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Asymptotic Distribution of the Log-Likelihood Function for Stochastic Processes

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Summary. Let $X_0, X_1, \dots, X_n$ be r.v.'s coming from a stochastic process whose finite dimensional distributions are of known functional form except that they involve a k-dimensional parameter. From the viewpoint of statistical inference, it is of interest to obtain the asymptotic distributions of the log-likelihood function and also of certain other r.v.'s closely associated with the likelihood function. The probability measures employed for this purpose depend, in general, on the sample size n. These problems are resolved provided the process satisfies some quite general regularity conditions. The results presented herein generalize previously obtained results for the case of Markovian processes, and also for i.n.n.i.d. r.v.'s. The concept of contiguity plays a key role in the various derivations.

1. Introduction

For $n \geqq 0$, integer, let $X_0, X_1, \dots, X_n$ be the first $n+1$ r.v.'s from a stochastic process each defined on the probability space $(\mathscr{X}, \mathscr{A}, P_\theta)$. The joint probability law of any finite set of such r.v.'s is assumed to have known functional form except that it depends on the k-dimensional parameter θ. The set of all possible values of θ, that is, the parameter space Θ, is assumed to be an open subset of $\mathbb{R}^k$, $k \geqq 1$. Under these and some additional suitable regularity conditions, one may write down the likelihood function of the above r.v.'s. Then, as is well known, it is of great importance, from statistical inference point of view, to determine the asymptotic distribution of the log-likelihood function. Also, the same for some other r.v.'s associated with the likelihood function. In both these cases the determination of the asymptotic distribution is required under sequences of probability measures which, in general, vary with the sampe size n.

For the special case that the above r.v.'s are coming from a stationary Markovian process which satisfies certain regularity conditions the problems

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stated have been dealt with in the monograph Roussas (1972). Also, the same problems have been resolved for the case that the r.v.'s under consideration are assumed to be independent but not necessarily identically distributed. This has been done in Philippou and Roussas (1973). In both these cases the concept of contiguity plays a key role in the discussions involved.

The results obtained herein are the natural extensions of those proved in the above cited two special cases.

This paper consists of seven sections. The necessary notation and the assumptions used throughout the paper are introduced in Sect. 2. Some comments on the assumptions, as well as some auxiliary results, are given in Sect. 3. The main results of the paper, Theorems 4.1–4.6, are given in the next section, whereas their proofs are deferred to Sect. 6. In Sect. 5, a series of lemmas required for the proofs of the main results, is presented. In the closing section, three examples are mentioned, where the assumptions made in this paper hold true. The last two of these examples, also provide some justification for undertaking this investigation.

Of the most recent works on the subjects the paper Prasad (1973) is especially noteworthy. Other papers dealing with the same problem are those of Rao (1966), Prakasa Rao (1974) and Crowder (1976). Also, the following papers, discussing related problems, are of interest, namely, Bhat (1974), Basawa, Feigin and Heyde (1976), Basawa and Scott (1977), (1978).

The methods used there, however, follow the classical line. At this point, it should also be mentioned that the new CLT for martingales obtained in Brown (1971) was very useful here.

In order to avoid unnecessary repetitions, it should be noted that all limits are taken as n tends to inifinity unless explicitly otherwise stated.

2. Notation and Assumptions

For $n \geqq 0$, integer, let $X_0, X_1, \ldots, X_n$ be r.v.'s defined on the probability space $(\mathscr{X}, \mathscr{A}, P_\theta)$, where the k-dimensional parameter $\theta \in \Theta$, an open subset of $\mathbb{R}^k$, $k \geqq 1$. Let $\mathscr{A}_n = \sigma(X_0, X_1, \ldots, X_n)$, the σ-field induced by the r.v.'s $X_0, X_1, \ldots, X_n$, and let $P_{n,\theta}$ be the restriction of P_θ to $\mathscr{A}_n$. It will be assumed in the sequel that, for each $n \geqq 0$, the probability measures in the family $\{P_{n,\theta};\ \theta \in \Theta\}$ are mutually absolutely continuous. Then, for $\theta, \theta^* \in \Theta$, let

$$q_n(\theta; \theta^*) = q_n(\mathbf{X}_n;\ \theta, \theta^*) = \frac{dP_{n,\theta^*}}{dP_{n,\theta}}, \qquad n \geqq 0, \tag{2.1}$$

be specified versions of the Radon-Nikodym derivatives involved, where $\mathbf{X}_n = (X_0, X_1, \ldots, X_n)$. Set

$$\begin{aligned} \varphi_n^2(\theta; \theta^*) = \varphi_n^2(\mathbf{X}_n; \theta, \theta^*) &= \frac{q_n(\theta; \theta^*)}{q_{n-1}(\theta; \theta^*)} \\ &= q_n(X_n \mid \mathbf{X}_{n-1}; \theta, \theta^*), \qquad n \geqq 1. \end{aligned} \tag{2.2}$$

The likelihood function, based on $\mathbf{X}_n$, is given by

$$L_n(\theta;\theta^*)=L_n(\mathbf{X}_n;\theta,\theta^*)=q_n(\theta;\theta^*)$$
$$=q_0(\theta;\theta^*)\prod_{j=1}^{n}\varphi_j^2(\theta;\theta^*), \tag{2.3}$$

so that

$$\Lambda_n(\theta;\theta^*)=\log L_n(\theta;\theta^*)=\log q_0(\theta;\theta^*)+2\sum_{j=1}^{n}\log\varphi_j(\theta;\theta^*). \tag{2.4}$$

Clearly, $\Lambda_n(\theta;\theta^*)$ is well defined with P_θ – probability 1 for all $\theta\in\Theta$.

It will be assumed in the following that, for each $\theta\in\Theta$, the random functions $\varphi_j(\theta;.)$, $j\geqq 1$, are differentiable in quadratic mean (q.m.) when the probability measure P_θ is used. Let $\dot\varphi_j(\theta)$, $j\geqq 1$, be the derivatives in q.m. involved evaluated at θ. Next, set

$$\Gamma_j(\theta)=4\mathscr{E}_\theta[\dot\varphi_j(\theta)\,\dot\varphi_j'(\theta)],\quad j\geqq 1,\qquad \bar\Gamma_n(\theta)=\frac{1}{n}\sum_{j=1}^{n}\Gamma_j(\theta), \tag{2.5}$$

and

$$\Delta_n(\theta)=2n^{-\frac{1}{2}}\sum_{j=1}^{n}\dot\varphi_j(\theta),\qquad n\geqq 1. \tag{2.6}$$

Assumptions

(A1) For each $n\geqq 0$, the (finite dimensional) probability measures $\{P_{n,\theta};\ \theta\in\Theta\}$ are mutually absolutely continuous.

(A2) (i) For each $\theta\in\Theta$, the random functions $\varphi_j(\theta;.)$ are differentiable in q.m. $[P_\theta]$ uniformly in $j\geqq 1$. That is, there are k-dimensional random vectors $\dot\varphi_j(\theta)$ – the q.m. derivatives of $\varphi_j(\theta;\theta^*)$ with respect to θ^* at θ – such that

$$\frac{1}{\lambda}|[\varphi_j(\theta;\theta+\lambda h)-1]-\lambda h'\dot\varphi_j(\theta)|\to 0 \tag{2.7}$$

in q.m. $[P_\theta]$, as $\lambda\to 0$, uniformly on bounded sets of $h\in\mathbb{R}^k$ and uniformly in $j\geqq 1$.

(ii) For $j\geqq 1$, $\dot\varphi_j(\theta)$ is $\mathscr{A}_j\times\mathscr{C}$-measurable, where $\mathscr{C}$ is the σ-field of Borel subsets of Θ.

(A3) (i) For each $\theta\in\Theta$ and each $t\in\mathbb{R}^k$, $[t'\dot\varphi_j(\theta)]^2$, $j\geqq 1$, are uniformly integrable with respect to P_θ. That is, uniformly in $j\geqq 1$,

$$\int_{\{[t'\dot\varphi_j(\theta)]^2>a\}}[t'\dot\varphi_j(\theta)]^2\,dP_\theta\to 0,\quad \text{as } a\to\infty. \tag{2.8}$$

(ii) For each $\theta\in\Theta$ and $n\geqq 1$, let the $k\times k$ covariance matrix $\bar\Gamma_n(\theta)$ be defined by (2.5). Then $\bar\Gamma_n(\theta)\to\bar\Gamma(\theta)$ (in any one of the standard norms in $\mathbb{R}^k$) and $\bar\Gamma(\theta)$ is positive definite, $\theta\in\Theta$.

(iii) For each $\theta\in\Theta$ and for the probability measure P_θ, the WLLN's holds for the sequence of r.v.'s

$$\{[t'\dot{\varphi}_j(\theta)]^2\},\quad j\geqq 1,\qquad \text{for each } t\in\mathbb{R}^k.$$

(iv) For each $\theta\in\Theta$ and each $t\in\mathbb{R}^k$,

$$\frac{1}{n}\sum_{j=1}^{n}[\mathscr{E}_\theta\{[t'\dot{\varphi}_j(\theta)]^2|\mathscr{A}_{j-1}\}-[t'\dot{\varphi}_j(\theta)]^2]\to 0 \tag{2.9}$$

in P_θ-probability.

(A4) For each $\theta\in\Theta$, let $q_0(\theta;.)$ be defined by (2.1). Then $q_0(\theta;.)$ is $\mathscr{A}_0\times\mathscr{C}$-measurable and continuous in P_θ-probability.

3. Some Comments on the Assumptions

In the first place, assumption (A3)(iii) means that, for each $\theta\in\Theta$ and each $t\in\mathbb{R}^k$,

$$\frac{1}{n}\sum_{j=1}^{n}\{[t'\dot{\varphi}_j(\theta)]^2-\mathscr{E}_\theta[t'\dot{\varphi}_j(\theta)]^2\}\to 0 \tag{3.1}$$

in P_θ-probability. Then, on the basis of assumption (A3)(ii) and relation (2.5), relation (3.1) may be reformulated equivalently as follows:

(A3)(iii′) For each $\theta\in\Theta$ and each $t\in\mathbb{R}^k$,

$$\frac{1}{n}\sum_{j=1}^{n}[t'\dot{\varphi}_j(\theta)]^2\to\tfrac{1}{4}t'\Gamma(\theta)\,t\qquad \text{in } P_\theta\text{-probability.} \tag{3.2}$$

Next, on the basis of relations (2.9) and (3.1), it is clear that assumption (A3)(iv) may be reformulated equivalently as follows:

(A3) (iv′) For each $\theta\in\Theta$ and each $t\in\mathbb{R}^k$, the r.v.'s

$$\{\mathscr{E}_\theta\{[t'\dot{\varphi}_j(\theta)]^2|\mathscr{A}_{j-1}\}\},\quad j\geqq 1,$$

satisfy the WLLN's when the probability measure P_θ is used.

Also, from relations (2.9) and (3.2), another equivalent reformulation of assumption (A3)(iv) is the following:

(A3) (iv″) For each $\theta\in\Theta$ and each $t\in\mathbb{R}^k$,

$$\frac{1}{n}\sum_{j=1}^{n}\mathscr{E}_\theta\{[t'\dot{\varphi}_j(\theta)]^2|\mathscr{A}_{j-1}\}\to\tfrac{1}{4}t'\Gamma(\theta)\,t\qquad \text{in } P_\theta\text{-probability.} \tag{3.3}$$

In the next section, a result regarding the asymptotic normality of the r.v. $\Delta_n(\theta)$, defined by relation (2.6), will be formulated. For this purpose, a certain martingale CLT (Theorem 2 in Brown (1971)) will be appropriate (see also Scott (1973)).

This theorem applies without the assumption that the underlying martingale sequence is stationary and ergodic. Both of these conditions are assumed in the more familiar martingale CLT (see, for example, Billingsley (1961) and Ibraginov (1963). Also, Roussas (1972), Theorem 2.2A, pp. 205–223.)

The applicability of the martingale CLT given by Brown is based on two conditions (expressed by relations (1) and (2) in Brown (1971)), the second of which is a familiar Lindeberg-type condition. It will be shown that these conditions hold true here on account of assumptions (A1)–(A4).

For each $\theta\in\Theta$ and each $t\in\mathbb{R}^k$ (both arbitrarily chosen and kept fixed thereafter), set

$$\sigma_j^2=\mathscr{E}_\theta\{[t'\dot{\varphi}_j(\theta)]^2\,|\,\mathscr{A}_{j-1}\},\quad j\geqq 1,$$

$$V_n^2=\sum_{j=1}^n \sigma_j^2,\quad s_n^2=\mathscr{E}_\theta(V_n^2),\quad n\geqq 1. \tag{3.4}$$

Then the following results hold true.

Lemma 3.1. *Under assumptions* (A1)–(A4),

$$V_n^2 s_n^{-2}\to 1 \quad \textit{in } P_\theta\textit{-probability}, \tag{3.5}$$

where V_n^2 *and* s_n^2 *are defined by* (3.4).

Proof. From relations (3.4), (2.5) and assumption (A3)(ii),

$$s_n^2=\frac{n}{4}t'\bar{\Gamma}_n(\theta)\,t \quad\text{and}\quad t'\bar{\Gamma}_n(\theta)\,t\to t'\bar{\Gamma}(\theta)t.$$

This result, along with relation (3.3), provides the desirable convergence in (3.5).

The lemma below provides the required Lindeberg condition for the martingale CLT. For its formulation, it will be convenient to set

$$Z_j=t'\dot{\varphi}_j(\theta),\quad j\geqq 1, \tag{3.6}$$

with θ and t being as above.

Lemma 3.2. *Under assumptions* (A1)–(A4) *and for every* $\varepsilon>0$,

$$s_n^{-2}\sum_{j=1}^n \int_{(|Z_j|\geqq \varepsilon s_n)} Z_j^2\,dP_\theta\to 0, \tag{3.7}$$

where s_n^2 *and* Z_j *are defined by* (3.4) *and* (3.6), *respectively.*

Proof. As was seen in the proof of Lemma 3.1, $\frac{4s_n^2}{n}\to t'\bar{\Gamma}(\theta)t$, so that $s_n\to\infty$. Now, for $\varepsilon>0$, let $a=a(\varepsilon)$ (>0) be sufficiently large so that

$$\int_{(Z_j^2>a)} Z_j^2\,dP_\theta<\varepsilon,\quad j\geqq 1.$$

This is possible by assumption (A3) (i). Next, there exists $N(\varepsilon)$ such that $n \geqq N(\varepsilon)$ implies $\varepsilon^2 s_n^2 > a$ and hence

$$\int_{(|Z_j| \geqq \varepsilon s_n)} Z_j^2 \, dP_\theta = \int_{(Z_j^2 \geqq \varepsilon^2 s_n^2)} Z_j^2 \, dP_\theta \leqq \int_{(Z_j^2 > a)} Z_j^2 \, dP_\theta < \varepsilon, \quad j \geqq 1.$$

Thus, with $4\tau_n = t' \bar{\Gamma}_n(\theta) t$ and $4\tau = t' \bar{\Gamma}(\theta) t$ (t nonzero), it follows that, for all sufficiently large n,

$$s_n^{-2} \sum_{j=1}^{n} \int_{(|Z_j| \geqq \varepsilon s_n)} Z_j^2 \, dP_\theta < \frac{4}{n\tau_n} n\varepsilon < \frac{8}{\tau} \varepsilon.$$

The proof of the convergence in (3.7) is completed.

4. Main Results

In this section, the main results of the paper are stated. Their proofs are deferred to a later section (Sect. 6) after some auxiliary lemmas have been established. These results provide an asymptotic expansion (in the probability sense) of the appropriate log-likelihood function, and its asymptotic normality. Also, they give the asymptotic normality of the r.v.'s $\Delta_n(\theta)$, $n \geqq 1$, as they are defined by (2.6).

It is to be noted that said results generalize Theorems 4.1–4.6, pp. 53–54 in Roussas (1972) which hold true for certain Markovian processes. Also, they generalize Theorems 3.1–3.6, pp. 457–458 in Philippou and Roussas (1973) which hold true for the i.n.n.i.d. case. As was the case in establishing the proofs of the theorems just cited, contiguity concepts and results will also play a central role here.

With $\theta \in \Theta$, let

$$\theta_n = \theta + h_n n^{-\frac{1}{2}}, \qquad h_n \to h \in \mathbb{R}^k, \tag{4.1}$$

and let $\Lambda_n(\theta)$ be defined by

$$\Lambda_n(\theta) = \Lambda_n(\theta; \theta_n), \tag{4.2}$$

where $\Lambda_n(\theta; \theta_n)$ is given by (2.4) with θ^* being replaced by θ_n. Then the log-likelihood function $\Lambda_n(\theta)$ assumes of the following expansion in P_θ-probability, namely.

Theorem 4.1. *Let $\Lambda_n(\theta)$ and $\Delta_n(\theta)$ be defined by* (4.2) *and* (2.6), *respectively* (*with θ_n given by* (4.1)). *Then, under assumptions* (A1)–(A4) *and for each $\theta \in \Theta$,*

$$\Lambda_n(\theta) - h' \Delta_n(\theta) \to -\tfrac{1}{2} h' \bar{\Gamma}(\theta) h \quad \text{in } P_\theta\text{-probability.}$$

The asymptotic distribution of the all important r.v.'s $\Delta_n(\theta)$, $n \geqq 1$, is provided by the following

Theorem 4.2. *Let $\Delta_n(\theta)$ be given by* (2.6). *Then, under assumptions* (A1)–(A4) *and for each $\theta \in \Theta$,*

$$\mathscr{L}[\Delta_n(\theta)|P_\theta] \Rightarrow N(0, \bar{\Gamma}(\theta)).$$

The asymptotic distribution of the log-likelihood $\Lambda_n(\theta)$ is also of interest. It follows from the two theorems just formulated. More precisely,

Theorem 4.3. *Let $\Lambda_n(\theta)$ and $\Delta_n(\theta)$ be as in Theorem* 4.1. *Then, under assumptions* (A1)–(A4) *and for each $\theta \in \Theta$,*

$$\mathscr{L}[\Lambda_n(\theta)|P_\theta] \Rightarrow N(-\tfrac{1}{2}h'\bar{\Gamma}(\theta)\,h,\ h'\bar{\Gamma}(\theta)\,h).$$

The following three results are versions of the three preceding theorems when the probability measure P_θ is replaced by the probability measure P_{θ_n}. Their proofs rely heavily on contiguity results.

Theorem 4.4. *Let $\Lambda_n(\theta)$, $\Delta_n(\theta)$ and θ_n be as in Theorem* 4.1. *Then, under assumptions* (A1)–(A4) *and for each $\theta \in \Theta$,*

$$\Lambda_n(\theta) - h'\Delta_n(\theta) \to -\tfrac{1}{2}h'\bar{\Gamma}(\theta)\,h \quad \text{in } P_{\theta_n}\text{-probability.}$$

Next, the appropriate version of Theorem 4.3 is as follows.

Theorem 4.5. *Let $\Lambda_n(\theta)$ and θ_n be as in Theorem* 4.1. *Then, under assumptions* (A1)–(A4) *and for each $\theta \in \Theta$,*

$$\mathscr{L}[\Lambda_n(\theta)|P_{\theta_n}] \Rightarrow N(\tfrac{1}{2}h'\bar{\Gamma}(\theta)\,h,\ h'\bar{\Gamma}(\theta)\,h).$$

Finally, under P_{θ_n}, the asumptotic distribution of $\Delta_n(\theta)$ is given by

Theorem 4.6. *Let $\Delta_n(\theta)$ and θ_n be as in Theorem* 4.1. *Then, under assumptions* (A1)–(A4) *and for each $\theta \in \Theta$,*

$$\mathscr{L}[\Delta_n(\theta)|P_{\theta_n}] \Rightarrow N(\bar{\Gamma}(\theta)\,h,\ \bar{\Gamma}(\theta)).$$

As has been already stated the proofs of the preceding theorems are presented in Sect. 6.

5. Some Auxiliary Results

In this section, those lemmas necessary for the proofs of the main results, Theorems 4.1–4.6, are gathered together. Most of these lemmas are the appropriate versions of Lemmas 5.1–5.5 in Roussas (1972) and/or Lemmas 4.1–4.6 in Philippou and Roussas (1973).

To this end, it would be advisable to introduce the following simplifying notation. Consider the quantity $\varphi_n^2(\theta;\theta^*) = \varphi_n^2(\mathbf{X}_n;\theta,\theta^*)$ and replace $\mathbf{X}_n$ by $\mathbf{X}_j$ and θ^* by θ_n defined by (4.1). Thus, $\varphi_j^2(\theta;\theta_n) = \varphi_j^2(\mathbf{X}_j;\theta,\theta_n)$ which is denoted by $\varphi_{nj}^2(\theta)$ or even φ_{nj}^2 for simplicity. That is,

$$\varphi_{nj} = \varphi_{nj}(\theta) = \varphi_j(\mathbf{X}_j;\theta,\theta_n). \tag{5.1}$$

Next, the quantities $\dot{\varphi}_j(\theta)$, $\Lambda_n(\theta)$ and $\bar{\Gamma}(\theta)$ are defined in assumption (A2)(i), by relation (4.2) and assumption (A3)(ii), respectively. Again, for convenience, set

$$\dot{\varphi}_j = \dot{\varphi}_j(\theta), \quad \Lambda_n = \Lambda_n(\theta), \quad \bar{\Gamma} = \bar{\Gamma}(\theta), \qquad 1 \leqq j \leqq n, \ n \geqq 1. \tag{5.2}$$

The following simple proposition is formulated and proved below for easy reference.

Proposition 5.1. *For* $1 \leqq j \leqq n$, *consider the* r.v.'s Z_j *and* Z_{nj} *defined on the probability space* $(\Omega, \mathscr{A}, P)$ *and suppose that*

$$\max(\mathscr{E}|Z_{nj} - Z_j|^2;\ 1 \leqq j \leqq n) \to 0, \qquad \mathscr{E} Z_j^2 \leqq M(<\infty),\ j \geqq 1.$$

Then

$$\max(\mathscr{E}|Z_{nj}^2 - Z_j^2|;\ 1 \leqq j \leqq n) \to 0.$$

Proof. It is an immediate consequence of the boundedness assumption and the Minkowski and Hölder inequalities.

In all that follows, the quantities $\varphi_{nj} = \varphi_{nj}(\theta)$, $\dot{\varphi}_j = \dot{\varphi}_j(\theta)$, $\Lambda_n = \Lambda_n(\theta)$ and $\bar{\Gamma} = \bar{\Gamma}(\theta)$ are defined by relations (5.1) and (5.2). All results to be dealt with in this section are understood to hold under assumptions (A1)–(A4) and for every $\theta \in \Theta$. Finally, for convenience again, the expression $\max\limits_j \alpha_{nj}$ will be used rather than the more complete expression $\max(\alpha_{nj};\ 1 \leqq j \leqq n)$ for several quantities denoted here by α_{nj}.

Lemma 5.1. $\max[\mathscr{E}_\theta |n(\varphi_{nj} - 1)^2 - (h' \dot{\varphi}_j)^2|;\ 1 \leqq j \leqq n] \to 0.$

Proof. By Proposition 5.1, it suffices to show that

$$\max_j \mathscr{E}_\theta |n^{\frac{1}{2}}(\varphi_{nj} - 1) - (h' \dot{\varphi}_j)|^2 \to 0, \quad \mathscr{E}_\theta (h' \dot{\varphi}_j)^2 \leqq M(<\infty), \qquad j \geqq 1. \tag{5.4}$$

The second part of (5.4) follows immediately from assumption (A3)(i). That is,

$$\mathscr{E}_\theta (h' \dot{\varphi}_j)^2 \leqq M, \qquad j \geqq 1. \tag{5.5}$$

From this inequality, it follows that

$$\mathscr{E}_\theta \|\dot{\varphi}_j\|^2 \leqq M, \qquad j \geqq 1,$$

where $\|.\|$ is the usual norm in $\mathbb{R}^k$ and M is a generic constant. Next, on account of the previous inequality,

$$\mathscr{E}_\theta |(h_n - h)' \dot{\varphi}_j|^2 \leqq \|h_n - h\|^2 M \to 0. \tag{5.6}$$

Thus,

$$\begin{aligned} \max_j \mathscr{E}_\theta^{\frac{1}{2}} |n^{\frac{1}{2}}(\varphi_{nj} - 1) - (h' \dot{\varphi}_j)|^2 &\leqq \max_j \mathscr{E}_\theta^{\frac{1}{2}} |n^{\frac{1}{2}}(\varphi_{nj} - 1) - (h_n' \dot{\varphi}_j)|^2 \\ &\quad + \max_j \mathscr{E}_\theta^{\frac{1}{2}} |(h_n - h)' \dot{\varphi}_j|^2 \to 0 \end{aligned} \tag{5.7}$$

by means of (5.6) amd also (2.7) applied with λ and h replaced by $n^{-\frac{1}{2}}$ and $h_n \to h \in \mathbb{R}^k$, respectively. This is the first relation in (5.4) so that the proof of the lemma is completed.

This lemma has the following corollaries.

Corollary 5.1. $\sum_{j=1}^{n} (\varphi_{nj}-1)^2 - \frac{1}{n} \sum_{j=1}^{n} (h' \dot{\varphi}_j)^2 \to 0$ *in P_θ-probability.*

Proof. For every $\varepsilon > 0$,

$$P_\theta\left[\left|\sum_{j=1}^{n} (\varphi_{nj}-1)^2 - \frac{1}{n}\sum_{j=1}^{n} (h'\dot{\varphi}_j)^2\right| > \varepsilon\right] \leqq \frac{1}{n\varepsilon} \sum_{j=1}^{n} \mathscr{E}_\theta |n(\varphi_{nj}-1)^2 - (h'\dot{\varphi}_j)^2|. \tag{5.8}$$

But, on account of the lemma, there exists $N = N(\varepsilon)$ such that $n \geqq N$ implies

$$\max_j \mathscr{E}_\theta |n(\varphi_{nj}-1)^2 - (h'\dot{\varphi}_j)^2| < \varepsilon^2.$$

Then, for $n \geqq N$, the right hand side of (5.8) is bounded by $\frac{1}{n\varepsilon} n\varepsilon^2 = \varepsilon$ which establishes the corollary.

Corollary 5.2. $\sum_{j=1}^{n} (\varphi_{nj}-1)^2 \to \frac{1}{4} h' \bar{\Gamma} h$ *in P_θ-probability.*

Proof. It is immediate from Corollary 5.1 and relation (3.2).

As has already been mentioned elsewhere most of the lemmas to be discused in this section are suitable versions of lemmas in Roussas (1972) and/or lemmas in Philippou and Roussas (1973). This fact implies that there will be many similarities in the proofs of the corresponding lemmas. However, all lemmas will be supplied with a proof for reasons of completeness. Some of these proofs will be reduced to only an outline. Having this in mind, let us consider the following lemma.

Lemma 5.2. $\max(|\varphi_{nj}-1|;\ 1 \leqq j \leqq n) \to 0$ *in P_θ-probability.*

Proof. Set $R_{nj} = n^{\frac{1}{2}}(\varphi_{nj}-1) - (h'\dot{\varphi}_j)$ and let $\varepsilon > 0$. Then working as in the proof of Lemma 4.3 in Philippou and Roussas (1973), one has

$$P_\theta(\max_j |\varphi_{nj}-1| > \varepsilon) \leqq P_\theta\left(\max_j |R_{nj}| > \frac{\varepsilon n^{\frac{1}{2}}}{2}\right) + P_\theta\left(\max_j |h'\dot{\varphi}_j| > \frac{\varepsilon n^{\frac{1}{2}}}{2}\right). \tag{5.9}$$

But

$$P_\theta\left(\max_j |R_{nj}| > \frac{\varepsilon n^{\frac{1}{2}}}{2}\right) \leqq \frac{4}{n\varepsilon^2} \sum_{j=1}^{n} \mathscr{E}_\theta |R_{nj}|^2,$$

and by means of (2.7), applied with λ and h replaced by $n^{-\frac{1}{2}}$ and h_n, respectively,

$$\mathscr{E}_\theta |R_{nj}|^2 \leqq \frac{\varepsilon^3}{8} \quad \text{for } n \geqq \text{ some } N = N(\varepsilon) \text{ and } 1 \leqq j \leqq n.$$

Thus,

$$P_\theta\left(\max_j |R_{nj}| \geqq \frac{\varepsilon n^{\frac{1}{2}}}{2}\right) \leqq \frac{\varepsilon}{2}, \qquad n \geqq N. \tag{5.10}$$

Next, on the basis of assumption (A3)(i) and working as in the proof of the latter part of Lemma 4.3 in Philippou and Roussas (1973), one has

$$P_\theta\left(\max_j |h' \dot{\varphi}_j| > \frac{\varepsilon n^{\frac{1}{2}}}{2}\right) \leqq \frac{\varepsilon}{2}, \qquad n \geqq N. \tag{5.11}$$

Then relations (5.9)–(5.11) establish the lemma.

Remark 5.1. At this point let us recall the following simple fact. Namely, if Y_n and Z_n, $n \geqq 1$, are r.v.'s defined on the probability space $(\Omega, \mathscr{A}, P)$ and such that

$$Y_n - Z_n \xrightarrow{P} 0 \quad \text{and } \{Y_n\} \text{ is bounded in probability,}$$

then $\{Z_n\}$ is also bounded in probability, as follows from the inequality

$$P(|Z_n| > 2M) \leqq P(|Y_n - Z_n| > M) + P(|Y_n| > M).$$

We now proceed with the following lemma.

Lemma 5.3. $\Lambda_n - 2\left[\sum_{j=1}^n (\varphi_{nj} - 1) - \frac{1}{2}\sum_{j=1}^n (\varphi_{nj} - 1)^2\right] \to 0$ *in P_θ-probability.*

Proof. From (5.5), it follows that $\left\{\frac{1}{n}\sum_{j=1}^n (h' \dot{\varphi}_j)^2\right\}$ is bounded in P_θ-probability. This fact, along with Remark 5.1 and Corollary 5.1, implies that $\left\{\sum_{j=1}^n (\varphi_{nj} - 1)^2\right\}$ is also bounded in P_θ-probability. Hence, by this result and Lemma 5.2,

$$(\max_j |\varphi_{nj} - 1|) \sum_{j=1}^n (\varphi_{nj} - 1)^2 \to 0 \quad \text{in } P_\theta\text{-probability.} \tag{5.12}$$

Next, following the proof of Lemma 4.4 in Philippou and Roussas (1973), one also obtains, by means of (5.12),

$$\sum_{j=1}^n \log \varphi_{nj}^2 - 2\left[\sum_{j=1}^n (\varphi_{nj} - 1) - \frac{1}{2}\sum_{j=1}^n (\varphi_{nj} - 1)^2\right] \to 0 \tag{5.13}$$

in P_θ-probability. Now, on account of (2.4), (5.1) and (5.2),

$$\Lambda_n = \log q_0(\theta; \theta_n) + \sum_{j=1}^n \log \varphi_{nj}^2,$$

and

$$\log q_0(\theta; \theta_n) \to 0 \quad \text{in } P_\theta\text{-probability,}$$

by assumption (A4) and relation (4.1). Thus,

$$\Lambda_n - \sum_{j=1}^{n} \log \varphi_{nj}^2 \to 0 \quad \text{in } P_\theta\text{-probability.} \tag{5.14}$$

Relations (5.13) and (5.14) give then the desired result.

By using standard arguments one can also show the following simple result. Namely,

Lemma 5.4. *For* $1 \leqq j \leqq n$, $\mathscr{E}_\theta \varphi_{nj}^2 = 1$.

A somewhat more complicated result referring to convergence in the first mean will now be established.

Lemma 5.5. $\max_j \mathscr{E}_\theta |n^{\frac{1}{2}}(\varphi_{nj}^2 - 1) - 2h' \dot{\varphi}_j| \to 0$.

Proof. Consider the identity

$$\begin{aligned} &n^{\frac{1}{2}}(\varphi_{nj}^2 - 1) - 2h' \dot{\varphi}_j \\ &\quad = \varphi_{nj}[n^{\frac{1}{2}}(\varphi_{nj} - 1) - h' \dot{\varphi}_j] + h' \dot{\varphi}_j(\varphi_{nj} - 1) + [n^{\frac{1}{2}}(\varphi_{nj} - 1) - h' \dot{\varphi}_j] \end{aligned}$$

and work as in the proof of Lemma 5.4 in Roussas (1972). One then obtains

$$\begin{aligned} \max_j \mathscr{E}_\theta |n^{\frac{1}{2}}(\varphi_{nj}^2 - 1) - 2h' \dot{\varphi}_j| &\leqq 2 \max_j \mathscr{E}_\theta^{\frac{1}{2}} |n^{\frac{1}{2}}(\varphi_{nj} - 1) - h' \dot{\varphi}_j|^2 \\ &\quad + [\max_j \mathscr{E}_\theta^{\frac{1}{2}} (h' \dot{\varphi}_j)^2] [\max_j \mathscr{E}_\theta^{\frac{1}{2}} |\varphi_{nj} - 1|^2]. \end{aligned} \tag{5.15}$$

From Lemma 5.1,

$$\max_j \mathscr{E}_\theta \left| (\varphi_{nj} - 1)^2 - \frac{1}{n} (h' \dot{\varphi}_j)^2 \right| \to 0, \tag{5.16}$$

so that

$$\max_j \mathscr{E}_\theta |\varphi_{nj} - 1|^2 \leqq \max_j \mathscr{E}_\theta \left| (\varphi_{nj} - 1)^2 - \frac{1}{n} (h' \dot{\varphi}_j)^2 \right| + \frac{1}{n} \max_j \mathscr{E}_\theta (h' \dot{\varphi}_j)^2 \to 0$$

on account of (5.16) and the second relation in (5.4). This result, along with (5.7) and (5.15), establishes the lemma.

Now, interpreting the expectation (conditional or not) of a random vector in the usual coordinatewise sense, one may formulate and prove the following result.

Lemma 5.6. *For* $j \geqq 1$, $\mathscr{E}_\theta(\dot{\varphi}_j | \mathscr{A}_{j-1}) = 0$ *a.s.* $[P_\theta]$ *(so that* $\mathscr{E}_\theta(h' \dot{\varphi}_j) = 0$ *for every* $h \in \mathbb{R}^k$*).*

Proof. By using standard arguments one can show that

$$\mathscr{E}_\theta(\varphi_{nj}^2 | \mathscr{A}_{j-1}) = 1 \quad \text{a.s. } [P_\theta], \qquad 1 \leqq j \leqq n.$$

On the basis of this result and Lemma 5.5, the desired result is established as in the proof of Lemma 5.5(i) in Roussas (1972).

For $j=1, \ldots, n$, define the r.v.'s ψ_{nj} as follows

$$\psi_{nj}=\psi_{nj}(\theta)=\mathscr{E}_\theta(\varphi_{nj} \mid \mathscr{A}_{j-1}). \tag{5.17}$$

Also, set

$$Y_j=Y_{nj}=(\varphi_{nj}-1)-n^{-\frac{1}{2}} h' \dot{\varphi}_j-(\psi_{nj}-1), \quad j=1, \ldots, n. \tag{5.18}$$

Then the following lemma is true; that is,

Lemma 5.7. *Let the* r.v.'s Y_j, $j=1, \ldots, n$ *be defined by* (5.18). *Then*

$$\sum_{j=1}^{n} Y_j \to 0 \quad \text{in } P_\theta\text{-probability.}$$

Proof. By well known properties of conditioning, Lemma 5.6 and relation (5.17), one has

$$\mathscr{E}_\theta(Y_{j+1} \mid Y_1, \ldots, Y_j)=0 \quad \text{a.s. } [P_\theta], \quad j \geqq 1, \tag{5.19}$$

and, clearly, $\mathscr{E}_\theta Y_1=0$ by means of Lemma 5.6 and relations (5.17) and (5.18). Then by means of a well known inequality (see, for example, Loève (1963), p. 386), one obtains

$$\begin{aligned} P_\theta\left(\left|\sum_{j=1}^{n} Y_j\right|>\varepsilon\right) &\leqq \frac{1}{\varepsilon^2} \sum_{j=1}^{n} \sigma_\theta^2(Y_j) \\ &\leqq \frac{2}{n\varepsilon^2} \sum_{j=1}^{n} [\mathscr{E}_\theta|n^{\frac{1}{2}}(\varphi_{nj}-1)-h' \dot{\varphi}_j|^2+\mathscr{E}_\theta|n^{\frac{1}{2}}(\psi_{nj}-1)|^2], \end{aligned} \tag{5.20}$$

the last inequality holding true by the c_r-inequality applied with $r=2$. At this point recall that, if X is a r.v. defined on the probability space $(\Omega, \mathscr{A}, P)$ and for which $\mathscr{E}X$ exists, and if $\mathscr{F}$ is a sub-σ-field of $\mathscr{A}$, then

$$|\mathscr{E}(X \mid \mathscr{F})|^r \leqq \mathscr{E}(|X|^r \mid \mathscr{F}) \quad \text{a.s.,} \quad r \geqq 1.$$

On the basis of this fact, one has

$$\mathscr{E}_\theta|n^{\frac{1}{2}}(\psi_{nj}-1)|^2=\mathscr{E}_\theta|n^{\frac{1}{2}} \mathscr{E}_\theta(\varphi_{nj}-1)|\mathscr{A}_{j-1}|^2 \leqq \mathscr{E}_\theta|n^{\frac{1}{2}}(\varphi_{nj}-1)-h' \dot{\varphi}_j|^2 \to 0,$$

uniformly in $j \geqq 1$. Therefore, for $\varepsilon>0$,

$$\mathscr{E}_\theta|n^{\frac{1}{2}}(\psi_{nj}-1)|^2<\frac{\varepsilon^3}{4}, \quad n \geqq \text{some } N=N(\varepsilon), \ 1 \leqq j \leqq n. \tag{5.21}$$

Also,

$$\mathscr{E}_\theta|n^{\frac{1}{2}}(\varphi_{nj}-1)-h' \dot{\varphi}_j|^2<\frac{\varepsilon^3}{4}, \quad n \geqq N, \ 1 \leqq j \leqq n. \tag{5.22}$$

On the basis of (5.21) and (5.22), relation (5.20) becomes as follows, for $n \geqq N$ and $1 \leqq j \leqq n$,

$$P_\theta\left(\left|\sum_{j=1}^{n} Y_j\right| > \varepsilon\right) < \varepsilon,$$

as was to be shown.

Now Corollary 5.2 states that

$$\sum_{j=1}^{n} (\varphi_{nj}-1)^2 \to \tfrac{1}{4} h' \bar{\Gamma} h \quad \text{in } P_\theta\text{-probability.}$$

It is also true that the above convergence is still true when the r.v.'s $(\varphi_{nj}-1)^2$, $j=1,\ldots,n$ are suitably conditioned. More precisely, the following lemma is true.

Lemma 5.8. $\sum_{j=1}^{n} \mathscr{E}_\theta[(\varphi_{nj}-1)^2 \mid \mathscr{A}_{j-1}] \to \frac{1}{4} h' \bar{\Gamma} h$ *in* P_θ*-probability.*

Proof. It can be seen that

$$P_\theta\left\{\left|\sum_{j=1}^{n} \mathscr{E}_\theta[(\varphi_{nj}-1)^2 \mid \mathscr{A}_{j-1}] - \frac{1}{n}\sum_{j=1}^{n} \mathscr{E}_\theta[(h'\dot{\varphi}_j)^2 \mid \mathscr{A}_{j-1}]\right| > \varepsilon\right\}$$
$$\leqq \frac{1}{n\varepsilon}\sum_{j=1}^{n} \mathscr{E}_\theta |n(\varphi_{nj}-1)^2 - (h'\dot{\varphi}_j)^2|.$$

From Lemma 5.1,

$$\mathscr{E}_\theta |n(\varphi_{nj}-1)^2 - (h'\dot{\varphi}_j)^2| < \varepsilon^2, \qquad n \geqq \text{ some } N = N(\varepsilon),\ 1 \leqq j \leqq n.$$

Therefore

$$\sum_{j=1}^{n} \mathscr{E}_\theta[(\varphi_{nj}-1)^2 \mid \mathscr{A}_{j-1}] - \frac{1}{n}\sum_{j=1}^{n} \mathscr{E}_\theta[(h'\dot{\varphi}_j)^2 \mid \mathscr{A}_{j-1}] \to 0$$

in P_θ-probability. This result, along with relation (3.3), applied with $t=h$, gives then

$$\sum_{j=1}^{n} \mathscr{E}_\theta[(\varphi_{nj}-1)^2 \mid \mathscr{A}_{j-1}] \to \tfrac{1}{4} h' \bar{\Gamma} h$$

in P_θ-probability, as was to be shown.

The following simple result will also be needed below. Namely,

Lemma 5.9. *For* $j=1,\ldots,n$, *let the* r.v.'s ψ_{nj} *be defined by* (5.17). *Then*

$$2\sum_{j=1}^{n} (\psi_{nj}-1) \to -\tfrac{1}{4} h' \bar{\Gamma} h \quad \textit{in } P_\theta\textit{-probability.}$$

Proof. It is the same as the proof of Lemma 5.5(ii) of Roussas (1972) on the basis of Lemma 5.6 and Lemma 5.8 herein.

Finally, this section is closed with the following

Lemma 5.10. *Let $\Delta_n=\Delta_n(\theta)$ be defined by* (2.6). *Then*

$$2\sum_{j=1}^{n}(\varphi_{nj}-1)-h'\Delta_n\to-\tfrac{1}{4}h'\Gamma h \quad \textit{in } P_\theta\textit{-probability.}$$

Proof. Taking into consideration the definition of the r.v.'s Y_j, $j=1,\ldots,n$ by (5.18), Lemma 5.7 states that

$$2\sum_{j=1}^{n}(\varphi_{nj}-1)-h'\Delta_n-2\sum_{j=1}^{n}(\psi_{nj}-1)\to 0$$

in P_θ-probability. This fact, in conjunction with Lemma 5.9, provides the desires result.

6. Proof of Main Results

The main results formulated in Sect. 4 can now be proved.

Proof of Theorem 4.1. From Lemma 5.3, Corollary 5.2 and Lemma 5.10, one has, respectively, the following convergences in P_θ-probability:

$$\Lambda_n-2\sum_{j=1}^{n}(\varphi_{nj}-1)+\sum_{j=1}^{n}(\varphi_{nj}-1)^2\to 0,$$

$$\sum_{j=1}^{n}(\varphi_{nj}-1)^2\to\tfrac{1}{4}h'\Gamma h,$$

and

$$2\sum_{j=1}^{n}(\varphi_{nj}-1)-h'\Delta_n\to-\tfrac{1}{4}h'\Gamma h.$$

From these expressions, one obtains in an obvious manner that

$$\Lambda_n-h'\Delta_n\to-\tfrac{1}{2}h'\Gamma h \quad \text{in } P_\theta\text{-probability,}$$

as was to be shown.

Proof of Theorem 4.2. Set $S_n=\sum_{j=1}^{n}t'\dot{\varphi}_j$, $n\geqq 1$. Then, by well known properties of conditioning and Lemma 5.6,

$$\mathscr{E}_\theta(S_n\,|\,S_1,\ldots,S_{n-1})=S_{n-1} \quad \text{a.s. } [P_\theta].$$

Thus, for each $\theta\in\Theta$, $\{S_n,\mathscr{A}_n\}$, $n\geqq 1$, is a martingale on the probability space $(\mathscr{X},\mathscr{A},P_\theta)$ with $S_0=0$. Consider the quantities defined in (3.4). Then, by (3.5) and (3.7) in Lemmas 3.1 and 3.2, Conditions (1) and (2) in Brown (1971) are satisfied. Therefore Theorem 2 in the reference just cited holds true. Namely,

$$\mathscr{L}[t'\Delta_n(t'\bar{\Gamma}_n t)^{-\frac{1}{2}}\,|\,P_\theta]\Rightarrow N(0,1).$$

This convergence and the fact that $t'\bar{\Gamma}_n t \to t'\bar{\Gamma} t$ give then

$$\mathscr{L}(t'\Delta_n \mid P_\theta) \Rightarrow N(0, t'\bar{\Gamma} t).$$

Since this last convergence is true for every $t \in \mathbb{R}^k$, one concludes that

$$\mathscr{L}(\Delta_n \mid P_\theta) \Rightarrow N(0, \bar{\Gamma}),$$

as was to be seen.

Proof of Theorem 4.3. Immediate from Theorems 4.1, 4.2 and the standard Slutsky theorems.

The proofs of Theorems 4.4–4.6 follow from those of the preceeding theorems and the proposition below about contiguity of the sequences of probability measures $\{P_\theta\}$ $(=\{P_{n,\theta}\})$ and $\{P_{\theta_n}\}$ $(=\{P_{n,\theta_n}\})$. More precisely, the following proposition holds true.

Proposition 6.1. *If $\{h_n^*\}$ is a bounded sequence in $\mathbb{R}^k$ and if $\theta_n^* = \theta + n^{-\frac{1}{2}} h_n^*$, then the sequences of probability measures $\{P_{n,\theta}\}$ and $\{P_{n,\theta_n}\}$ are contiguous.*

Proof. See Proposition 6.1 in Roussas (1972).

We may now complete the proofs of the remaining theorems.

Proof of Theorem 4.4. By the preceeding proposition the sequences $\{P_\theta\}$ and $\{P_{\theta_n}\}$ are contiguous. This fact, along with Theorem 4.1 and the definition of contiguity, provides the required proof.

Proof of Theorem 4.5. As in the proof of Theorem 4.5 in Roussas (1972), it is a consequence of Theorem 4.3, the contiguity of the sequences $\{P_\theta\}$ and $\{P_{\theta_n}\}$ and Corollary 7.2, p. 35, in Roussas (1972).

This section is concluded with the proof of the last theorem, namely,

Proof of Theorem 4.6. Once again, one refers to the proof of the corresponding theorem in Roussas (1972). Namely, the assumptions of Lemma 7.1, p. 36, in the reference just cited are fulfilled with $P_n = P_\theta$, $P_n' = P_{\theta_n}$, $T_n = \Delta_n$ and $\Gamma = \bar{\Gamma}$. This is so because of Theorems 4.2 and 4.1 herein. Then Theorem 7.2 in Roussas (1972), p. 38, gives the desirable result.

7. Examples

In this closing section, three examples are mentioned, where assumptions (A 1)–(A4) of the present paper are met. Relevant details can be found in Stamatelos (1976).

Example 7.1. The r.v.'s X_n, $n \geqq 0$, constitute a Gaussian process such that

$$\mathscr{E}_\theta X_n = \theta, \quad \sigma_\theta^2(X_n) = 1, \quad C_\theta(X_m, X_n) = \rho^{|m-n|}, \quad |\rho| < 1.$$

Then the process is a stationary Markov process (see, for example, Doob (1953), Example 4, p. 218 and pp. 223–224).

Example 7.2. The r.v.'s X_n, $n \geqq 0$, satisfy the relation

$$X_n = \alpha X_{n-1} + u_n,$$

where $|\alpha| < 1$ is known and the r.v.'s u_n, $n \geqq 1$, are independently distributed as $N(\theta, 1)$, and $X_0 = 0$.

Example 7.3. The r.v.'s X_n, $n \geqq 0$, are coming from a Gaussian processes such that:

(i) $\mathscr{E}_\theta X_n = \theta$ or 2θ according as n is odd or even,
(ii) $C_\theta(X_m, X_n) = c$ (known) for all m and n $(m \neq n)$,
(iii) $\sigma_\theta^2(X_n) = 2c$ for all n.

This process is neither Markovian nor stationary.

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