^\infty p(t,0,x)\,E^{x;\,t,0}\{e^{-\beta A(t\wedge R)}\}\,dx\\ &=2\int_0^\infty E^x\{e^{-\beta A(R)}\,p(t-R,0,0);\,R<t\}\,dx\end{aligned}$$

since $p(t,0,x)=p(t,x,0)$. Here $A(s)=\int_0^s f_\varepsilon(B_u)\,du$ as before. Applying (2.5) with $h(u)=p(t-u,0,0)=[2\pi(t-u)]^{-1/2}$ on $0\leqq u<t$ results in

$$E^x\{e^{-\beta A(R)}\,p(t-R,0,0);\,R<t\}\to(\cosh\sqrt{2\beta})^{-1}\,E^x\{p(t-R,0,0);\,R<t\}$$

as $\varepsilon\downarrow 0$. Moreover, as in (2.12),

$$\begin{aligned}E^x\{e^{-\beta A(R)}\,p(t-R,0,0);\,R<t\}&\leqq E^x\{p(t-R,0,0);\,R<t\}\\ &=\int_0^t \frac{1}{\sqrt{2\pi(t-s)}}\,P^x(R\in ds)=p(t,x,0).\end{aligned}$$

This estimate allows us to pass to the limit as $\varepsilon\downarrow 0$ in (2.14) and obtain

$$E\left\{\exp\left[-\beta\int_{G_t}^{t} f_\varepsilon(B_s)\,ds\right]\right\}\to(\cosh\sqrt{2\beta})^{-1}\,2\int_0^\infty p(t,x,0)\,dx=(\cosh\sqrt{2\beta})^{-1}.$$

Combining this with (2.7) and (2.8) completes the proof of (1.3).

(2.15) *Remark.* The argument leading from (2.4) to (1.3) is quite general. Suppose for example that for each $\varepsilon>0$, $A^\varepsilon(t)$ is a continuous additive functional satisfying

$$(2.16)\quad E^x\{e^{-\beta A^\varepsilon(R)}\}=g(\beta)\qquad \text{for}\quad x\geqq\varepsilon>0$$

where g is independent of x and ε. Thus A^ε is now playing the role of $\varepsilon^{-2}A^\varepsilon$ in (2.4) and what follows. Then (2.5) remains true with $(\cosh\sqrt{2\beta})^{-1}$ replaced by $g(\beta)$. If, in addition, A^ε satisfies

$$(2.17)\quad [A^\varepsilon(t)-A^\varepsilon(G_t)]\circ r_t=A^\varepsilon(t\wedge R),$$

then the above argument may be repeated word for word to obtain

$$(2.18)\quad \lim_{\varepsilon\downarrow 0} E^0\{e^{-\beta[A^\varepsilon(D(t))-A^\varepsilon(G(t))]}\}=(g(\beta))^2.$$

We shall make use of this remark in Section 6. Moreover, it follows from (2.13) and standard approximation theorems for additive functionals of Brownian motion that (2.17) holds for all finite continuous additive functionals of Brownian motion.

3. The Case $a>0$

In the introduction we pointed out that if $a>0$, then

$$(3.1)\quad \lim_{\varepsilon\downarrow 0} \varepsilon^{-1} S(t; a, \varepsilon)=l^a(D_t)-l^a(G_t) \stackrel{\text{def}}{=} Z(a, t),$$

where l^a is the local time at a for $|B|$; that is, the local time for the reflecting Brownian motion. We normalize l^a so that for $x\geqq 0$, $a\geqq 0$, and $\alpha>0$

$$(3.2)\quad E^x \int_0^\infty e^{-\alpha s}\, d l_s^a = u^\alpha(|x-a|)+u^\alpha(x+a),$$

where for $\alpha>0$

$$(3.3)\quad u^\alpha(x)=\frac{1}{\sqrt{2\alpha}}\, e^{-|x|\sqrt{2\alpha}}=\int_0^\infty e^{-\alpha s}(2\pi s)^{-1/2}\, e^{-x^2/2s}\, ds$$

is the α-potential kernel for B. Of course, the right side of (3.2) is just the α-potential kernel of $|B|$, the reflecting Brownian motion.

In this section we are going to study the distribution of $Z(a, t)$ under P^0. We begin with the following elementary lemma which will also be used in the next section.

(3.4) **Lemma.** *Let X be a Hunt process and let T be a terminal time for X. Let A be a continuous additive functional of X with $A_T<\infty$. Define for α, $\beta\geqq 0$.*

$$(3.5)\quad \varphi(x)=\varphi(x; \beta, \alpha)=E^x\{e^{-\beta A(T)}\, e^{-\alpha T}\}.$$

Then

$$(3.6)\quad \varphi(x; \beta, \alpha)=\varphi(x; 0, \alpha)-\beta E^x \int_0^T e^{-\alpha s}\, \varphi(X_s; \beta, \alpha)\, dA_s.$$

In particular if $\psi(x)=\varphi(x; \beta, 0)$, then

$$(3.7)\quad \psi(x)=1-\beta E^x \int_0^T \psi(X_s)\, dA_s.$$

Proof. Since A is continuous and $A_T<\infty$, one has

$$\begin{aligned} e^{-\beta A(T)} &= 1-\beta \int_0^T e^{-\beta(A(T)-A(s))}\, dA_s \\ &= 1-\beta\int_0^T e^{-\beta A(T)\circ\theta_s}\, dA_s. \end{aligned}$$

If we multiply this by $e^{-\alpha T}$ which is equal to $e^{-\alpha(s+T\circ\theta_s)}$ on $\{s<T\}$ and take expectations, we obtain (3.6).

We shall now apply (3.4) with $X=|B|$, $A=l^a$, and $T=R=\inf\{t: |B_t|=0\}$. Let

$$(3.8)\quad \varphi(x)=\varphi(x;\beta,\alpha)=E^x\{\exp[-\beta l^a(R)-\alpha R]\}.$$

If $x\geqq 0$ it is well known that

$$(3.9)\quad \varphi(x;0,\alpha)=E^x\{e^{-\alpha R}\}=e^{-x\sqrt{2\alpha}},$$

and so from (3.6) we obtain for $x\geqq 0$

$$\begin{aligned}(3.10)\quad \varphi(x)&=e^{-x\sqrt{2\alpha}}-\beta E^x\int_0^R e^{-\alpha s}\,\varphi(X_s)\,dl_s^a\\ &=e^{-x\sqrt{2\alpha}}-\beta\varphi(a)\,E^x\int_0^R e^{-\alpha s}\,dl_s^a.\end{aligned}$$

But (3.2) and (3.9) imply that for $\alpha>0$

$$(3.11)\quad E^x\int_0^R e^{-\alpha s}\,dl_s^a=\frac{1}{\sqrt{2\alpha}}\,[e^{-\sqrt{2\alpha}\,|x-a|}-e^{-\sqrt{2\alpha}\,(x+a)}].$$

Substitute this into (3.10), set $x=a$, and solve to find

$$(3.12)\quad \varphi(a)=\frac{\sqrt{2\alpha}\,e^{-a\sqrt{2\alpha}}}{\sqrt{2\alpha}+\beta[1-e^{-2a\sqrt{2\alpha}}]}.$$

Finally substituting (3.11) and (3.12) into (3.10) we have

$$(3.13)\quad \varphi(x;\beta,\alpha)=\frac{(\beta+\sqrt{2\alpha})\,e^{-x\sqrt{2\alpha}}-\beta\,e^{-\sqrt{2\alpha}\,(a+|x-a|)}}{\sqrt{2\alpha}+\beta(1-e^{-2a\sqrt{2\alpha}})}.$$

In deriving (3.13) we assumed $\alpha>0$. If we let $\alpha\to 0$ we find

$$\begin{aligned}(3.14)\quad \varphi(x;\beta,0)&=E^x\{e^{-\beta l^a(R)}\}\\ &=\begin{cases}\dfrac{1}{1+2a\beta}, & \text{if } x\geqq a\\[2ex] \left(1-\dfrac{x}{a}\right)+\dfrac{x}{a}\,\dfrac{1}{1+2a\beta}, & \text{if } x<a.\end{cases}\end{aligned}$$

Hence we have established the following well known results. If $x\geqq a$, then $l^a(R)$ has an exponential distribution with parameter $\dfrac{1}{2a}$ under P^x, while if $0<x<a$ the distribution of $l^a(R)$ under P^x is a mixture of unit mass at the origin and this exponential with weights $1-\dfrac{x}{a}$ and $\dfrac{x}{a}$ respectively.

Recalling the definition of $Z(a, t)$ in (3.1), the following limit theorem is an immediate consequence of (2.18) and (3.14)

$$\text{(3.15)} \quad \lim_{a \downarrow 0} E\{e^{-\beta Z(a,t)/a}\} = \left(\frac{1}{1+2\beta}\right)^2.$$

The limiting distribution in (3.15) is the convolution of two exponential distributions with parameter 1/2 and should be compared with the limit theorem of (1.3).

We are now going to look at the distribution of $Z(a, t)$. Let

$$H(a, t; \beta) = E\{e^{-\beta Z(a,t)}\}.$$

Then, arguing as in Section 2,

$$\begin{aligned} H(a, t; \beta) &= E\{e^{-\beta[l^a(t) - l^a(G_t)]}\, e^{-\beta l^a_R \circ \theta_t}\} \\ &= E\{e^{-\beta[l^a(t) - l^a(G_t)]}\, \varphi(X_t; \beta, 0)\} \end{aligned}$$

where $\varphi(x; \beta, \alpha)$ is defined in (3.8). Using the reversal argument which led to (2.14) we obtain

$$H(a, t; \beta) = 2 \int_0^\infty \varphi(x; \beta, 0)\, E^x\left\{\frac{e^{-\beta l^a(R)}}{\sqrt{2\pi(t-R)}}; R < t\right\} dx.$$

Taking the Laplace transform in t yields in light of (3.8)

$$\text{(3.16)} \quad \int_0^\infty e^{-\alpha t} H(a, t; \beta)\, dt = \frac{2}{\sqrt{2\alpha}} \int_0^\infty \varphi(x; \beta, 0)\, \varphi(x; \beta, \alpha)\, dx.$$

Finally substituting (3.13) and (3.14) into the above, one obtains after a straightforward but tedious calculation

$$\begin{aligned} \text{(3.17)} \quad & \int_0^\infty e^{-\alpha t} E^0\{e^{-\beta Z(a,t)}\}\, dt \\ &= \frac{1}{\alpha} \frac{\sqrt{2\alpha} + \beta(1 + e^{-2a\sqrt{2\alpha}})}{\sqrt{2\alpha} + \beta(1 - e^{-2a\sqrt{2\alpha}})} - \frac{\sqrt{2}\beta}{(1 + 2a\beta)\,\alpha^{3/2}}. \end{aligned}$$

This formula contains the distribution of $Z(a, t)$ via a double Laplace transform. However, it does not seem to be easily invertible. On the other hand letting $\beta \to \infty$ in (3.17) gives

$$\text{(3.18)} \quad \int_0^\infty e^{-\alpha t} P[Z(a, t) = 0]\, dt = \frac{\cosh a\sqrt{2\alpha}}{\alpha \sinh a\sqrt{2\alpha}} - \frac{1}{\sqrt{2}\, a\, \alpha^{3/2}}.$$

But $Z(a, t) = l^a(D_t) - l^a(G_t) = 0$ if and only if $M_t = \sup\{|B_s| : G_t < s < D_t\} < a$. Also the right side of (3.18) can be inverted explicitly – see for example pp. 136, 98, and 99 of [7] – to obtain

(3.19) $$P[Z(a,t)=0]=P[M_t<a]$$
$$=\sqrt{\frac{2}{\pi a^2}}\sum_{n\geq 1} e^{-n^2\pi^2 t/2a^2}\int_0^t e^{n^2\pi^2 s/2a^2}\frac{ds}{\sqrt{s}}$$
$$=1-\sqrt{\frac{2t}{a^2\pi}}+\sum_{n\geq 1}\frac{4an}{\sqrt{2\pi}}\int_{1/t}^{\infty} e^{-2a^2n^2 s}\frac{ds}{\sqrt{s}}.$$

The last equality in (3.19) is obtained using the Poisson summation formula or the theta function identity at the bottom of p. 99 of [7]. This formula should be compared with Theorem 7 in [3].

Thus the distribution of $Z(a,t)$ has an atom at the origin whose mass is given by (3.19). If we subtract off this atom what remains of the double Laplace transform in (3.17) can be inverted in β as the difference of two exponentials. However, the dependence on α seems to be too complicated for explicit inversion.

(3.20) *Remark.* Once again the argument leading to (3.16) is quite general. Let A be a continuous additive functional of $|B|$ with $A_t<\infty$ for $t<\infty$. Define

$$\varphi(x;\beta,\alpha)=E^x\{\exp[-\beta A(R)-\alpha R]\}$$

where $R=\inf\{t\colon |B_t|\}=0$. Then exactly as above one finds

(3.21) $$\int_0^\infty e^{-\alpha t} E\{e^{-\beta[A(D_t)-A(G_t)]}\}\,dt=\frac{2}{\sqrt{2\alpha}}\int_0^\infty \varphi(x;\beta,0)\,\varphi(x;\beta,\alpha)\,dx.$$

4. Distributions of Additive Functionals

In the preceding two sections, we solved two particular problems of the following general type. Given a *regular* diffusion X with state space a subinterval I of the real line, and given a continuous additive functional A of X and a terminal time T for X, determine the distribution of A_T under P^x.

In most cases, the terminal time T will be a first passage time, and we shall always use the notation $T_a=\inf\{t\colon X_t=a\}$ for $a\in I$. We shall always suppose that death occurs only at the endpoints of I. That is, if ζ denotes the lifetime of X, then a.s. on $\{\zeta<\infty\}$, $X_{\zeta-}$ is an endpoint of I.

As before, let

$$\varphi(x)=\varphi_\beta(x)=E^x\exp\{-\beta A_T\}.$$

According to (3.7), if $A_T<\infty$ almost surely P^x, then

(4.1) $$1-\varphi(x)=\beta E^x\int_0^T \varphi(X_t)\,dA_t.$$

If $dA_t=f(X_t)\,dt$, and we let V denote the potential operator for the process X killed at time T, then (4.1) leads to

(4.2) $$1-\varphi(x)=\beta V(f\varphi)(x)$$

at any x for which $P^x(A_T=\infty)=0$. For bounded continuous f, knowing the generator for X, we could then write down a differential equation for φ. For a complete solution we would need boundary conditions. Proposition (4.6) could provide them in certain cases.

We shall not, in fact, follow this direct path. One reason is that for the examples we have in mind, the differential equations require awkward substitutions and unwieldy calculations. The other, more important, reason is that we wish to point out that within the classical framework of Bessel processes, some rather interesting facts can be observed when one computes first passage time distributions, and these computations lead to formulas for $\varphi(x)$ in particular cases.

With X and A as above, let (τ_t) denote the right continuous inverse of A and let $\tilde{X}_t = X_{\tau(t)}$ denote the process obtained from X by a time change based on A. If T is a terminal time for X, then A_T is a terminal time for $\tilde{X}$. If $T=T_a$ and a is in the support of A, then A_T is just the first passage time to a for the process $\tilde{X}$. This reduces the calculation of the distribution of A_{T_a} to the calculation of a first passage distribution for the time changed diffusion $\tilde{X}$. In order to apply the standard techniques for calculating first passage distributions for $\tilde{X}$ it is necessary to describe $\tilde{X}$ in terms of its differential generator and appropriate boundary conditions. Such boundary conditions are written down in (4.7) and (4.11).

We now fix a scale function $s=s(x)$ for our regular diffusion X on I. Thus s is a strictly increasing function on I such that if $a<x<b$ and $a, b \in I^0$ (here I^0 is the interior of I), then

$$P^x[T_a<T_b]=\frac{s(b)-s(x)}{s(b)-s(a)}.$$

We shall write the generator Γ of X in the form $\Gamma=\frac{1}{2}\frac{d}{dm}\frac{d}{ds}$ together with appropriate boundary conditions. The measure m is a Radon measure on I and is called the speed measure of X. Note that our definition is slightly different from that in [6]. Ito and McKean write the generator in the form $\frac{d}{dm}\frac{d}{ds}$ so that their speed measure is twice ours. For example, our m for Brownian motion in natural scale is Lebesgue measure. This normalization is more convenient for the applications we have in mind.

If J is a subinterval of I with $\bar{J} \subset I^0$, then the potential operator for the suprocess obtained by killing X at $T=T(J)=\inf\{t\colon X_t \notin J\}$ is given by $V_J f(x)=E^x \int_0^{T(J)} f(X_t)\,dt$. It is well known that V_J has a symmetric density $v_J(x,y)$ with respect to m so that $V_J f(x)=\int v_J(x,y) f(y)\, m(dy)$. Under our normalization, if $J=(\alpha,\beta)$, then for $\alpha \leqq x \leqq y \leqq \beta$,

$$v_J(x,y)=\frac{2[s(x)-s(\alpha)]\,[s(\beta)-s(y)]}{s(\beta)-s(\alpha)}.$$

For each $y \in I$ we denote by l_t^y the local time at y normalized so that it is a density for the occupation time relative to m. That is, almost surely simultaenously for all subintervals $J \subset I$ and t one has

$$\int_0^t 1_J(X_s)\,ds = \int_J l_t^y\, m(dy).$$

If, in addition, $\bar{J} \subset I$ then

$$(4.3) \quad v_J(x,y) = E^x(l_{T(J)}^y)$$

for all $x, y \in J$ where, as before, $T(J)$ is the hitting time of J^c. It is a standard fact that every additive functional A of X that is finite on $[0,\zeta)$ is continuous and has a unique representation

$$(4.4) \quad A_t = \int_I l_t^y\, \mu_A(dy)$$

where μ_A is a Radon measure on I. In particular, if $dA_t = f(X_t)\,dt$ then $d\mu_A = f\,dm$.

(4.5) **Lemma.** *Let a be an interior point of I. Then there exist two non-negative numbers K_a^+, K_a^- such that*

(i) *for* $x \in I \cap (-\infty, a)$, $\quad P^x\{T_a = \infty\} = K_a^-(s(a) - s(x))$,

(ii) *for* $x \in I \cap (a, \infty)$, $\quad P^x\{T_a = \infty\} = K_a^+(s(x) - s(a))$.

Proof. The function $\xi(x) = P^x\{T_a = \infty\}$ for $x \in I \cap (-\infty, a)$ is harmonic for X killed at T_a in the sense that if $x \in (\alpha, \beta) \subset I \cap (-\infty, a)$ then, by an obvious application of the strong Markov property

$$\xi(x) = \xi(\alpha)\frac{(s(\beta) - s(x))}{s(\beta) - s(\alpha)} + \xi(\beta)\frac{(s(x) - s(\alpha))}{s(\beta) - s(\alpha)}.$$

Thus ξ is an affine function of s on every subinterval $(\alpha, \beta) \subset I \cap (-\infty, a)$ and consequently on $I \cap (-\infty, a)$ also. Since $\xi(a) = 0$, (i) is immediate. The proof of (ii) is exactly the same.

We remark that K_a^- and K_a^+ can be determined from s alone provided the endpoints of I are not elastic boundaries.

(4.6) **Proposition.** *Let $[a, b]$ be a subinterval of I such that a is an interior point of I. Let A_t be an additive functional of X such that μ_A is carried by $[a, b]$. If $h(x) = E^x A_\infty$ is finite, it satisfies*

$$(4.7) \quad \lim_{c \downarrow a} (h(c) - h(a))/(s(c) - s(a)) = K_a^-\, h(a) - 2\mu_A(\{a\}).$$

Proof. For sufficiently small $y > a$, there exists a unique $x \in I \cap (-\infty, a)$ with $s(y) - s(a) = s(a) - s(x)$. Let $J = (x, y)$ and $T = T(J^c)$. We may assume y chosen so close to a that x and y are interior points of I. One has $T < \zeta$, P^a a.s. and so

$$\begin{aligned} h(a) = E^a A_\infty &= E^a A_T + E^a A_\infty \circ \theta_T \\ &= E^a A_T + E^a h(X_T). \end{aligned}$$

Because of the choice of x and y, this leads to

$$h(a)=E^a A_T+\tfrac{1}{2}(h(x)+h(y)).$$

Since A is carried by $[a,b]$ and $x<a$, $A(T_a)=0$, a.s. P^x, and so

$$h(x)=E^x A_\infty=E^x A(T_a)+E^x A_\infty\circ\theta_{T_a}=P^x\{T_a<\infty\}\,E^a A_\infty.$$

Substituting in the preceding equation gives

$$(*)\quad h(y)-h(a)-h(a)\,P^x\{T_a=\infty\}+2E^a A_T=0.$$

In view of (4.3) and (4.4)

$$\begin{aligned}E^a A_T&=\int_{[a,y)} E^a l_T^z\,\mu_A(dz)=\int_{[a,y)} v_J(a,z)\,\mu_A(dz)\\&=\int_{[a,y)} 2(s(y)-s(z))(s(a)-s(x))/(s(y)-s(x))\,\mu_A(dz)\\&=\int_{[a,y)} (s(y)-s(z))\,\mu_A(dz)\end{aligned}$$

by the choice of x and y above. Using the dominated convergence theorem one sees that $E^a A_T/(s(y)-s(a))\to\mu_A\{a\}$ as $y\downarrow a$. The relation (4.7) is then apparent from $(*)$, using (4.5) and the fact that $s(y)-s(a)=s(a)-s(x)$.

(4.8) **Corollary.** *Let A and X satisfy the hypotheses of* (4.6) *except for finiteness of $E^x A_\infty$. If the support of μ_A is precisely $[a,b]$ then the diffusion $\tilde X$ obtained from X by time change based on A has the same scale function as X, its speed measure is μ_A, and $\tilde X$ can die only at the endpoints of $[a,b]$. The endpoint a is a reflecting or elastic barrier for $\tilde X$ at which death is possible, as summarized by the boundary condition* (4.9) *for a function u in the domain of the generator $\tilde\Gamma$ of $\tilde X$:*

$$(4.9)\quad \lim_{x\downarrow a}\frac{u(x)-u(a)}{s(x)-s(a)}=K_a^-\,u(a)+2\mu_A(\{a\})\,\tilde\Gamma u(a),$$

where K_a^- is as defined in (4.5).

Proof. All assertions except those in the last sentence are standard facts in diffusion theory. The boundary condition at a is the same for $\tilde X$ as for $\tilde X$ killed at a point η, with $a<\eta<b$. If $\tilde V_\eta$ denotes the potential operator for this killed subprocess, then $u=\tilde V_\eta f$, for a bounded, positive continuous f, implies that $\tilde\Gamma u(a)=-f(a)$. On the other hand

$$\tilde V_\eta f(x)=E^x\int_0^{T_\eta} f(X_t)\,dA_t.$$

The additive functional $f(X_t)\,dA_t$ corresponds to the measure $f\,d\mu_A$. Since μ_A is a Radon measure, $\int_{[a,\eta)} f\,d\mu_A<\infty$. This shows, by (4.3) and (4.4) that $E^x\int_0^{T_\eta} f(X_t)\,dA_t<\infty$ for all $x\in[a,\eta)$. Thus $f(X_t)\,dA_t$ satisfies the hypotheses of (4.6) for the process X killed at T_η. We now apply (4.7) to obtain (4.9).

(4.10) *Remark.* Trivial changes apply in the statements of (4.6) and (4.8) if one wants to consider the boundary condition at b, an interior point of I. One gets, for u in the domain of the generator $\tilde{\Gamma}$ of $\tilde{X}$

$$(4.11) \quad \lim_{x \uparrow b} \frac{u(x)-u(b)}{s(x)-s(b)} = -K_b^+ u(b) - 2\mu_A(\{b\}) \tilde{\Gamma} u(b)$$

where K_b^+ is as defined in (4.5).

We turn now to the specific problems that we wish to solve. The solutions are written down in (5.14).

(4.12) *Problems.* Let $(B_\cdot)$ denote a standard Brownian motion on the line and let $a \geqq 0$. Let γ be a real number. We wish to find formulas for the P^x distributions of

(a) $\int_0^{T_0} |B_s|^\gamma \, ds,$

(b) $\int_0^{T_0} |B_s|^\gamma \, 1_{(0,a)}(B_s) \, ds,$

(c) $\int_0^{T_a} |B_s|^\gamma \, ds \qquad (a>0).$

The method is clear, following the outline provided above. Let $A_t = \int_0^t |B_s|^\gamma \, ds$ (cases (a), (c)) or $\int_0^t |B_s|^\gamma \, 1_{(0,a)}(B_s) \, ds$ (case (b)). Let X_t be the process obtained from $|B_t|$ by the time changed based on A. Then as explained earlier the quantities of interest in (a), (b), and (c) are just first passage times for X. We identify X as follows:

(4.13) In (a) and (c), X has state space $[0, \infty)$ or $(0, \infty)$, scale function $s(x)=x$ and speed measure $x^\gamma dx$. In (b), X has state space $[0, a]$ or $(0, a]$, scale function $s(x)=x$ and speed measure $x^\gamma \, 1_{(0,a)}(x) \, dx$. The endpoint a is a reflecting barrier: for $c>a$, $P^c\{T_a=\infty\}=0$, and so in (4.11), $K_a^+=0$, and of course $\mu_A(\{a\})=0$.

(4.14) *Remark.* For these diffusions one knows ([6], p. 130) that the endpoint 0 is an exit point if $\int_0^y s(x) \, m(dx) < \infty$ for sufficiently small y and an entrance point if $\int_0^y m[0,x] \, s(dx) < \infty$ for sufficiently small y. In this particular case we see that 0 is an exit point if $\gamma+2>0$ and an entrance point if $\gamma+1>0$. That is, problems (4.12) (a) and (b) are trivial unless $\gamma+2>0$.

5. Bessel Processes

Throughout this section, X will denote a Bessel process with index v: the state space is either $[0, \infty)$ or $(0, \infty)$ and the generator Γ agrees on $(0, \infty)$ with the differential operator

$$(5.1) \quad \Lambda = \frac{1}{2}\left(\frac{d^2}{dx^2} + \frac{2v+1}{x}\frac{d}{dx}\right).$$

The index ν may be any real number. Sometimes X is described in terms of another parameter

(5.2) $d=2\nu+2.$

With the index ν as in (5.1), the modulus of a d-dimensional Brownian motion is a Bessel process with $\nu=\frac{1}{2}(d-2)$.

The scale function s and the speed measure m, restricted to $(0,\infty)$, are determined from the obvious formulas $s'(x)=Kx^{-(2\nu+1)}$, $m(dx)=K^{-1}x^{2\nu+1}\,dx$ where K is any convenient positive constant. For application to the problems listed in Section 4, we normalize so that

(5.3) $$\begin{aligned} s(x)&=(-\operatorname{sgn}\nu)(2|\nu|)^{2\nu}x^{-2\nu} && \text{if } \nu\neq 0\\ &=\log x && \text{if } \nu=0 \end{aligned}$$

and

$$\begin{aligned} m(dx)&=[x/(2|\nu|)]^{2\nu+1}\,dx && \text{if } \nu\neq 0\\ &=x\,dx && \text{if } \nu=0. \end{aligned}$$

The classification of boundary points in terms of properties of the product measure $s(dx)\,m(dy)$ as set out in [6], p. 130, gives the following information:

(5.4) (i) The endpoint ∞ is a natural boundary;
(ii) $\nu\geqq 0$: 0 is an entrance but not exit point;
$-1<\nu<0$: 0 is an entrance and exit point;
$\nu\leqq -1$: 0 is an exit but not entrance point.

In the case $-1<\nu<0$, we complete the definition of X by requiring that 0 be a reflecting point. Thus for all $\nu>-1$, 0 is an entrance point, and if u is in the domain of the generator Γ, the quantity $(u(x)-u(0))/\int_0^x s(t)\,m(dt)$ converges to $\Gamma u(0)$ as $x\to 0$. The boundary conditions that we shall need for u in the domain of Γ are then

(5.5) (i) $\nu\leqq -1$: $u(0+)=0$
(ii) $\nu>-1$, $\nu\neq 0$: $(u(x)-u(0))/x^2$ has a finite limit as $x\downarrow 0$.

Given $\varepsilon\geqq 0$, let $T_\varepsilon=\inf\{t\colon X_t=\varepsilon\}$ and let $\beta\geqq 0$. Let

(5.6) $\varphi(x)=\varphi_\beta(x)=E^x\exp(-\beta T_\varepsilon).$

This gives, of course, the Laplace transform of the distribution relative to P^x of the first passage time through ε. We obtain the following explicit formulas for φ, involving the usual modified Bessel functions I_ν and K_ν. The proofs are deferred until Section 8.

(5.7) **Proposition.** (i) *Let* $\varepsilon=0$ *and* $\nu<0$. *Then*

$$\varphi(x)=\Gamma(|\nu|)^{-1}\,2^{\nu+1}(\sqrt{2\beta}\,x)^{-\nu}K_\nu(\sqrt{2\beta}\,x).$$

An explicit formula for the density is given in Remark (5.9)(ii).

(ii) *Let* $\varepsilon > 0$. *For arbitrary* ν *and for* $x \geqq \varepsilon$

$$\varphi(x) = (\varepsilon/x)^\nu K_\nu(\sqrt{2\beta}\, x)/K_\nu(\sqrt{2\beta}\, \varepsilon).$$

(iii) *Let* $\varepsilon > 0$. *For* $0 < x \leqq \varepsilon$ *and arbitrary* ν

$$\varphi(x) = (\varepsilon/x)^\nu I_{\theta\nu}(\sqrt{2\beta}\, x)/I_{\theta\nu}(\sqrt{2\beta}\, \varepsilon) \quad \textit{where } \theta = \begin{cases} -1 & \textit{if } \nu \leqq -1 \\ 1 & \textit{if } \nu > -1. \end{cases}$$

(iv) *Let* $\varepsilon > 0$ *and* $\nu > -1$. *Then*

$$\varphi(0) = (\varepsilon\sqrt{2\beta})^\nu / [2^\nu I_\nu(\varepsilon\sqrt{2\beta})\, \Gamma(\nu+1)].$$

(5.8) **Proposition.** *Let* $\nu > 0$. *Let* $\psi(x) = E^x\left\{\exp\left[-\beta \int_0^\infty 1_{(0,\varepsilon)}(X_s)\, ds\right]\right\}$ *denote the Laplace transform of the* P^x *distribution of the total time* X *spends in* $(0, \varepsilon)$. *Then for* $0 < x < \varepsilon$,

$$\text{(i)} \quad \psi(x) = [2\nu(\varepsilon\sqrt{2\beta})^{\nu-1}(x\sqrt{2\beta})^{-\nu} I_\nu(x\sqrt{2\beta})]/I_{\nu-1}(\varepsilon\sqrt{2\beta})$$

and

$$\text{(ii)} \quad \psi(0) = (\varepsilon\sqrt{2\beta})^{\nu-1}/[2^{\nu-1}\Gamma(\nu)\, I_{\nu-1}(\varepsilon\sqrt{2\beta})].$$

(5.9) *Remarks.* Certain very curious facts are evident from the formulas above.

(i) The total time spent by a Bessel process with index $\nu > 0$ in the interval $(0, \varepsilon)$ has the same distribution, starting from 0, as the first passage time through ε for the Bessel process with index $\nu - 1$. This fact was noticed by Ciesielski and Taylor [2] in the special case where $X_t = |B_t|$, the absolute value of a d-dimensional Brownian motion with $d \geqq 3$.

(ii) It was shown in [5] that if X is a Bessel process with index $\nu > 0$ and if $L_\varepsilon = \sup\{t : X_t = \varepsilon\}$ then the distribution of L_ε, starting from 0, has Laplace transform

$$E^0 \exp[-\beta L_\varepsilon] = [2^{\nu-1}\Gamma(\nu)]^{-1}(\varepsilon\sqrt{2\beta})^\nu K_\nu(\varepsilon\sqrt{2\beta})$$

and that the P^0 distribution of L_ε had density $\varepsilon^{-2} f_\nu(t/\varepsilon^2)$ where

$$f_\nu(t) = [2^\nu \Gamma(\nu)\, t^{\nu+1}]^{-1} e^{-1/2t}, \qquad t > 0.$$

In view of (5.7)(i), the P^0 distribution of L_ε is identical to the P^ε distribution of T_0 for a Bessel process with index $-\nu$. Recall that $K_\nu = K_{-\nu}$ for all ν. We point out the obvious fact that the above density gives an explicit inversion of the formula (5.7(i).

In [5] it is likewise proven that for a Bessel process with index $\nu > 0$, if $0 < a < b$, then the P^0 distribution of $L_b - L_a$ has Laplace transform

$$E^0 \exp[-\beta(L_b - L_a)] = b^\nu K_\nu(b\sqrt{2\beta})/[a^\nu K_\nu(a\sqrt{2\beta})]$$

and therefore, by (5.7)(ii), the P^0 distribution of $L_b - L_a$ for a Bessel process with index $\nu > 0$ is identical to the P^b distribution of T_a for a Bessel process with index $-\nu$.

The second group of results on Bessel processes involves a modification by means of a reflecting barrier. More precisely, let $\bar{X}$ be the diffusion on $(0, \varepsilon]$ or $[0, \varepsilon]$ that has generator given by (5.1) on $(0, \varepsilon)$ and has a reflecting barrier at ε. The boundary condition at 0 is to be the same as for the original Bessel process X. We suppose that the index $\nu < 0$. Making use of (5.3) one sees that for $c > \varepsilon$,

$$P^c\{T_\varepsilon = \infty\} = \lim_{x \to \infty} \frac{s(c) - s(\varepsilon)}{s(x) - s(\varepsilon)} = 0.$$

In view of (4.8), X may be considered as the process X subjected to time change via the additive functional $A_t = \int_0^t 1_{(0,\varepsilon)}(X_s)\, ds$.

(5.10) **Proposition.** *Let* $\bar{T} = \inf\{t\colon \bar{X}_t = 0\}$ *and let* $\nu < 0$. *Then for* $0 < x \leqq \varepsilon$,

$$\text{(i)} \quad E^x \exp[-\beta \bar{T}] = \Gamma(|\nu|)^{-1} 2^{\nu+1} (x\sqrt{2\beta})^{-\nu} [K_{\nu+1}(\varepsilon\sqrt{2\beta})\, I_{-\nu}(x\sqrt{2\beta}) + I_{-\nu-1}(\varepsilon\sqrt{2\beta})\, K_\nu(x\sqrt{2\beta})] / I_{-\nu-1}(\varepsilon\sqrt{2\beta}).$$

In particular

$$\text{(ii)} \quad E^\varepsilon \exp[-\beta \bar{T}] = \Gamma(|\nu|)^{-1} 2^{\nu+1} (\varepsilon\sqrt{2\beta})^{-(\nu+1)} / I_{-\nu-1}(\varepsilon\sqrt{2\beta}).$$

The proof is given in Section 8.

(5.11) *Remarks.* Once again, the distribution that first appeared in (5.7(iv) and (5.8)(ii) recurs in (5.10)(ii). That is, the P^ε distribution of $\bar{T}$ for the Bessel process with index $\nu < 0$ and with a reflecting barrier at ε is identical to the P^0 distribution of T_ε for a Bessel process with index $-(\nu+1)$, and to the P^0 distribution of the total time spend in $(0, \varepsilon)$ by a Bessel process with index $-\nu$.

In our applications of the formulas of this section to problems (4.12), we require only a change of scale. If X is a Bessel process with index ν with scale function and speed measure normalized as in (5.3), the process $Y_t = s(X_t)$ is a process in natural scale, the state space being the positive half line if $\nu < 0$, the real line if $\nu = 0$ and the negative half line if $\nu > 0$. The speed measure, m_Y for Y is given, in case $\nu \neq 0$, by

$$(5.12) \quad m_Y(dy) = (-y \operatorname{sgn} \nu)^{-2-1/\nu}\, dy = |y|^{-2-1/\nu}\, dy.$$

It follows that if a diffusion Z on $(0, \infty)$ or $[0, \infty)$ is in natural scale and its speed measure is $z^\gamma dz$ with $\gamma + 2 > 0$ and 0 a reflecting point if $\gamma > -1$, then we may compute the P^z distribution of $T_a = \inf\{t\colon Z_t = a\}$ from the formulas in (5.7) by making the substitutions

$$(5.13) \quad \begin{aligned} \nu &= -(\gamma+2)^{-1} \\ x &= 2|\nu|\, z^{-1/2\nu} = (1 + \tfrac{1}{2}\gamma)^{-1} z^{1+\frac{1}{2}\gamma} \\ \varepsilon &= (1 + \tfrac{1}{2}\gamma)^{-1} a^{1+\frac{1}{2}\gamma}. \end{aligned}$$

The same procedure is valid if Z is modified so that the point a is a reflecting barrier. Taking (4.13) and (4.14) into account we make the substitutions (5.13) in (5.7) and (5.10) and obtain the following result. We write down only the most interesting cases.

(5.14) **Proposition.** *The solutions to problems* (4.12) *are as follows: For a standard Brownian motion on the line, for $a>0$ and $\gamma+2>0$, one obtains, setting $\lambda=(\gamma+2)^{-1}$,*

(a) $$E^z \exp\left[-\beta \int_0^{T_0} |B_s|^\gamma ds\right]$$
$$=2\Gamma(\lambda)^{-1}(\lambda\sqrt{2\beta|z|^{1/\lambda}})^\lambda K_\lambda(2\lambda\sqrt{2\beta|z|^{1/\lambda}}) \qquad (z \neq 0);$$

(b) $$E^a \exp\left[-\beta \int_0^{T_0} |B_s|^\gamma 1_{(0,a)}(|B_s|)\, ds\right]$$
$$=\Gamma(\lambda)^{-1}(\lambda\sqrt{2\beta a^{1/\lambda}})^{\lambda-1}/I_{\lambda-1}(2\lambda\sqrt{2\beta a^{1/\lambda}});$$

(c) $$E^0 \exp\left[-\beta \int_0^{T_a} |B_s|^\gamma ds\right]$$
$$=\Gamma(1-\lambda)^{-1}(\lambda\sqrt{2\beta a^{1/\lambda}})^{-\lambda}/I_{-\lambda}(2\lambda\sqrt{2\beta a^{1/\lambda}})$$

provided $\gamma+1>0$ (that is, $0<\lambda<1$).

6. Applications to Excursions

We now possess all the necessary preparations to obtain the limiting results on distributions of additive functionals over excursions. Firstly, recall that (B_s) denotes a standard Brownian motion on the line.

(6.1) **Theorem.** *Fix $t>0$ and $\gamma>-2$. Recall that $D_t=\inf\{s>t\colon B_s=0\}$ and $G_t=\sup\{s<t\colon B_s=0\}$ denote the endpoints of the excursion interval straddling t. Let $\lambda=(\gamma+2)^{-1}$. Then as $a\downarrow 0$, the limiting distribution of $a^{-(\gamma+2)} \int_{(G_t,D_t)} |B_s|^\gamma 1_{(0,a)}(|B_s|)\, ds$ under P^0 exists and the limiting distribution is the convolution square of the distribution having Laplace transform $\psi(\beta)=\Gamma(\lambda)^{-1}(\lambda\sqrt{2\beta})^{\lambda-1}/I_{\lambda-1}(2\lambda\sqrt{2\beta})$.*

Proof. Use (2.18) with the result of (5.14)(b).

We point out here that results of the same type can be obtained, at least in principle, for excursions of quite general regular diffusions. As is known ([6], p. 149) a regular diffusion on an interval I has a transition density $p(t,x,y)$ relative to the speed measure m that is symmetric in x and y, is jointly continuous on $(0,\infty)\times I\times I$, and satisfies the Chapman-Kolmogorov equation identically.

Let 0 be the left endpoint of I and assume that 0 is in I and is a regular point for the diffusion X. In applying the discussion in Section 2 to this case we need a general argument for the validity of (2.12)

(6.2) **Lemma.** *Let $R=\inf\{s\colon X_s=0\}$. Then for all $x>0$ and $t>0$*

$$p(t,x,0)=E^x[p(t-R,0,0); R<t].$$

Proof. We derive this from standard facts about dual Markov processes. If the potential operators of a pair of dual processes $(Y, \hat{Y})$ are given by $U(x, dy) = u(x, y)\,\xi(dy)$ and $\hat{U}(dx, y) = \xi(dx)\,u(x, y)$ with u excessive in x and coexcessive in y, then the potential operators for the pair of subprocesses obtained by killing at $T = T_B$ where T_B is the hitting time of a Borel set B, are given by a density $v(x, y)$ that satisfies

$$(6.3)\quad u(x, y) = v(x, y) + E^x[u(Y_T, y);\ T < \infty].$$

Moreover $v(x, y) = 0$ if either x is regular for B or y is coregular for B. We apply (6.3) to the space time process $Y_t = (\tau_t, X_t)$ on $\mathbb{R} \times I$ where τ_t is uniform motion to the right at speed one. This process is in duality with $\hat{Y}_t = (\hat{\tau}_t, X_t)$ relative to the measure $\xi = l \times m$ on $\mathbb{R} \times I$ where l is Lebesgue measure, m is the speed measure of X, and $\hat{\tau}$ is uniform motion to the left at speed 1. Let $B = \mathbb{R} \times \{0\}$. Then $T_B = \inf\{t\colon X_t = 0\} = R$. Moreover $u((r, x), (t, y)) = 1_{(r, \infty)}(t)\, p(t - r, x, y)$ and so

$$\begin{aligned} E^{(r, x)}[u((\tau_R, X_R), (t, y))\colon R < \infty] &= E^x[u((r + R, 0), (t, y)); R < \infty)] \\ &= E^x[p(t - r - R, 0, y);\ r + R < t]. \end{aligned}$$

Now specializing (6.3) to the case $r = 0$, $y = 0$ gives

$$p(t, x, 0) = v((0, x), (t, 0)) + E^x[p(t - R, 0, 0);\ R < t]$$

for all $t > 0$. But $(t, 0)$ is regular for $B = \mathbb{R} \times \{0\}$ for $\hat{Y} = (\hat{\tau}, X)$ and so $v((0, x), (t, 0)) = 0$, establishing (6.2).

We illustrate this by examining the excursions away from 0 for a Bessel process X with index ν, $-1 < \nu < 0$, as described in Section 5. We continue to use the notation (G_t, D_t) for the excursion straddling t. Because of (2.15), as modified above, the problem reduces to calculating

$$(6.4)\quad \varphi(a; \beta) = E^a \exp\left[-\beta \int_0^{T_0} X_s^\gamma\, 1_{(0, a)}(X_s)\, ds\right].$$

The computation is based on the results of Section 5. Let $K = (2|\nu|)^{-\gamma}$ and

$$A_t = K \int_0^t X_s^\gamma\, 1_{(0, a)}(X_s)\, ds.$$

Under time change via A, X transforms to a diffusion $\bar{X}$ with scale function

$$s(x) = (2|\nu|)^{2\nu} x^{-2\nu} \qquad \text{on } (0, a)$$

and speed measure $\bar{m}$, where

$$\bar{m}(dx) = (2|\nu|)^{-\gamma} x^\gamma\, 1_{(0, a)}(x)\, [x/(2|\nu|)]^{2\nu + 1}\, dx.$$

Moreover, the point a is a reflecting point for $\bar{X}$. Let $Y_t = s(\bar{X}_t)$ be this latter process in natural scale. Its speed measure is

$$m_Y(dy) = 1_{(0, s(a))}(y)\, y^{-2 - (\gamma + 2)/2\nu}\, dy.$$

Then

$$\varphi(a;\beta)=E^a\exp[-\beta K^{-1}A_{T_0}]=\psi(s(a);\beta K^{-1})$$

where $\psi(y;\alpha)=\bar{E}^y\exp[-\alpha\bar{T}_0]$, the bars now referring to the process Y. The term $[-2-(\gamma+2)/2\nu]$ now plays the role formerly played by γ in (5.13). We substitute this for γ in (5.14) and obtain finally, as in (6.1),

(6.5) **Proposition.** *Let* $\lambda=2|\nu|/(\gamma+2)$ *where* $\gamma>-2$ *and* $-1<\nu<0$. *Then*

(i) $$\varphi(a;\beta)=\Gamma(\lambda)^{-1}\left(\frac{\lambda}{2|\nu|}\sqrt{2\beta a^{\gamma+2}}\right)^{\lambda-1}\Big/I_{\lambda-1}\left(\frac{\lambda}{|\nu|}\sqrt{2\beta a^{\gamma+2}}\right),$$

(ii) $$\lim_{a\to 0}E^0\exp\left[-\frac{\beta}{a^{\gamma+2}}\int_{(G_t,D_t)}X_s^\gamma 1_{(0,a)}(X_s)\,ds\right]$$
$$=\left(\Gamma(\lambda)^{-1}\left(\frac{\lambda}{2|\nu|}\sqrt{2\beta}\right)^{\lambda-1}\Big/I_{\lambda-1}\left(\frac{\lambda}{|\nu|}\sqrt{2\beta}\right)\right)^2.$$

7. Area of an Excursion

It would be interesting to be able to compute the distribution, over the excursion straddling t, of an additive functional of the type considered in Section 6. Explicit formulas are available for the particular case of the length of an excursion (e.g. [3]), and in Section 3 we examined the local time at a point, obtaining the Laplace transform in t of their distributions.

If one tries to imitate the methods of Section 3 in case $A_s=\int_0^s|B_u|^\gamma du$, where $\gamma>-2$, one obtains from (3.21)

(7.1) $$\int_0^\infty e^{-\alpha t}E^0 e^{-\beta[A(D_t)-A(G_t)]}\,dt=\frac{2}{\sqrt{2\alpha}}\int_0^\infty\varphi(x;\beta,\alpha)\,\varphi(x;\beta,0)\,\mathrm{d}x$$

where

(7.2) $$\varphi(x;\beta,\alpha)=\mathrm{E}^x\exp\{-\beta A_R-\alpha R\}$$

with $R=\inf\{s:B_s=0\}$.

For general $\gamma>-2$, we are unable even to calculate explicit formulas for $\varphi(x;\beta,\alpha)$. In the case $\gamma=1$, corresponding to the "area" additive functional, we can compute $\varphi(x;\beta,\alpha)$ but the integral in (7.1) does not seem tractable.

The calculation of $\varphi(x;\beta,\alpha)$ can be based on Lemma (3.4), but it is simpler to give an argument that relies only on the spatial homogeneity of Brownian motion.

Fix x and $y>0$, and let $T=\inf\{t:B_t=y\}$. The P^{x+y} distribution of (A_T-yT,T) is, by spatial homogeneity, the same as the P^x distribution of (A_R,R). Therefore

$$\begin{aligned}\varphi(x+y;\beta,0) &= E^{x+y}\exp\{-\beta A_R\}\\ &= E^{x+y}\exp\{-\beta A_T-\beta A_R\circ\theta_T\}\\ &= E^{x+y}\exp\{-\beta A_T\}\,E^y\exp\{-\beta A_R\}\\ &= E^{x+y}\exp\{-\beta(A_T-yT)-\beta yT\}\,\varphi(y;\beta,0)\\ &= \varphi(x;\beta,\beta y)\,\varphi(y;\beta,0).\end{aligned}$$

We obtain then, for $\beta>0$

$$(7.3)\quad \varphi(x;\beta,\alpha)=\varphi\left(x+\frac{\alpha}{\beta};\beta,0\right)\Big/\varphi\left(\frac{\alpha}{\beta};\beta,0\right).$$

However, $\varphi(x;\beta,0)=E^x\exp\{-\beta A_R\}$ has been computed in (5.14). Setting $\gamma=1$, and hence $\lambda=1/3$, in (5.14)(a) we have

$$(7.4)\quad \varphi(x;\beta,0)=2\left(\frac{\sqrt{2\beta}}{3}\right)^{1/3}\sqrt{x}\,K_{1/3}(\tfrac{2}{3}\sqrt{2\beta}\,x^{3/2})/\Gamma(1/3).$$

Substituting this in (7.3) gives, for $\alpha>0$ and $\beta>0$

$$(7.5)\quad \varphi(x;\beta,\alpha)=\sqrt{1+\frac{\beta x}{\alpha}}\,K_{1/3}(\tfrac{2}{3}\sqrt{2}\,\beta^{-1}[x\beta+\alpha]^{3/2})/K_{1/3}(\tfrac{2}{3}\sqrt{2}\,\beta^{-1}\alpha^{3/2}).$$

The argument given above breaks down if $\gamma\neq 1$, but application of (3.6), for general γ, leads to a differential equation for $\varphi(x;\beta,\alpha)$. We have been unable to solve this equation explicitly.

8. Calculations

We collect in this section the calculations necessary to derive the formulas listed in (5.7), (5.8), and (5.10). Let X denote the Bessel process of index ν as described in Section 5. Let $\varphi(x)=\varphi_\beta(x)=E^x(e^{-\beta T(\varepsilon)})$ where $T(\varepsilon)=T_\varepsilon=\inf\{t\colon X_t=\varepsilon\}$. If $T_\varepsilon<\infty$, it follows from (3.7) with $A_t=t$ that $1-\varphi$ is in the domain of the generator of X killed at T_ε. In particular

$$(8.1)\quad \varphi''(x)+\frac{2\nu+1}{x}\varphi'(x)-2\beta\varphi(x)=0$$

on any open subinterval of $(0,\infty)$ not containing the point ε and on which $P^x(T^\varepsilon<\infty)=1$.

We shall use [1] has a handy reference for the properties of Bessel functions needed here. Using 9.1.52 and the remarks preceding 9.6.41 of [1] the following three functions are solutions of (8.1) on $(0,\infty)$

$$(8.2)\quad \begin{aligned}\varphi_1(x)&=x^{-\nu}I_{-\nu}(x\sqrt{2\beta}),\\ \varphi_2(x)&=x^{-\nu}K_{-\nu}(x\sqrt{2\beta}),\\ \varphi_3(x)&=x^{-\nu}I_\nu(x\sqrt{2\beta}).\end{aligned}$$

If v is not an integer any pair of these functions is linearly independent. If v is an integer $\varphi_1 = \varphi_3$. Also $K_v = K_{-v}$ for all v and so $\varphi_2(x) = x^{-v} K_v(x\sqrt{2\beta})$. Of course, I_v and K_v are the usual modified Bessel functions of the first and third kind. We summarize some of the properties of these functions that we shall need.

(8.3) φ_1 and φ_3 approach infinity as $x \to \infty$, while φ_2 is decreasing and vanishes at ∞.

(8.4) If $v \geqq 0$, φ_2 is unbounded at 0, while if $v < 0$, $\varphi_2(x) \to \Gamma(|v|)\, 2^{-(v+1)} (\sqrt{2\beta})^v$ as $x \to 0$.

(8.5) If $v \neq -1, -2, \ldots$, then

$$\Gamma(v+1)(x/2)^{-v} I_v(x) = 1 + \frac{x^2}{4(v+1)} + 0(x^4) \text{ as } x \to 0.$$

(8.6) If $v < 0$, $\varphi_1(x) \to 0$ as $x \to 0$.

(8.7) If $v > -1$, $\varphi_3(x) \to \dfrac{(\sqrt{2\beta})^v}{2^v \Gamma(v+1)}$ as $x \to 0$.

(8.8) $\varphi_1'(x) = \sqrt{2\beta}\, x^{-v} I_{-v-1}(x\sqrt{2\beta})$; $\quad \varphi_2'(x) = -\sqrt{2\beta}\, x^{-v} K_{-v-1}(x\sqrt{2\beta})$.

We now turn to the proof of (5.7). We begin with (5.7)(ii). If $x \geqq \varepsilon > 0$ and $v \leqq 0$, $P^x(T_\varepsilon < \infty) = 1$. Since the only solutions of (8.1) that are bounded at infinity are multiples of φ_2, $\varphi(x) = c\,\varphi_2(x)$ if $x > \varepsilon$. Since $\varphi(\varepsilon+) = 1$ we obtain $\varphi(x) = \varphi_2(x)/\varphi_2(\varepsilon)$ proving (5.7)(ii) when $v \leqq 0$. If $v > 0$, then $P^x(T_\varepsilon = \infty) > 0$ if $x > \varepsilon$, and so one can not apply (8.1) directly. However, if $r > x$ and $T = T_\varepsilon \wedge T_r$ we may apply (3.7) to T. Thus $\psi_r(x) = E^x(e^{-\beta T})$ is a solution of (8.1) on (ε, r) satisfying $\psi_r(\varepsilon+) = 1$ and $\psi_r(r-) = 1$. Consequently $\psi_r(x) = A_2 \varphi_2(x) - A_1 \varphi_1(x)$ where

$$A_j = [\varphi_j(r) - \varphi_j(\varepsilon)]\, [\varphi_1(r)\, \varphi_2(\varepsilon) - \varphi_1(\varepsilon)\, \varphi_2(r)]^{-1}, \quad j = 1, 2.$$

But $T_r \uparrow \infty$ as $r \to \infty$, and by (8.3), $A_1 \to [\varphi_2(\varepsilon)]^{-1}$ and $A_2 \to 0$ as $r \to \infty$. Therefore we obtain for $x > \varepsilon$

$$\varphi(x) = \lim_{r \to \infty} \psi_r(x) = \varphi_2(x)/\varphi_2(\varepsilon).$$

Thus (5.7)(ii) is verified in all cases.

If $v < 0$ and we let $\varepsilon \downarrow 0$ in (5.7)(ii) we obtain, using (8.4), the result stated in (5.7)(i).

Now consider (5.7)(iii). If $v > -1$ and $0 < x < \varepsilon$, then $P^x(T_\varepsilon < \infty) = 1$. Thus φ is a solution of (8.1). But $\varphi(0+) > 0$ in this case. If $-1 < v < 0$, φ_1 and φ_3 are linearly independent, and hence, (8.6) implies that $\varphi = c\varphi_3$. If $v \geqq 0$, φ_2 and φ_3 are linearly independent, and (8.4) implies $\varphi = c\varphi_3$. In either case $\varphi(\varepsilon-) = 1$, and so $\varphi(x) = \varphi_3(x)/\varphi_3(\varepsilon)$ establishing (5.7)(iii) when $v > -1$. If $v \leqq -1$, 0 is an exit but not entrance point and so $P^x(T_\varepsilon = \infty) > 0$ if $0 < x < \varepsilon$. But ζ and $\zeta \wedge T_\varepsilon$ are finite, and so $\psi(x) = E^x(e^{-\beta\zeta})$ and $\psi_\varepsilon(x) = E^x\{e^{-\beta(\zeta \wedge T_\varepsilon)}\}$ satisfy (8.1) on $(0, \varepsilon)$. A simple calculation shows that

$$\varphi(x) = (\psi_\varepsilon(x) - \psi(x))/(1 - \psi(\varepsilon))$$

and so φ satisfies (8.1) on $(0,\varepsilon)$. Clearly $\varphi(0+)=0$ and $\varphi(\varepsilon-)=1$. Hence (8.4) and (8.6) imply that $\varphi(x)=\varphi_1(x)/\varphi_1(\varepsilon)$ completing the proof of (5.7)(iii).

Finally if $\nu>-1$ and we let $x\downarrow 0$ in (5.7)(iii) we obtain (5.7)(iv) because of (8.7). Thus Proposition 5.7 is completely established.

We turn next to the proof of (5.10). Now $\bar{T}<\infty$ because $\nu<0$ and so the solution φ satisfies (8.1) on $(0,\varepsilon)$. Since $\nu<0$, the endpoint 0 is an exit point and we have the boundary condition

(8.9) $\quad \varphi(0+)=1.$

Moreover, at the endpoint ε, the reflection condition is

(8.10) $\quad \varphi'(\varepsilon)=0.$

From (8.6), $\varphi_1(0)=0$ so (8.9) implies that

(8.11) $\quad \varphi=A\varphi_1+B\varphi_2$

where

(8.12) $\quad B^{-1}=\varphi_2(0)=\Gamma(|\nu|)\,2^{-(\nu+1)}(2\beta)^{\nu/2}.$

Making use of (8.8), one sees that (8.10) leads to

$$A\sqrt{2\beta}\,\varepsilon^{-\nu}I_{-\nu-1}(\varepsilon\sqrt{2\beta})-B\sqrt{2\beta}\,\varepsilon^{-\nu}K_{-\nu-1}(\varepsilon\sqrt{2\beta})=0.$$

But $K_\nu=K_{-\nu}$ and so

$$A=BK_{\nu+1}(\varepsilon\sqrt{2\beta})/I_{-\nu-1}(\varepsilon\sqrt{2\beta}).$$

Substitution of this last formula and (8.12) into (8.11) yields (5.10)(i). Letting $x\uparrow\varepsilon$ in (5.10)(i) and using the Wronskian relation ([1], 9.6.15), we obtain (5.10)(ii).

We turn finally to the proof of (5.8). Here X is a Bessel process with $\nu>0$. If we time change X by means of the additive functional $A_t=\int_0^t 1_{(0,\varepsilon)}(X_s)\,ds$ we obtain a diffusion $\tilde{X}$ on $[0,\varepsilon]$ whose generator on $(0,\varepsilon)$ is given by the differential operator Λ defined in (5.1), having 0 as an entrance but not exit point, and, as we shall show, satisfying the boundary condition at ε

(8.13) $\quad u'(\varepsilon)=-2\nu\varepsilon^{-1}u(\varepsilon)$

for u in the domain of the generator of $\tilde{X}$. To see this, observe that for $c>\varepsilon$ one has (here T_a denotes the hitting time of a by X)

$$\begin{aligned}P^c(T_\varepsilon=\infty)&=\lim_{x\to\infty}P^c(T_x<T_\varepsilon)\\&=\lim_{x\to\infty}\frac{s(c)-s(\varepsilon)}{s(x)-s(\varepsilon)}=\frac{s(c)-s(\varepsilon)}{-s(\varepsilon)},\end{aligned}$$

because $s(x) \to 0$ as $x \to \infty$. (See (5.3).) Recalling the definition of K_ε^+ in (4.5)(ii) we have $K_\varepsilon^+ = (-s(\varepsilon))^{-1}$, and so from (4.11)

$$\lim_{x \uparrow \varepsilon} \frac{u(x)-u(\varepsilon)}{s(x)-s(\varepsilon)} = \frac{u(\varepsilon)}{s(\varepsilon)}$$

if u is in the domain of the generator of $\tilde{X}$. But $\dfrac{du}{dx} = \dfrac{du}{ds}\dfrac{ds}{dx}$ and so $u'(\varepsilon) = \dfrac{s'(\varepsilon)}{s(\varepsilon)} u(\varepsilon)$. Finally from (5.3), $s'(\varepsilon)/s(\varepsilon) = -2\nu/\varepsilon$ establishing the boundary condition (8.13).

Now the proof of (5.8) is easy. The function $\psi(x) = E^x\{\exp[-\beta A_\infty]\}$ satisfies (4.2) since $A_\infty < \infty$ and consequently ψ is a solution of (8.1) on $(0, \varepsilon)$. Since $\nu > 0$, (8.4) implies that $\psi(x) = A\varphi_3(x)$. In the present situation (4.2) may be written

$$1 - \psi(x) = E^x \int_0^\infty \psi(X_t)\, dA_t = E^x \int_0^\infty \psi(\tilde{X}_t)\, dt = \tilde{U}\psi(x)$$

where $\tilde{U}$ is the potential operator for $\tilde{X}$. Thus $1-\psi$ is in the domain of the generator of $\tilde{X}$, and so from (8.13)

$$\text{(8.14)} \quad \frac{\varepsilon}{2\nu}\psi'(\varepsilon) + \psi(\varepsilon) = 1.$$

Now $\psi'(x) = A\varphi_3'(x) = A\sqrt{2\beta}\, x^{-\nu} I_{\nu+1}(x\sqrt{2\beta})$ according to ([1], 9.6.28). Substituting this into (8.14) and using the recurrence relation ([1], 9.6.26) leads directly to (5.8)(i). Letting $x \to 0$ and using (8.7) we obtain (5.8)(ii).

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