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Goodness-of-Fit Test Statistics that Dominate the Kolmogorov Statistics

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Summary. Two statistics are proposed for the simple goodness-of-fit problem. These are derived from a general principle for combining dependent test statistics that has been discussed elsewhere by the authors. It is shown that these statistics are relatively optimal in the sense of Bahadur efficiency and consequently, are more efficient than any weighted Kolmogorov statistic at every alternative. A curious pathology occurs: Under certain alternatives, the sequence of statistics has a Bahadur efficacy or exact slope only in the weak sense of convergence in law.

1. Introduction

Goodness-of-fit statistics are presented here that are more Bahadur efficient than the corresponding (weighted) Kolmogorov statistics at every alternative. The statistics are obtained by applying the principle discussed in Berk and Jones (1978) and turn out to be relatively optimal combinations of the family of statistics $\{F_n(x)\colon 0 \leqq x \leqq 1\}$ where F_n is the empirical distribution function of the sample. A combination of test statistics is relatively optimal if at each alternative, the Bahadur efficiency of the combination is (at least) the maximum of the efficiencies of the component statistics. The goodness-of-fit statistics are given below, after some notation is introduced.

We consider the usual reduced goodness-of-fit problem: $X_1, X_2, \ldots$ are iid observations in $[0, 1]$ with common df F and under H_0, F is uniform (on $[0, 1]$). We consider primarily the one-sided alternative: $F(x) > x$ for some $0 \leqq x < 1$, although as indicated below, our discussion applies to the two-sided case as well. Let F_n be the empirical df for $X_1, \ldots, X_n$. We let P_0 denote the null distribution of $X_1, X_2, \ldots, P_F$, its distribution under F and let

$$G_n(t, x) = P_0(F_n(x) \geqq t),$$
$$L^+_{n,x} = G_n(F_n(x), x)$$

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and

$$
\begin{aligned}
K^+(t,x) &= t\log\frac{t}{x} + (1-t)\log\frac{1-t}{1-x} && \text{if } 0<x<t<1 \\
&= 0 && \text{if } 0\leqq t\leqq x\leqq 1 \\
&= \infty && \text{otherwise.}
\end{aligned}
$$

Clearly $L_{n,x}^+$ is the level attained by $F_n(x)$, large values being significant. We use the following basic facts about the binomial distribution: Let

$$K_n^+(t,x) = -n^{-1}\log G_n(t,x). \tag{1.1}$$

Then

$$K_n^+(t,x) \geqq K^+(t,x) \tag{1.2}$$

and

$$\lim_n K_n^+(t,x) = K(t,x). \tag{1.3}$$

Cf. Chernoff (1952). As in Berk and Jones (1978), we describe (1.3) by saying $K(\cdot, x)$ is the index (of large deviation) of $\{F_n(x)\colon n\geqq 1\}$. It follows that

$$-n^{-1}\log L_{n,x}^+ \to K^+(F(x),x)\,[P_F]; \tag{1.4}$$

i.e., that $\{F_n(x)\colon n\geqq 1\}$ has Bahadur efficacy $K^+(F(x),x)$ (or exact slope $2K^+(F(x),x)$) under F. Note that G_n is decreasing in t and increasing in x; by (1.1) and (1.3), the opposite is true of K_n^+ and K^+. Moreover, it is clear that K^+ increases continuously to $\log 1/x$ as $t\uparrow 1$ and increases continuously to $+\infty$ as $x\downarrow 0$ (on the range $0<x<t<1$; the monotonicity is, in fact, strict).

The goodness-of-fit statistics we consider are the (one-sided) minimum attained level statistic

$$M_n^+ = \inf_x L_{n,x}^+,$$

so that

$$-n^{-1}\log M_n^+ = \sup_x K_n^+(F_n(x),x) \tag{1.5}$$

and the corresponding maximum index statistic

$$
\begin{aligned}
R_n^+ &= \sup_x K^+(F_n(x),x) \\
&= \sup\left\{F_n(x)\log\frac{F_n(x)}{x} + [1-F_n(x)]\log\frac{1-F_n(x)}{1-x}\colon F_n(x)>x\right\}.
\end{aligned} \tag{1.6}
$$

The motivation for studying these particular combinations of $\{F_n(x)\colon 0\leqq x\leqq 1\}$ may be found in Berk and Jones (1978). It is not hard to see that these statistics are measurable. As shown below, these statistics are relatively optimal but they do not have (pointwise) Bahadur efficacies at every alternative.

The following considerations will be of use. Let $X_{(1)}, \ldots, X_{(n)}$ be the order statistics for $X_1, \ldots, X_n$. We recall that

$$G_n(k/n, x) = P_0(nF_n(x) \geqq k) = P_0(X_{(k)} \leqq x). \tag{1.7}$$

Let $\sup_{x|k}$ denote supremum for $X_{(k)} \leqq x < X_{(k+1)}$. Since $F_n(x) = k/n$ on this range,

$$\sup_{x|k} K^+(F_n(x), x) = K^+(k/n, X_{(k)}),$$

so that

$$R_n^+ = \max_{1 \leqq k \leqq n} K^+(k/n, X_{(k)}). \tag{1.8}$$

(Note that for $x < X_{(1)}$, $F_n(x) = 0$, so that $K^+(F_n(x), x) = 0$ also.) Similarly

$$M_n^+ = \min_{1 \leqq k \leqq n} G_n(k/n, X_{(k)}), \tag{1.9}$$

or

$$-\frac{1}{n} \log M_n^+ = \max_{1 \leqq k \leqq n} K_n^+(k/n, X_{(k)}) \geqq R_n^+. \tag{1.10}$$

We see from (1.7) that $G_n(k/n, X_{(k)})$ is the level attained by $X_{(k)}$, small values being significant. Hence (1.9) shows that M_n^+ is also the minimum attained level for the order statistics. (Note that rejecting for small values of $X_{(k)}$ is plausible since our alternative includes all distributions stochastically smaller than the uniform.)

2. Relative Optimality

The theorem of this section establishes that $\{M_n^+\}$ and $\{R_n^+\}$ are relatively optimal for the family $\{F_n(x): 0 \leqq x \leqq 1\}$. This is then used in Section 3 to establish that these statistics are superior in the Bahadur sense to the Kolmogorov statistics. We present first two lemmas. The first is actually a special case of a theorem of Hoeffding (1965), Theorem 2.1. As the proof is short, we include it.

Lemma 2.1. *For every integer $n \geqq 1$ and integer $k \leqq n$,*

$$K^+(k/n, x) \leqq K_n^+(k/n, x) \leqq K^+(k/n, x) + (\log 3\pi n)/2n. \tag{2.1}$$

Proof. The left inequality is (1.2). Also,

$$P_0(nF_n(x) \geqq k) \geqq P_0(nF_n(x) = k) = \binom{n}{k} x^k (1-x)^{n-k};$$

hence

$$K_n^+(k/n, x) \leqq \frac{k}{n} \log \frac{1}{x} + \left(1 - \frac{k}{n}\right) \log \frac{1}{1-x} - \frac{1}{n} \log \binom{n}{k}.$$

An application of Stirling's formula for $n!$ shows that

$$-n^{-1}\log\binom{n}{k}\leqq\frac{k}{n}\log\frac{k}{n}+\left(1-\frac{k}{n}\right)\log\left(1-\frac{k}{n}\right)+(\log 3\pi n)/2n. \quad \|$$

It is an immediate consequence of (2.1) that

$$R_n^+\leqq -n^{-1}\log M_n^+\leqq R_n^+ +(\log 3\pi n)/2n \qquad [P_F] \tag{2.2}$$

for every F. This relation implies that most aspects of asymptotic behavior of interest to us are the same for $\{R_n^+\}$ and $\{-n^{-1}\log M_n^+\}$.

The next lemma is Theorem 7.4 of Bahadur (1971).

Lemma 2.2. *Let L be the level attained by a statistic. Then under any null distribution, $P(L\leqq u)\leqq u$, $0\leqq u\leqq 1$.*

Theorem 2.3. *For $r\geqq 0$*

$$\lim_n\,[-n^{-1}\log P_0(-n^{-1}\log M_n^+\geqq r)]=\lim_n\,[-n^{-1}\log P_0(R_n^+\geqq r)]=r \tag{2.3}$$

while

$$\liminf_n\,[-n^{-1}\log M_n^+]=\liminf_n R_n^+\geqq\sup_x K^+(F(x),x) \qquad [P_F]. \tag{2.4}$$

Thus $\{M_n^+ : n\geqq 1\}$ and $\{R_n^+ : n\geqq 1\}$ are relatively optimal: If L_n denotes the level attained by M_n^+ and $\tilde{L}_n$, the level attained by R_n^+,

$$\liminf_n\,[-n^{-1}\log L_n]=\liminf_n\,[-n^{-1}\log \tilde{L}_n]\geqq\sup_x K^+(F(x),x) \qquad [P_F]. \tag{2.5}$$

Proof. Since $K^+(t,x)$ is continuous in t, for $0<r<\log 1/x$, there is a unique $\tau=\tau(r,x)$ in $(x,1)$ for which $K^+(\tau,x)=r$ and $\{t: K^+(t,x)\geqq r\}=[\tau,\infty)$. Using also (1.8), we see that

$$\begin{aligned}P_0(M_n^+\leqq e^{-nr})&\geqq P_0(R_n^+\geqq r)\\ &\geqq P_0(K^+(F_n(x),x)\geqq r)=P_0(F_n(x)\geqq\tau), \qquad x<e^{-r}.\end{aligned}$$

Since $-n^{-1}\log P_0(F_n(x)\geqq\tau)=K_n^+(\tau,x)\to K^+(\tau,x)=r$,

$$\begin{aligned}&\limsup_n\,[-n^{-1}\log P_0(M_n^+\leqq e^{-nr})]\\ &\quad\leqq\limsup\,[-n^{-1}\log P_0(R_n^+\geqq r)]\leqq r, \qquad r\geqq 0.\end{aligned} \tag{2.6}$$

Furthermore, using (1.8) and (1.7),

$$P_0(R_n^+\geqq r)\leqq P_0(M_n^+\leqq e^{-nr})\leqq\sum_{k=1}^{n}P_0(G_n(k/n,X_{(k)})\leqq e^{-nr})\leqq n e^{nr}.$$

The last inequality follows from Lemma 2.2 since $G_n(k/n, X_{(k)})$ is the level attained by $X_{(k)}$. Thus also

$$\liminf_n\,[-n^{-1}\log P_0(R_n^+\geqq r)]\geqq\liminf_n\,[-n^{-1}\log P_0(M_n^+\leqq e^{-nr})]\geqq r, \tag{2.7}$$

which establishes (2.3). The pointwise relation (2.4) follows from (1.6) and (2.2) and then (2.5) follows from (2.2)–(2.4), cf. Lemma 2.5 of Berk and Jones (1978). ‖

It is shown below in Section 4 that there are alternatives for which $\{R_n^+\}$ and $\{-n^{-1}\log M_n^+\}$ do not converge pointwise or in probability. At such alternatives these sequences do not have Bahadur efficacies in the usual sense.

3. The Kolmogorov Statistic

We compare the statistics discussed above with the Kolmogorov statistic $D_n^+ = \sup_x (F_n(x)-x)$. It follows from results of Abrahamson (1967) (see also Bahadur (1971), (5.23) and Example 8.2) that the Bahadur efficacy of $\{D_n^+\}$ at F is

$$b^+(F) = \inf_x K^+(d^+(F)+x, x),$$

where $d^+(F) = \sup_x (F(x)-x)$. Taking F right continuous, $d^+(F) = F(y)-y$ for some $y \in [0, 1)$. Thus

$$b^+(F) = \inf_x K^+(F(y)-y+x, x) \leqq K^+(F(y), y) \leqq \sup_x K^+(F(x), x) \tag{3.1}$$

and we see from (2.5) that $\{R_n^+\}$ and $\{M_n^+\}$ are at least as good in the Bahadur sense as $\{D_n^+\}$. Regarding the possibility of equality in (3.1), note that $K^+(d+x, x)$, considered as a function of x, is the sum of a convex decreasing function $[(d+x)\log(1+d/x)]$ and a convex increasing function $[(1-x-d)\log(1-d/(1-x))]$ and is therefore convex. Thus $h(x) = K^+(F(y)-y+x, x)$ attains its minimum at $x=y$ iff $0<y<1$ and $h'(y)=0$, which is to say, iff $0<y<1$ and

$$\log \frac{F(y)}{1-F(y)} \frac{1-y}{y} = \frac{F(y)}{y} + \frac{1-F(y)}{1-y} - 2. \tag{3.2}$$

This is then necessary and sufficient that the first inequality in (3.1) be an equality.

Clearly only for certain alternatives will (3.2) hold. Assuming it does, let us check whether the second inequality in (3.1) can also be an equality. Assume that F has a continuous derivative f. Then $K^+(F(x), x)$ is non-negative and vanishes at 0 and 1, thus if it attains its maximum at y, its derivative vanishes there. That entails

$$f(y) \log \frac{F(y)}{1-F(y)} \frac{y}{1-y} = \frac{F(y)}{y} - \frac{1-F(y)}{1-y}. \tag{3.3}$$

Since y maximizes $F(x)-x$, $f(y)=1$, thus (3.3) and (3.2) imply that $F(y)=y$, i.e., that $d^+(F)=0$. Thus, at least for smooth alternatives,

$$b^+(F) < \sup_x K^+(F(x), x) \quad \text{if } d^+(F) > 0. \tag{3.4}$$

That is, $\{M_n^+\}$ and $\{R_n^+\}$ are more (Bahadur) efficient than $\{D_n^+\}$.

It is not hard to see from Abrahamson (1967) that this conclusion persists for the weighted Kolmogorov statistics $D_{nw}^+ = \sup_x (F_n(x) - x)\, w(x)$, $w \geqq 0$. See also Section 5 below.

4. Pointwise Behavior

We study in more detail the pointwise behavior of $\{R_n^+\}$.

Theorem 4.1. *If for some* $\gamma > 0$,

$$F(x) < \left[\log \frac{1}{x} \left(\log_2 \frac{1}{x}\right)^{1+\gamma}\right]^{-1}, \qquad x < \gamma \tag{4.1}$$

and

$$1 - F(x) < \left[\log \frac{1}{1-x} \left(\log_2 \frac{1}{1-x}\right)^{1+\gamma}\right]^{-1}, \qquad x > 1 - \gamma, \tag{4.2}$$

then

$$\lim_n R_n^+ = \lim_n \left[-n^{-1} \log M_n^+\right] = \sup_x K^+(F(x), x). \tag{4.3}$$

Before proving this theorem, we establish some preliminary results. In the following lemmas, as in Theorem 4.1, there is no assumption of continuity for F. However, results for discontinuous F are consequences of those for continuous F, because of the following. Let $U_1, \ldots, U_n$ be iid uniform random variables and let H_n be their empirical df. Then $X_i^* = F^{-1}(U_i) \sim F$ and the empirical df of $X_1^*, \ldots, X_n^*$ is $H_n(F(x))$. Below we consider certain "distribution-free" statistics, as for example $D_n^+(F) = \sup_x (F_n(x) - F(x))$ or

$$V_n = \sup_x F_n(x)/F(x). \tag{4.4}$$

These quantities are distribution-free only when F is continuous, of course. However, in general

$$\sup_x F_n(x)/F(x) \sim \sup_x H_n(F(x))/F(x) \leqq \sup_x H_n(x)/x.$$

Thus an upper bound for a probability such as $P(V_n \geqq v)$ established for continuous F is a fortiori an upper bound when F is discontinuous. Keeping this remark in mind, we will be able to assume in the sequel, if necessary, that F is continuous (or even uniform).

Lemma 4.2. *For* V_n *as defined in* (4.4),

$$P_F(V_n \geqq v) \leqq 1/v, \qquad v > 1 \tag{4.5}$$

while if F is continuous,

$$P_F(V_n \geqq v) = 1/v, \qquad v > 1. \tag{4.6}$$

That is, for continuous F and every n, $V_n \sim 1/U$, where U is a uniform random variable.

Proof. Robbins (1954) gives a proof of (4.6); (4.5) is a consequence of the remarks made above.

The next lemma is a sharpened two-sided version of results given by Lai (1975), Theorem 1.

Lemma 4.3. *For $\varepsilon > 0$,*

$$P_F(\sup_{F \geqq \varepsilon} F_n(x)/F(x) \geqq v) \leqq \exp\{-nK^+(\varepsilon v, \varepsilon)\}, \tag{4.7}$$

$$P_F(\inf_{F \geqq \varepsilon} F_n(x)/F(x) \leqq u) \leqq \exp\{-nK^+(1-\varepsilon u, 1-\varepsilon)\}. \tag{4.8}$$

Consequently,

$$P_F\left(\sup_{F \leqq 1-\varepsilon} \frac{1-F_n(x)}{1-F(x)} \geqq v\right) \leqq \exp\{-nK^+(\varepsilon v, \varepsilon)\}, \tag{4.9}$$

$$P_F\left(\inf_{F \leqq 1-\varepsilon} \frac{1-F_n(x)}{1-F(x)} \leqq u\right) \leqq \exp\{-nK^+(1-\varepsilon u, \varepsilon)\}. \tag{4.10}$$

Proof of Lemma 4.3. For F continuous, Kiefer (1973) has noted that for each n, $[1-F_n(x)]/[1-F(x)]$ is a martingale in x. An equivalent statement is that $F_n(x)/F(x)$ is a reverse martingale in x:

$$E_F(F_n(x)/F(x) \mid F_n(u)\colon u \geqq y) = F_n(y)/F(y), \qquad y \geqq x.$$

We may establish (4.7) as follows. For $s>0$, $\exp\{\varepsilon s H_n(x)/x\}$ is a reverse submartingale, thus

$$\begin{aligned} P_0(\sup_{F \geqq \varepsilon} F_n(x)/F(x) \geqq v) &\leqq P(\sup_{x \geqq \varepsilon} H_n(x)/x \geqq v) \\ &\leqq E\exp\{sH_n(\varepsilon) - s\varepsilon v\}, \qquad s>0. \end{aligned}$$

Thus a bound for the left hand side of (4.7) is $\inf_{s>0} E\exp\{sH_n(\varepsilon) - s\varepsilon v\} = \exp\{-nK^+(\varepsilon v, \varepsilon)\}$ (Bernstein's inequality). Relation (4.8) is proved similarly, by considering the reverse submartingale $\exp(-\varepsilon s H_n(x)/x\}$. The remaining relations follow on replacing $F_n(x)$ by $1-F_n(1-x)$, which is the empirical df for $1-X_1, \ldots, 1-X_n$ and which have df $1-F(1-x)$. ||

We note in passing that Lemma 4.3 provides a quick proof of the Glivenko-Cantelli theorem. As remarked above, we may suppose that F is uniform. Then

$$\begin{aligned} \sup_x |F_n(x)-x| &\leqq \sup_{x<\varepsilon} |F_n(x)-x| + \sup_{x \geqq \varepsilon} x\,|F_n(x)/x - 1| \\ &\leqq F_n(\varepsilon) + \varepsilon + \sup_{x \geqq \varepsilon} |F_n(x)/x - 1|. \end{aligned}$$

By Lemma 4.3, $P(\sup_{x \geqq \varepsilon} |F_n(x)/x - 1| > \delta \text{ i.o.}) = 0$ for $\delta > 0$; thus $\sup_{x \geqq \varepsilon} |F_n(x)/x - 1| \to 0$ wpl. Since also $F_n(\varepsilon) \to \varepsilon$ wpl,

$$\limsup_n \sup_x |F_n(x) - x| \leqq 2\varepsilon, \text{ wpl.}$$

Since ε is arbitrary, the result follows.

Proof of Theorem 4.1. For $0<\alpha<1$ and $\delta>0$, let

$$A_n=(\sup\{|F_n(x)-F(x)|/F(x)[1-F(x)]: n^{-\alpha}\leqq F(x)\leqq 1-n^{-\alpha}\}\leqq\delta). \tag{4.11}$$

On A_n, $F_n\leqq(1+\delta)F$ and $1-F_n\leqq(1+\delta)(1-F)$, so

$$K^+(F_n(x), x)\leqq(1+\delta)[K^+(F(x), x)+\log(1+\delta)] \quad \text{on } A_n.$$

For $\alpha<1$, Lemma 4.3 shows that $P_F(A_n^c \text{ i.o.})=0$ for all $\delta>0$. Thus

$$\limsup_n \sup\{K^+(F_n(x), x): n^{-\alpha}\leqq F(x)\leqq 1-n^{-\alpha}\}$$
$$\leqq\sup_x K^+(F(x), x) \quad [P_F]. \tag{4.12}$$

We consider next $F(x)<n^{-\alpha}$. Inverting (4.1), we see that for small x,

$$\log 1/x<2/F(x)[\log 1/F(x)]^{1+\gamma}, \tag{4.13}$$

thus for large n and $F(x)<n^{-\alpha}$,

$$K^+(F_n(x), x)\leqq F_n(x)\log 1/x\leqq 2F_n(x)/F(x)[\log 1/F(x)]^{1+\gamma},$$

hence

$$\sup_{F<n^{-\alpha}} K^+(F_n(x), x)\leqq 2V_n/(\alpha\log n)^{1+\gamma}. \tag{4.14}$$

We show that $V_n/(\log n)^{1+\alpha}\to 0$ $[P_F]$, using a device discussed by Kiefer (1972). Let $\{b_n\}$ be a sequence of constants for which nb_n (ultimately) increases with n and let

$$B_n=(V_n\geqq b_n), \qquad C_n=(2V_{2^{n+1}}\geqq b_{2^n}).$$

Since $nF_n(x)$ and consequently nV_n also increase with n, if for some k between 2^n and 2^{n+1}, B_k occurs, then C_n occurs as well. Thus

$$P_F(B_n \text{ i.o.})\leqq P_F(C_n \text{ i.o.}).$$

Taking $b_n=\varepsilon(\log n)^{1+\alpha}$, by Lemma 4.2

$$P_F\, C_n\leqq 2/\varepsilon(\log 2^n)^{1+\gamma}\leqq 4/\varepsilon n^{1+\gamma},$$

so that $P_F(C_n \text{ i.o.})=0$ and consequently $P_F(B_n \text{ i.o.})=0$. That is,

$$(\log n)^{-(1+\gamma)}\sup_x F_n(x)/F(x)\to 0 \quad [P_F].$$

From (4.14) we see that this entails

$$\lim_n \sup_{F<n^{-\alpha}} K^+(F_n(x), x)=0 \quad [P_F]. \tag{4.15}$$

A similar argument shows that

$$\lim_{n} \sup_{F>1-n^{\alpha}} K^{+}(F_n(x), x)=0 \qquad [P_F] \tag{4.16}$$

which, together with (4.12) and (4.15) shows that

$$\limsup_{n} R_n^{+} \leqq \sup_{x} K^{+}(F_n(x), x) \qquad [P_F].$$

In view of (2.4) and (2.2), the theorem is proved. ||

Remark. It is clear from an examination of the above proof that it goes through if the right-hand side of (4.1) is replaced by

$$1/\log 1/x \log_2 1/x \ldots \log_k 1/x(\log_{k+1} 1/x)^{1+\gamma}$$

for some $\gamma>0$, with a corresponding expression in (4.2). The right-hand side of (4.13) becomes

$$2/F \log 1/F \ldots \log_{k-1} 1/F(\log_k 1/F)^{1+\gamma}$$

and in (4.14), the right-hand side is a multiple of

$$V_n/\log n \ldots \log_{k-1} n(\log_k n)^{1+\gamma}.$$

The same argument, with the same choice of b_n shown that this last quantity tends to zero wpl. That is, for integer $k\geqq 1$ and $\gamma>0$,

$$\log_2 n+\ldots+\log_k n+(1+\gamma)\log_{k+1} n$$

is an upper class boundary for $\log \sup_{x} F_n(x)/F(x)$.

We show next by example that (4.1) cannot be entirely dispensed with in Theorem 4.1. We consider

$$F(x)=[1+\log 1/x]^{-1} \tag{4.17}$$

for which $\sup_{x} K^{+}(F(x), x)=1$, the maximum occurring at $x=0$. Since (4.2) holds, it follows from the proof of Theorem 4.1 (cf. (4.12) and (4.16)) that

$$\begin{aligned}\limsup_{n} \sup_{F\geqq n^{-\alpha}} & K^{+}(F_n(x), x) \\ & \leqq \sup_{x} K^{+}(F(x), x)=1 \qquad [P_F].\end{aligned} \tag{4.18}$$

Also, wpl

$$\begin{aligned}\sup_{F<n^{-\alpha}} K^{+}(F_n(x), x) &= \sup_{F<n^{-\alpha}} F_n(x)\log 1/x+o(1) \\ &= \sup_{F<n^{-\alpha}} F_n(x)/F(x)+o(1)=V_n+o(1),\end{aligned} \tag{4.19}$$

where, to obtain the last equality, we use (4.7) to conclude that

$$\sup_{F \geqq n^{-\alpha}} F_n(x)/F(x) \to 1 \qquad [P_F]. \tag{4.20}$$

Since $V_n \sim 1/U$ for every n and this random variable is never less than one, (4.18) and (4.19) show that R_n^+ and consequently also $-n^{-1}\log M_n$ converge in law under F to $1/U$.

It then follows that

$$P_F(\liminf_n R_n^+ = 1, \limsup R_n^+ = \infty) = 1 \tag{4.21}$$

(and the same is true of $-n^{-1}\log M_n$): For example, for $v > 1$

$$\begin{aligned} 0 < 1 - 1/v &= \lim_n P_F(R_n^+ \leqq v) \\ &\leqq P_F(R_n^+ \leqq v \text{ i.o.}) = P_F(\liminf_n R_n^+ \leqq v) \end{aligned}$$

and by the Hewitt-Savage zero-one law, this last probability must be 0 or 1; thus it is 1. It follows from (2.3) that for $\tilde{L}_n$

$$P_F(\liminf_n [-n^{-1}\log \tilde{L}_n] = 1, \limsup_n [-n^{-1}\log \tilde{L}_n] = \infty) = 1 \tag{4.22}$$

and likewise for L_n. Thus for this F, R_n^+ and M_n^+ have no exact slope in the pointwise sense.

It is easy to see that they do not converge in probability either. In view of (4.18) and (4.19), it suffices to show that $\{V_n\}$ does not converge in probability. But $F_{2n} = \frac{1}{2}(F_n + F_n^*)$, where F_n^* is the empirical df for $X_{n+1}, \ldots, X_{2n}$. Thus $V_{2n} \geqq \frac{1}{2}(V_n + V_n^*)$ where $V_n^* = \sup_x F_n^*(x)/F(x)$ and thus $V_{2n} - V_n \geqq \frac{1}{2}(V_n^* - V_n) \sim \frac{1}{2}(1/U^* - 1/U)$, where U^* and U are independent uniform random variables. It follows that $\{V_{2n} - V_n\}$ does not converge to zero in probability, hence that $\{V_n\}$ does not converge in probability. It follows from (2.3) and the foregoing that $\{-n^{-1}\log L_n\}$ converges in law but not in probability to $1/U$ (and likewise L_n). It is not clear in general how one is to compare (or obtain efficiencies from) Bahadur efficacies that are random and only limits in law of $\{-n^{-1}\log L_n\}$. In the present case, making a comparison with D_n^+ presents no problem, since by (3.1), the Bahadur efficacy of the latter does not exceed one, so that (4.22) shows R_n^+ to be more efficient in the Bahadur sense for almost every sample sequence.

This example is a somewhat delicate but natural example of a statistic sequence that does not have a pointwise Bahadur efficacy. Similar behavior is exhibited by the smallest (resp., largest) order statistic sequence, small (resp., large) values being significant: Under H_0, the index of $nX_{(1)}$ is easily computed to be $\rho(t) = \log(1 - e^{-t})$, while under a continuously differentiable (at zero) alternative F, $nX_{(1)}$ converges in law to an exponential random variable with parameter $f(0)$.

To complete the discussion of pointwise behavior, we note that if $1/F$ does not exceed $(\log 1/x)^{1-\alpha}$ near zero, where $\alpha < 1$ (and similarly for x near one), or more generally, when $\sup_x K^+(F(x), x) = \infty$, (2.3) shows that the Bahadur efficacy of $\{R_n^+\}$ is ∞.

5. Two-Sided Statistics

It is clear that the above reasoning similarly provides "lower" one-sided statistics: M_n^-, the minimum attained level for $\{F_n(x): 0 \leqq x \leqq 1\}$ with *small* values being significant and R_n^-, the corresponding maximum index. Letting $K_n^-(t,x) = K_n^+(1-t, 1-x)$ and similarly for K^-,

$$-\frac{1}{n}\log M_n^- = \sup_x K_n^-(F_n(x), x)$$

and

$$\begin{aligned} R_n^- &= \sup_x K^-(F_n(x), x) \\ &= \sup\left\{F_n(x)\log\frac{F_n(x)}{x} + [1-F_n(x)]\log\frac{1-F_n(x)}{1-x}: F_n(x) < x\right\}. \end{aligned}$$

In analogy with the two-sided Kolmogorov statistic $D_n = \max\{D_n^+, D_n^-\}$ we get the two sided statistics

$$M_n = \min\{M_n^+, M_n^-\}$$

and

$$R_n = \max\{R_n^+, R_n^-\} = \sup_x K(F_n(x), x),$$

where $K(t,x) = t\log t/x + (1-t)\log(1-t)/(1-x)$ if $0 \leqq t$, $x \leqq 1$ and is ∞ otherwise. (A poor analogy with $D_n = \sup_x |F_n(x) - x|$ would be the minimum attained level and maximum index statistics for $\{|F_n(x) - x|: 0 < x < 1\}$. This is because $|F_n(x) - x|$ is a suboptimal combination of $F_n(x) - x$ and $x - F_n(x)$.)

The previous discussion on relative efficiency carries over to R_n and M_n vis-a-vis D_n: Letting $d(F) = \sup_x |F(x) - x|$, the Bahadur efficacy of $\{D_n\}$ is (cf. Abrahamson (1967), Bahadur (1971))

$$b(F) = \inf_x K(d(F) + x, x) \leqq \sup_x K(F(x), x)$$

and the last quantity is the (pointwise) Bahadur efficacy of $\{R_n\}$ and $\{M_n\}$ when such exists. More generally, Abrahamson (1967) showed that the Bahadur efficacy of the weighted Kolmogorov statistic

$$D_{nw} = \sup_x |F_n(x) - x|\, w(x), \qquad w \geqq 0$$

is

$$b_w(F) = \inf_x K(d_w(F)/w(x) + x, x),$$

where $d_w(F) = \sup_x |F(x) - x| w(x)$. Supposing that $d_w(F) = (F(y) - y)\, w(y)$ for some $0 \leqq y < 1$,

$$b_w(F) = \inf_x K((F(y)-y)w(y)/w(x)+x, x)$$
$$\leqq K(F(y), y) \leqq \sup_x K(F(x), x),$$

so that R_n and M_n are better than the weighted statistics as well. This is to be expected since $F_n(x)$ and $(F_n(x)-x)w(x)$ have the same attained level (if $w(x)>0$).

Abrahamson (1967) showed that the Kuiper statistics $D_n^* = D_n^+ + D_n^-$ has Bahadur efficacy $\inf_x K(d^*(F)+x, x)$, where $d^*(F) = d^+(F) + d^-(F)$; consequently that it too is at least as efficient as the two-sided Komogorov statistic at every alternative. No conclusive comparison between D_n^* and R_n can be made; one does not dominate the other as regards Bahadur efficacy. For example, if F is the point mass at $y \varepsilon (0, 1)$, $d^*(F)=1$ and the Bahadur efficacy of $\{D_n^*\}$ is ∞, while that of $\{R_n\}$ is $\log 1/y$. On the other hand, if F is stochastically larger (or smaller) than the uniform distribution, $d^*(F)=d(F)$ and consequently the Bahadur efficacy of $\{R_n\}$ is typically the larger. It seems plausible that $\{R_n\}$ is more efficient at most alternatives of interest. Also, it is not clear that D_n^* dominates the weighted Kolmogorov statistics as does R_n and, of course, there are no onesided versions of D_n^*.

6. Limiting Null Distribution

We note also that the limiting null distribution of R_n may be found. By expanding $K(F_n(x), x)$ in a Taylor series about $F(x)$, one sees that under P_0,

$$R_n = \tfrac{1}{2} \sup_x (F_n(x)-x)^2/x(1-x)\,[1+0_p(n^{-\frac{1}{2}})].$$

The limiting null distribution of

$$\Delta_n^2 = \tfrac{1}{2} \sup_x (F_n(x)-x)^2/x(1-x)$$

has been obtained by Jaeschke (1977), whose results may be paraphrased as follows: Under P_0, $n\Delta_n^2 - \log_2 n - \frac{1}{2}\log_3 n$ converges in law to Z, where

$$P(Z<z) = \exp\{-2\pi^{-\frac{1}{2}} e^{-z}\}, \qquad -\infty<z<\infty.$$

Consequently,

$$P_0(nR_n < \log_2 n + \tfrac{1}{2}\log_3 n + z) \to \exp\{-2\pi^{-\frac{1}{2}} e^{-z}\}, \qquad -\infty<z<\infty. \tag{6.1}$$

The corresponding one-sided result is

$$P_0(nR_n^+ < \log_2 n + \tfrac{1}{2}\log_3 n + z) \to \exp\{-\pi^{-\frac{1}{2}} e^{-z}\}, \qquad -\infty<z<\infty. \tag{6.2}$$

Unfortunately, (2.2) is not sharp enough to give a corresponding result for M_n. However, for its "approximate version" $\tilde{M}_n = \exp\{-nR_n\}$, we have

$$P_0(\tilde{M}_n(\log_2 n)^{\frac{1}{2}} \log n > t) \to \exp\{-2\pi^{-\frac{1}{2}} t\}, \qquad t>0 \tag{6.3}$$
$$P_0(\tilde{M}_n^+ (\log_2 n)^{\frac{1}{2}} \log n > t) \to \exp\{-\pi^{-\frac{1}{2}} t\}, \qquad t>0. \tag{6.4}$$

These latter limiting distributions are simple exponential distributions.

We note finally a paper by Tusnády (1977), which discusses a sequence of goodness-of-fit statistics that for testing a simple null hypothesis is Bahadur optimal against every alternative. The statistics are based on (increasingly finer) finite partitions of the sample space and are, in fact, likelihood ratio statistics for the corresponding multinomial distributions. This result supports Karl Pearon's principle of partitioning the sample space, but not use of the chi-squared statistic. Unhappily, Tusnády's results shed no light on how to partition the sample space.

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