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Convergence Rates in the Law of Large Numbers when Extreme Terms are Excluded

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1. Introduction and a Theorem

For a sequence $x_1, \ldots, x_n$ of n real numbers, let $f_{nr}(x_1, \ldots, x_n) = x_j$ if $|x_j|$ is the r-th maximum of $|x_1|, \ldots, |x_n|$. More precisely let $m_n(j)$, $1 \leqq j \leqq n$, be the number of x_i's satisfying either $|x_i| > |x_j|$, $1 \leqq i \leqq n$, or $|x_i| = |x_j|$, $1 \leqq i \leqq j$, and let $f_{nr}(x_1, \ldots, x_n) = x_j$ if $m_n(j) = r$. Let $\{X_n, n = 1, 2, \ldots\}$ be a sequence of i.i.d. random variables with common distribution function $F(x)$ and put $\mathscr{F}(x) = P(|X_1| > x)$. Write

$$X_n^{(k)} = f_{nk}(X_1, \ldots, X_n), \qquad S_n = \sum_{i=1}^{n} X_i = {}^{(0)}S_n$$

and

$${}^{(r)}S_n = S_n - \sum_{k=1}^{r} X_n^{(k)} \quad \text{for} \quad 1 \leqq r \leqq n.$$

One of the authors [5], [6] has shown the strong law of large numbers for ${}^{(r)}S_n$. In this paper, we consider the rates of convergence in it. The main theorem we are going to prove is the following.

Theorem 1. (I) *Let $r \geqq 0$ be an integer and let $0 < \alpha < 2$, $t \geqq 1$. If there exists a sequence $\{a_n\}$ of constants such that for every $\varepsilon > 0$*

$$\sum_{n=1}^{\infty} n^{t-2} P(|{}^{(r)}S_n - a_n| > n^{1/\alpha} \varepsilon) < \infty, \tag{1}$$

then

$$\int_0^{\infty} x^{\alpha(r+t)-1} \mathscr{F}(x)^{r+1} \, dx < \infty. \tag{2}$$

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(II) *Conversely, if* (2) *holds then* (1) *holds for every* $\varepsilon>0$. *In this case the sequence* $\{a_n\}$ *may be chosen according to the formula* $a_n=n\int_{|x|\leqq n\tau} x\,dF(x)$, *where* $\tau>0$ *is an arbitrary constant. In particular, if* $0<\alpha<1$ *then the constants* a_n *may be chosen to be zero. If* $1\leqq\alpha<2$ *and* $E|X_1|<\infty$ *then* a_n *may be chosen to be* nEX_1.

Remark 1. If (2) holds, then applying the dominated convergence theorem to the right hand side of the relationship

$$x^{\alpha(r+t)}\mathscr{F}(x)^{r+1}=\alpha(r+t)\int_0^x y^{\alpha(r+t)-1}\mathscr{F}(y)^{r+1}\left(\frac{\mathscr{F}(x)}{\mathscr{F}(y)}\right)^{r+1}dy,$$

we have

$$\lim_{x\to\infty} x^{\alpha(r+t)}\mathscr{F}(x)^{r+1}=0,$$

that is, $\mathscr{F}(x)=o(x^{-\alpha(r+t)/(r+1)})$ as $x\to\infty$. Therefore if either $1<\alpha<2$ or $\alpha=1$, $t>1$ then (2) implies $E|X_1|<\infty$.

Remark 2. Our theorem extends a result of Baum-Katz [1] who studied the case $r=0$.

In the last section, we shall apply the above theorem to obtain a result on ruled sums.

2. Proof of (I) in Theorem 1

We begin with some lemmas.

Lemma 1. *Let* $r\geqq 0$ *be an integer and* $0<\alpha<2$. *If* (1) *holds for every* $\varepsilon>0$ *then there exists a sequence* $\{c_n\}$ *such that*

$$\lim_{n\to\infty}\frac{{}^{(r)}S_n-c_n}{n^{1/\alpha}}=0 \quad \textit{in probability} \tag{3}$$

and

$$\sum_{n=1}^{\infty} n^{t-2}P(|{}^{(r)}S_n-c_n|>n^{1/\alpha}\varepsilon)<\infty. \tag{4}$$

Proof. We first prove (3). When $r=0$, the lemma is known. (See Baum-Katz [1].) Suppose that $r\geqq 1$. For each n let $(\pi(1),\ldots,\pi(n))$ be random permutation of $(1,\ldots,n)$ such that $P((\pi(1),\ldots,\pi(n))=(i_1,\ldots,i_n))=1/n!$ for every permutation $(i_1,\ldots,i_n)$ of $(1,\ldots,n)$. Suppose that $(\pi(1),\ldots,\pi(n))$ is independent of $\{X_n\}$. Let $\bar{X}_n^{(k)}=f_{nk}(X_{\pi(1)},\ldots,X_{\pi(n)})$ and ${}^{(r)}\bar{S}_n=S_n-\sum_{k=1}^{r}\bar{X}_n^{(k)}$. It suffices to prove

$$\lim_{n\to\infty}\frac{{}^{(r)}\bar{S}_n-c_n}{n^{1/\alpha}}=0 \quad \text{in probability,} \tag{5}$$

because ${}^{(r)}\bar{S}_n$ and ${}^{(r)}S_n$ are identically distributed.

In order to prove (5), we use the concentration function of ${}^{(r)}\bar{S}_n/n^{1/\alpha}$ defined by

$$C_n^{(r)}(h)=\sup_x P\left(\left|\frac{{}^{(r)}\bar{S}_n}{n^{1/\alpha}}-x\right|\leqq h\right), \qquad h>0.$$

It is easy to see that $\lim_{n\to\infty} C_n^{(r)}(h)=1$ for every $h>0$ if and only if (5) holds for a sequence $\{c_n\}$. Suppose that there exist an increasing sequence $\{n_i\}$ of integers and constants $\varepsilon>0$ and $d_0<1$ such that $C_{n_i}^{(r)}(\varepsilon)\leqq d_0$ for $i\geqq 1$, Choose d_1 and ρ such that $d_0<d_1<\rho^r<1$. We may assume $n_i/\rho\leqq n_{i+1}$. For $n_i\leqq n\leqq[n_i/\rho']$, where $\rho'=(1+\rho)/2$, and for an arbitrary x we have

$$P\left(\left|\frac{{}^{(r)}\bar{S}_n}{n^{1/\alpha}}-x\right|\leqq h,\ \bar{X}_n^{(k)}=\bar{X}_{n_i}^{(k)},\ 1\leqq k\leqq r\right)$$

$$=\frac{n_i}{n}\cdot\frac{n_i-1}{n-1}\cdots\frac{n_i-r+1}{n-r+1}\left(\left|\frac{{}^{(r)}\bar{S}_n}{n^{1/\alpha}}-x\right|\leqq h\right)$$

$$\geqq d_1 P\left(\left|\frac{{}^{(r)}\bar{S}_n}{n^{1/\alpha}}-x\right|\leqq h\right).$$

On the other hand the probability on the left hand side of the above relationship is equal to

$$P\left(\left|\frac{{}^{(r)}\bar{S}_{n_i}+S_n-S_{n_i}}{n^{1/\alpha}}-x\right|\leqq h,\ \bar{X}_n^{(k)}=\bar{X}_{n_i}^{(k)},\ 1\leqq k\leqq r\right)$$

$$\leqq P\left(\left|\frac{{}^{(r)}\bar{S}_{n_i}+S_n-S_{n_i}}{n^{1/\alpha}}-x\right|\leqq h\right)$$

$$=\int_{-\infty}^{\infty} P\left(\left|\frac{{}^{(r)}S_{n_i}}{n^{1/\alpha}}+y-x\right|\leqq h\right) F^{(n-n_i)*}(n^{1/\alpha}\,dy)$$

$$\leqq\int_{-\infty}^{\infty} P\left(\left|\frac{{}^{(r)}S_{n_i}}{n_i^{1/\alpha}}+\left(\frac{n}{n_i}\right)^{1/\alpha}(y-x)\right|\leqq\frac{h}{\rho^{1/\alpha}}\right) F^{(n-n_i)*}(n^{1/\alpha}\,dy)$$

$$\leqq C_{n_i}^{(r)}\left(\frac{h}{\rho^{1/\alpha}}\right).$$

Hence we have for $n_i\leqq n\leqq[n_i/\rho']$

$$d_1\, C_n^{(r)}(h)\leqq C_{n_i}^{(r)}\left(\frac{h}{\rho^{1/\alpha}}\right)$$

and therefore by choosing $h=\rho^{1/\alpha}\varepsilon$ we obtain

$$d_1\, C_n^{(r)}(\rho^{1/\alpha}\varepsilon)\leqq C_{n_i}^{(r)}(\varepsilon)<d_0$$

so that

$$C_n^{(r)}(\rho^{1/\alpha}\varepsilon)<d_0/d_1 \qquad \text{for}\ \ n_i\leqq n\leqq[n_i/\rho'].$$

Thus we have for every sequence $\{a_n\}$

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n} P\left(\left|\frac{{}^{(r)}\bar{S}_n - a_n}{n^{1/\alpha}}\right| > \rho^{1/\alpha}\varepsilon\right)
&\geqq \sum_{i=1}^{\infty} \sum_{n=n_i}^{[n_i/\rho']} \frac{1}{n} P\left(\left|\frac{{}^{(r)}\bar{S}_n - a_n}{n^{1/\alpha}}\right| > \rho^{1/\alpha}\varepsilon\right) \\
&\geqq \sum_{i=1}^{\infty} \sum_{n=n_i}^{[n_i/\rho']} \frac{1}{n} \{1 - C_n^{(r)}(\rho^{1/\alpha}\varepsilon)\} \\
&\geqq \left(1 - \frac{d_0}{d_1}\right) \sum_{i=1}^{\infty} \frac{\rho'}{n_i} \left(\frac{n_i}{\rho'} - n_i\right) = \infty,
\end{aligned}$$

which is a contradiction and concludes (3).

In order to prove that (1) implies (4), it suffices to show

$$P(|{}^{(r)}S_n - c_n| > n^{1/\alpha}\varepsilon) \leqq P\left(|{}^{(r)}S_n - a_n| > n^{1/\alpha}\frac{\varepsilon}{4}\right) \tag{6}$$

for all large n. If $(-\varepsilon, \varepsilon) \supset \left(\frac{a_n - c_n}{n^{1/\alpha}} - \frac{\varepsilon}{4}, \frac{a_n - c_n}{n^{1/\alpha}} + \frac{\varepsilon}{4}\right)$ then (6) is trivially valid, and in the other case, using (3), we have for large n

$$\begin{aligned}
P(|{}^{(r)}S_n - c_n| > n^{1/\alpha}\varepsilon) &\leqq P\left(|{}^{(r)}S_n - c_n| < n^{1/\alpha}\frac{\varepsilon}{2}\right) \\
&\leqq P\left(|{}^{(r)}S_n - a_n| > n^{1/\alpha}\frac{\varepsilon}{4}\right),
\end{aligned}$$

which is no more than (6).

Lemma 2. *If* $({}^{(r)}S_n - c_n)/n^{1/\alpha} \to 0$ *in probability, then*

$$\lim_{x \to \infty} x\mathscr{F}(x^{1/\alpha}) = 0 \tag{7}$$

and

$$\lim_{n \to \infty} \frac{S_n - c_n}{n^{1/\alpha}} = 0 \quad \textit{in probability.} \tag{8}$$

Proof. It suffices to show (7). In fact, (7) implies

$$P(|X_n^{(k)}| > \varepsilon n^{1/\alpha}) \sim \frac{1}{k!} (n\mathscr{F}(\varepsilon n^{1/\alpha}))^k \to 0$$

(see [5]), and therefore for every $k \geqq 1$ $\lim_{n \to \infty} X_n^{(k)}/n^{1/\alpha} = 0$ in probability. Together with the assumption this implies (8).

Suppose that (7) does not hold. Then there exist a constant $c > 0$ and an increasing sequence $\{n_i\}$ of positive integers such that $n_i \mathscr{F}(n_i^{1/\alpha}) > c$. Choose $x_i > n_i^{1/\alpha}$ so as to satisfy

$$n_i \mathscr{F}(x_i) \leqq c/2 \leqq n_i \mathscr{F}(x_i - 0).$$

For fixed $\xi>0$ let $X_1(\xi), \ldots, X_n(\xi)$ denote a sequence of i.i.d. random variables each having the distribution function $F_\xi(x)$ where

$$F_\xi(x)=P(X_1\leqq x \,|\, |X_1|\leqq \xi).$$

Let $S_n(\xi)=\sum_{k=1}^{n} X_k(\xi)$ and

$$f_n(\xi,\varepsilon)=P\left(\left|\frac{S_{n-r}(\xi)-c_n}{n^{1/\alpha}}\right|>\varepsilon\right).$$

Then it is easy to see that

$$P\left(\left|\frac{{}^{(r)}S_n-c_n}{n^{1/\alpha}}\right|>\varepsilon \,|\, X_n^{(r)}\right)=f_n(|X_n^{(r)}|,\varepsilon) \quad \text{a.s.} \tag{9}$$

On the other hand

$$\begin{aligned} P(|X_{n_i}^{(r)}|\geqq x_i) &= \sum_{k=r}^{n_i}\binom{n_i}{k}\mathscr{F}(x_i-0)^k(1-\mathscr{F}(x_i-0))^{n_i-k} \\ &\geqq \sum_{k=r}^{n_i}\binom{n_i}{k}\left(\frac{c}{2n_i}\right)^k\left(1-\frac{c}{2n_i}\right)^{n_i-k} \\ &\geqq \binom{n_i}{r}\left(\frac{c}{2n_i}\right)^r\left(1-\frac{c}{2n_i}\right)^{n_i-r}\sim\frac{1}{r!}\left(\frac{c}{2}\right)^r e^{-c/2}. \end{aligned} \tag{10}$$

If we choose $\varepsilon_i\downarrow 0$ such that

$$\lim_{i\to\infty} P\left(\left|\frac{{}^{(r)}S_{n_i}-c_{n_i}}{n_i^{1/\alpha}}\right|>\varepsilon_i\right)=0$$

then it follows from (9) that

$$\begin{aligned} &\inf_{\xi\geqq x_i} f_{n_i}(\xi,\varepsilon_i)\, P(|X_{n_i}^{(r)}|\geqq x_i) \\ &\quad\leqq \int_0^\infty f_{n_i}(\xi,\varepsilon_i)\, P(|X_{n_i}^{(r)}|\in(\xi,\xi+d\xi)) \\ &\quad= P\left(\left|\frac{{}^{(r)}S_{n_i}-c_{n_i}}{n_i^{1/\alpha}}\right|>\varepsilon_i\right)\to 0 \end{aligned}$$

as $i\to\infty$ and therefore by (10) there exists a sequence $\{\xi_i\}$ such that $\xi_i\geqq x_i$ and $\lim_{i\to\infty} f_{n_i}(\xi_i,\varepsilon_i)=0$. This implies

$$\lim_{i\to\infty}\frac{S_{n_i-r}(\xi_i)-c_{n_i}}{n_i^{1/\alpha}}=0 \quad \text{in probability.}$$

If follows from the well-known necessary condition for the weak law of large numbers (see, e.g. Gnedenko-Kolmogorov [3], §27) that for every $\varepsilon>0$

$$\lim_{i\to\infty}(n_i-r)\{F_{\xi_i}(-n_i^{1/\alpha}\varepsilon)+1-F_{\xi_i}(n_i^{1/\alpha}\varepsilon)\}=0$$

and therefore

$$\lim_{i\to\infty} n_i\{\mathscr{F}(n_i^{1/\alpha}\varepsilon)-\mathscr{F}(\xi_i)\}=0.$$

Since $n_i\mathscr{F}(\xi_i)\leqq n_i\mathscr{F}(x_i)\leqq c/2$, we have for $0<\varepsilon<1$

$$\begin{aligned} n_i\{\mathscr{F}(n_i^{1/\alpha}\varepsilon)-\mathscr{F}(\xi_i)\} &\geqq n_i\mathscr{F}(n_i^{1/\alpha})-n_i\mathscr{F}(\xi_i)\\ &\geqq c-c/2=c/2. \end{aligned}$$

This contradiction proves (7) and therefore the lemma.

Let zs return to the proof of (I). We see from (8) that the weak law of large numbers holds for the truncated random variables, that is, for every $\delta>0$

$$\lim_{n\to\infty}\frac{1}{n^{1/\alpha}}\left\{\sum_{k=1}^{n}X_k I(|X_k|\leqq n^{1/\alpha}\delta)-c_n\right\}=0 \quad \text{in probability.} \tag{11}$$

(See, e.g. Gnedenko-Kolmogorov [3], §24.) For a subset $\sigma=(\sigma_1,\dots,\sigma_{r+1})$ of $I_n=\{1,\dots,n\}$ let $A(\sigma,\varepsilon)$ denote the event

$$\begin{aligned} A(\sigma,\varepsilon)=\{&|X_i|>2n^{1/\alpha}\varepsilon \text{ for } i\in\sigma,\ |X_i|\leqq 2n^{1/\alpha}\varepsilon \text{ for } i\in\sigma'\\ &\text{and } |\sum_{j\in\sigma'}X_j-c_n|<n^{1/\alpha}\varepsilon\}, \end{aligned}$$

where σ' is the complement of σ in I_n. Then we have

$$P(A(\sigma,\varepsilon))\geqq\mathscr{F}(2n^{1/\alpha}\varepsilon)\{P(|\sum_{j\in\sigma'}X_j-c_n|<n^{1/\alpha}\varepsilon)-1+(1-\mathscr{F}(2n^{1/\alpha}\varepsilon))^{n-r+1}\}.$$

From (8) it follows easily that $(c_{n+k}-c_n)/n^{1/\alpha}\to 0$ for each k. Thus

$$\begin{aligned} &P(|\sum_{j\in\sigma'}X_j-c_n|<n^{1/\alpha}\varepsilon)\\ &\geqq P\left(\left|\sum_{k=1}^{n-r-1}X_k I(|X_k|>2(n-r-1)^{1/\alpha}\varepsilon)-c_{n-r-1}\right|<(n-r-1)^{1/\alpha}\frac{\varepsilon}{2}\right)\\ &\to 1 \quad \text{as } n\to\infty, \end{aligned}$$

because of (11). From (7) we have

$$\lim_{n\to\infty}(1-\mathscr{F}(2n^{1/\alpha}\varepsilon))^{n-r-1}=1.$$

It is easy to see that

$$\left\{\left|\frac{{}^{(r)}S_n-c_n}{n^{1/\alpha}}\right|>\varepsilon\right\}\supset\bigcup_{\sigma}A(\sigma,\varepsilon),$$

where the union on the right hand side extends over all subsets $\sigma \subset I_n$ of size $r+1$. Since $A(\sigma, \varepsilon)'$ s are disjoint, we have

$$\infty > \sum_{n=1}^{\infty} n^{t-2} P\left(\left|\frac{{}^{(r)}S_n - c_n}{n^{1/\alpha}}\right| > \varepsilon\right)$$

$$\geqq \text{const.} \cdot \sum_{n=1}^{\infty} n^{t-2} \binom{n}{r+1} \mathscr{F}(2n^{1/\alpha}\varepsilon)^{r+1}$$

$$\sim \text{const.} \cdot \sum_{n=1}^{\infty} n^{t+r-2} \mathscr{F}(2n^{1/\alpha}\varepsilon)^{r+1}.$$

Thus

$$\infty > \int_0^{\infty} x^{t+r-2} \mathscr{F}(x^{1/\alpha})^{t+1} \, dx = \int_0^{\infty} x^{\alpha(r+t)-1} \mathscr{F}(x)^{r+1} \, dx,$$

which completes the proof of (I).

3. Proof of (II) in Theorem 1

Lemma 3. *Let $a(\cdot)$ be a positive nonincreasing right-continuous function and let $b(t) = \inf\{s; a(s) < t\}$. The function $b(t)$ is then positive nonincreasing left-continuous and satisfies*

$$-\int_0^{\infty} f(t) \, da(t) = \int_{a(\infty)}^{a(+0)} f(b(s)) \, ds$$

for every nonnegative Baire function $f(\cdot)$.

Proof. See, e.g. Meyer [4], p. 108.

As was shown in Remark 1, (2) implies

$$\lim_{x \to \infty} x^{\alpha} \mathscr{F}(x) = 0. \tag{12}$$

Define ψ by $\psi(x) = (x^{\alpha}/\mathscr{F}(x))^{1/2}$, $x \geqq 0$. ψ is a right-continuous nondecreasing function with $\psi(0) = 0$ and $\psi(\infty) = \infty$. Let $\varphi(x) = \inf\{y; \psi(y) > x\}$, $x \geqq 0$. Then φ is also right-continuous nondecreasing and satisfies $\varphi(0) = 0$. It is seen from (12) that $\lim_{x \to \infty} x^{\alpha}/\psi(x) = 0$ and therefore $\lim_{x \to \infty} \varphi(x)/x^{1/\alpha} = 0$. By choosing $a(x) = \psi(x)^{-2r-2}$ in Lemma 3, we have

$$b(y) = \inf\{x; \psi(x)^{-2r-2} < y\} = \inf\{x; \psi(x) > y^{-1/(2r+2)}\} = \varphi(y^{-1/(2r+2)})$$

and therefore

$$\begin{aligned} -\int_0^{\infty} f(x) \, d(\psi(x)^{-2r-2}) &= \int_0^{\infty} f(\varphi(y^{-1/(2r+2)})) \, dy \\ &= (2r+2) \int_0^{\infty} f(\varphi(x)) \, x^{-2r-3} \, dx \end{aligned} \tag{13}$$

for every Baire function $f(\cdot)\geqq 0$. Suppose that (2) holds. Let $f(x)=x^{\alpha(2r+t+1)}$, then $f(0)=0$ and

$$\lim_{x\to\infty} f(x)\,\psi(x)^{-2r-2}=\lim_{x\to\infty} x^{\alpha(r+t)}\,\mathscr{F}(x)^{r+1}=0.$$

Thus integrating by parts, we find that the left hand side of (13) is equal to

$$\begin{aligned}
&-\int_0^\infty x^{\alpha(2r+t+1)}\,d(\psi(x)^{-2r-2})\\
&\quad=\alpha(2r+t+1)\int_0^\infty x^{\alpha(2r+t+1)-1}\left(\frac{\mathscr{F}(x)}{x^\alpha}\right)^{r+1}dx\\
&\quad=\alpha(2r+t+1)\int_0^\infty x^{\alpha(r+t)-1}\,\mathscr{F}(x)^{r+1}\,dx,
\end{aligned}$$

that is

$$\begin{aligned}
&\int_0^\infty \varphi(x)^{\alpha(2r+t+1)}\,x^{-2r-3}\,dx\\
&\quad=\frac{\alpha(2r+t+1)}{2r+2}\int_0^\infty x^{\alpha(r+t)-1}\,\mathscr{F}(x)^{r+1}\,dx<\infty. \qquad (14)
\end{aligned}$$

Remark 3. Mori [5], [6] used (14) to prove the strong law of large numbers for ${}^{(r)}S_n$. In those papers (14) was proved under the unnecessary assumption that $\mathscr{F}(x)$ is positive and differentiable.

Proof of (II). Given $\varepsilon>0$ choose δ such that $0<\delta<\varepsilon/(2r+t+1)$. Define S'_n and S''_n by

$$S'_n=\sum_{k=1}^n X_k\,I(|X_k|<n^{1/\alpha}\delta)$$

and

$$S''_n=\sum_{k=1}^n X_k\,I(|X_k|<\varphi(n)),$$

respectively. Let $\tau>0$ be fixed and $a_n=n\int_{|x|\leqq n\tau} x\,dF(x)$. We have

$$\sum_{n=1}^\infty n^{t-2}\,P(|{}^{(r)}S_n-a_n|>3n^{1/\alpha}\varepsilon)\leqq\sum_{(1)}+\sum_{(2)}+\sum_{(3)},$$

where

$$\sum_{(1)}=\sum_{n=1}^\infty n^{t-2}\,P(|{}^{(r)}S_n-S'_n|>n^{1/\alpha}\varepsilon), \qquad (15)$$

$$\sum_{(2)}=\sum_{n=1}^\infty n^{t-2}\,P(|S'_n-S''_n|>n^{1/\alpha}\varepsilon) \qquad (16)$$

and

$$\sum_{(3)} = \sum_{n=1}^{\infty} n^{t-2} P(|S_n'' - a_n| > n^{1/\alpha} \varepsilon).$$

At first we show that (15) is finite. Since $n\mathscr{F}(n^{1/\alpha}\delta) \to 0$, we have

$$P(|{}^{(r)}S_n - S_n'| > n^{1/\alpha}\varepsilon) \leqq P(|{}^{(r)}S_n - S_n'| > n^{1/\alpha} r\delta)$$
$$\leqq P(|X_n^{(r+1)}| > n^{1/\alpha}\delta) \sim \frac{1}{(r+1)!} (n\mathscr{F}(n^{1/\alpha}\delta))^{r+1},$$

and therefore

$$\sum_{(1)} \leqq \text{const.} \cdot \sum_{n=1}^{\infty} n^{r+t-1} \mathscr{F}(n^{1/\alpha}\delta)^{r+1} < \infty,$$

because

$$\int_0^{\infty} x^{r+t-1} \mathscr{F}(x^{1/\alpha})^{r+1}\, dx = \alpha \int_0^{\infty} x^{\alpha(r+t)-1} \mathscr{F}(x)^{r+1} dx < \infty.$$

To show that (16) is finite, let n be so large that $n^{1/\alpha}\delta > \varphi(n)$. Let N_n denote the number of X_j such that $|X_j| > \varphi(n)$, $j \leqq n$. Then we have

$$P(|S_n' - S_n''| > n^{1/\alpha}\varepsilon) \leqq P(n^{1/\alpha}\delta N_n > n^{1/\alpha}\varepsilon) = P(N_n > \varepsilon/\delta)$$
$$\leqq P(N_n \geqq s),$$

where $s = 2r + t + 1$. It is easy to see that $x\mathscr{F}(\varphi(x)) \leqq \varphi(x)^\alpha / x \leqq x\mathscr{F}(\varphi(x) - 0)$. This shows that

$$P(N_n \geqq s) \sim \text{const.} \cdot (n\mathscr{F}(\varphi(n)))^s \leqq \text{const.} \cdot \left(\frac{\varphi(n)^\alpha}{n}\right)^s.$$

Thus by (14) we have

$$\sum_{(2)} \leqq \text{const.} \cdot \sum_{n=1}^{\infty} n^{t-2-s} \varphi(n)^{\alpha s} < \infty.$$

Finally we show that $\sum_{(3)} < \infty$. Let, for a fixed n, $X_i' = X_i I(|X_i| \leqq \varphi(n))$ $1 \leqq i \leqq n$, and let $X_1'', \ldots, X_n''$ be independent copy of $X_1', \ldots, X_n'$. Let $Y_i = X_i' - X_i''$ be the symmetrization of X_i' and write $T_n = \sum_{i=1}^{n} Y_i$, $t_n^2 \equiv E T_n^2 = n E Y_1^2$. By (2) and Lemma 4 of Mori [6] we have

$$t_n^2 = 2n \int_{|y| \leqq \varphi(n)} y^2\, dF(y) = o(n\varphi(n)^{2-\alpha}). \tag{17}$$

Let $\eta = \varepsilon n^{1/\alpha} / 2t_n$ and $c = 2\varphi(n)/t_n$. By Prokhorov's inequality [7] we have

$$P\left(\frac{|T_n|}{n^{1/\alpha}}>\frac{\varepsilon}{2}\right)=P\left(\frac{|T_n|}{t_n}>\eta\right)\leqq 2\exp\left\{-\frac{\eta}{2c}\operatorname{arcsinh}\left(\frac{\eta c}{2}\right)\right\}$$
$$=2\exp\left\{-\frac{\varepsilon n^{1/\alpha}}{8\varphi(n)}\operatorname{arcsinh}\left(\frac{\varepsilon n^{1/\alpha}\varphi(n)}{2t_n^2}\right)\right\}.$$

If $\operatorname{arcsinh}\left(\frac{\varepsilon n^{1/\alpha}\varphi(n)}{2t_n^2}\right)\geqq\frac{1}{\varepsilon}$, then

$$P\left(\frac{|T_n|}{n^{1/\alpha}}>\frac{\varepsilon}{2}\right)\leqq 2\exp\left\{-\frac{n^{1/\alpha}}{8\varphi(n)}\right\}. \tag{18}$$

If $\operatorname{arcsinh}\left(\frac{\varepsilon n^{1/\alpha}\varphi(n)}{2t_n^2}\right)<\frac{1}{\varepsilon}$, then

$$\operatorname{arcsinh}\left(\frac{\varepsilon n^{1/\alpha}\varphi(n)}{2t_n^2}\right)\geqq C_\varepsilon\frac{\varepsilon n^{1/\alpha}\varphi(n)}{2t_n^2},$$

since $\operatorname{arcsinh} x\geqq C_\varepsilon x$ for $0\leqq x\leqq\sinh\frac{1}{\varepsilon}$, where $C_\varepsilon=\left(\varepsilon\sinh\frac{1}{\varepsilon}\right)^{-1}$. Thus from (17) we have

$$P\left(\frac{|T_n|}{n^{1/\alpha}}>\frac{\varepsilon}{2}\right)\leqq 2\exp\left\{-K_\varepsilon\frac{n^{2/\alpha}}{t_n^2}\right\}\leqq 2\exp\left\{-K_\varepsilon\left(\frac{n^{1/\alpha}}{\varphi(n)}\right)^{2-\alpha}\right\}, \tag{19}$$

where $K_\varepsilon=\frac{1}{16}C_\varepsilon\varepsilon^2$. (18) and (19) prove that

$$P\left(\frac{|T_n|}{n^{1/\alpha}}>\frac{\varepsilon}{2}\right)\leqq \text{const.}\cdot\left(\frac{\varphi(n)}{n^{1/\alpha}}\right)^{s\alpha} \tag{20}$$

for every large n. On the other hand, by Mori [6], (2) implies that

$$\lim_{n\to\infty}\frac{S_n-a_n}{n^{1/\alpha}}=0 \quad \text{in probability,}$$

where $a_n=n\int_{|x|\leqq n\tau} x\,dF(x)$. This in turn shows

$$\lim_{n\to\infty}\frac{S_n''-a_n}{n^{1/\alpha}}=0 \quad \text{in probability.}$$

Therefore

$$\lim_{n\to\infty}\frac{\operatorname{med}(S_n'')-a_n}{n^{1/\alpha}}=0, \tag{21}$$

and for large n we have from (20), (21) and the symmetrization inequality that

$$P(|S_n''-a_n|>\varepsilon n^{1/\alpha})\leqq P(|S_n''-\operatorname{med}(S_n'')|>\tfrac{1}{2}\varepsilon n^{1/\alpha})$$
$$\leqq 2P(|T_n|>\tfrac{1}{2}\varepsilon n^{1/\alpha})\leqq\text{const.}\cdot\left(\frac{\varphi(n)}{n^{1/\alpha}}\right)^{s\alpha}.$$

Consequently

$$\sum_{(3)} \leqq \text{const.} \cdot \sum_{n=1}^{\infty} n^{t-s-2}\, \varphi(n)^{s\alpha} < \infty.$$

This completes the proof of (II).

4. An Application to Ruled Sums

Following Baum-Katz-Stratton [2], we define the ruled sum $S_{(n)} = \sum_{i \in (n)} X_i$, where a rule (n) is some collection of n distinct positive integers for each n. We let $\mathbf{R}$ denote the set of all rules. If $(n) = \{1, 2, \ldots, n\}$, then $S_{(n)} = S_n$. In this last section, as an application of Theorem1, we shall give the law of large numbers for ${}^{(r)}S_{(n)}$, where ${}^{(r)}S_{(n)}$ is the ruled sum from which extreme terms are excluded and defined in the way similar to the one when we defined ${}^{(r)}S_n$.

Theorem 2. *Let* $r \geqq 0$ *be an integer and let* $0 < \alpha < 2$. *The following two statements are equivalent:*

(i) *For some sequence of real numbers* $\{a_n\}$,

$$\lim_{n \to \infty} \frac{{}^{(r)}S_{(n)} - a_n}{n^{1/\alpha}} = 0 \quad \text{a.s.} \qquad \textit{for all } (\,) \in \mathbf{R}.$$

(ii) $\int_0^{\infty} x^{\alpha(r+2)-1}\, \mathscr{F}(x)^{r+1}\, dx < \infty.$

Remark 4. The case $r=0$, $\alpha=1$ has been proved by Baum-Katz-Stratton [2].

Proof. We first note that

$$\sum_{n=1}^{\infty} P(|{}^{(r)}S_n - a_n| > n^{1/\alpha}\varepsilon) < \infty$$

is equivalent to

$$\sum_{n=1}^{\infty} P(|{}^{(r)}S_{(n)} - a_n| > n^{1/\alpha}\varepsilon) < \infty, \tag{22}$$

because $P(|{}^{(r)}S_n - a_n| > n^{1/\alpha}\varepsilon) = P(|{}^{(r)}S_{(n)} - a_n| > n^{1/\alpha}\varepsilon)$. Therefore it follows from Theorem 1 with $t=2$ that (22) is equivalent to (ii). Furthermore it is seen by the first Borel-Cantelli lemma that (22) implies (i). Hence it remains to show that (i) implies (22). We note that (i) means

$$P(|{}^{(r)}S_{(n)} - a_n| > n^{1/\alpha}\varepsilon, \text{ i.o.}) = 0. \tag{23}$$

If we take a particular rule $(\,)$ such that $(n) \cap (m) = \emptyset$ if $n \neq m$, then ${}^{(r)}S_{(1)}$, ${}^{(r)}S_{(2)}, \ldots, {}^{(r)}S_{(n)}$ are independent. Thus (22) follows from (23) by the second Borel-Cantelli lemma. This completes the proof.

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