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The Riesz Decomposition for Vector-Valued Amarts

G. A. Edgar* and L. Sucheston**

Department of Mathematics, Ohio State University, Columbus, Ohio 43210, U.S.A.

Let $(\Omega, \mathscr{F}, P)$ be a probability space, $\mathbb{N}=\{1,2,\ldots\}$, and let $(\mathscr{F}_n)_{n\in\mathbb{N}}$ be an increasing sequence of σ-algebras contained in $\mathscr{F}$. A *stopping time* is a mapping $\tau\colon \Omega\to\mathbb{N}\cup\{\infty\}$, such that $\{\tau=n\}\in\mathscr{F}_n$ for all $n\in\mathbb{N}$. The collection of bounded stopping times is denoted by T; under the natural ordering T is a directed set "filtering to the right". (The notation and the terminology of the present note are close to that of our longer article [7].)

Let $\mathscr{E}$ be a Banach space and consider a sequence $(X_n)_{n\in\mathbb{N}}$ of $\mathscr{E}$-valued random variables *adapted* to $(\mathscr{F}_n)$, i.e., such that $X_n\colon\Omega\to\mathscr{E}$ is $\mathscr{F}_n$-strongly measurable. EX (expectation of X) is the Pettis integral of X; $E_A X$ denotes $E(1_A\cdot X)$. The sequence (X_n) is called an *amart* iff each X_n is Pettis integrable and $\lim_T E(X_\tau)$ exists in the strong topology of $\mathscr{E}$.

Remark. Also the term "asymptotic martingale" introduced in [4] is in use. We at present feel that the notion is so simple and basic that it merits a short name. There are further several varieties of amarts; e.g. "weak amarts" and "weak sequential amarts" introduced in [3], and "hyperamarts", appearing in our paper on the continuous parameter, see [7], Part B. Clearly the term "amart" is better suited to bear added qualifiers than "asymptotic martingale".

The *real* Riesz decomposition theorem for amarts [7] asserts that an amart X_n can be uniquely written as a sum of a martingale Y_n, and an amart Z_n that converges to zero in nearly all possible ways: $Z_n\to 0$ a.e. and in L^1, and $Z_\tau\to 0$ in L^1. The Banach valued version of this theorem is the main result of the present note.

The *Pettis norm* of a random variable X is

$$\|X\| = \sup_{\substack{f\in\mathscr{E}'\\ |f|\leq 1}} E|f(X)|. \tag{1}$$

It is known [9] that $\|X\|$ is equal to the *semivariation* of the measure $E_A X$, i.e.,

$$\|X\| = \sup |\textstyle\sum \alpha_i E_{A_i} X| \tag{2}$$

* To whom offprint requests should be sent

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where the supremum is taken over all finite collections of scalars with $|\alpha_i| \leqq 1$ and all partitions of Ω into finitely many disjoint sets. One also has

$$\|X\| \leqq 4 \sup_{A \in \mathscr{F}} |E_A X|. \tag{3}$$

This is a known result, which can be proved as follows: Let $Y \cdot P$ be the measure defined by

$$(Y \cdot P)(A) = \int_A Y dP.$$

Then

$$\begin{aligned}\|X\| &= \sup_{|f| \leq 1} \{\text{variation}\,[f(X) \cdot P]\} \leqq \sup_{|f| \leq 1} 4 \sup_A \left|\int_A f(X)\, dP\right| \\ &= 4 \sup_A \left[\sup_{|f| \leq 1} \left|f\left(\int_A X\, dP\right)\right|\right] = 4 \sup_A \left|\int_A X\, dP\right|.\end{aligned}$$

A *potential* is an amart that converges to zero in the Pettis norm. A sequence of adapted random variables is sait to be *of class* (*B*) iff

$$\sup_T E|X_\tau| < \infty. \tag{B}$$

We prove

Theorem 1 (Riesz decomposition). *Let $\mathscr{E}$ be a Banach space with the Radon-Nikodym property and let $(X_n, \mathscr{F}_n)$ be an $\mathscr{E}$-valued amart such that*

$$\liminf E|X_n| < \infty. \tag{4}$$

(i) *X_n can be uniquely written as the sum of a martingale Y_n and a potential Z_n. $(Z_\tau)_{\tau \in T}$ converges to zero in Pettis norm.*

(ii) *If $\mathscr{E}'$ is separable and $(X_n, \mathscr{F}_n)$ is of class* (B), *then $Z_n \to 0$* a.e. *weakly.*

Proof of (i). The uniqueness of the decomposition follows from the observation that if an arbitrary sequence X_n can be written as

$$X_n = Y_n + Z_n = Y_n' + Z_n'$$

where Y_n and Y_n' are martingales and

$$\lim_n E_A Z_n = \lim_n E_A Z_n'$$

for each $A \in \bigcup_m \mathscr{F}_m$, then the martingale $Y_n' - Y_n = Z_n - Z_n' = U_n$ satisfies for $A \in \mathscr{F}_m$

$$\lim_n E_A U_n = E_A U_m = 0.$$

Hence $U_m = 0$ a.e.; thus $Z_n = Z_n'$ a.e. and $Y_n = Y_n'$ a.e.

It will be useful to state for the purpose of reference the vector amart convergence theorem due to [4].

Theorem 2 ([4]; *a proof is also given in* [7], *Section* 5). *Let $\mathscr{E}$ have the Radon-Nikodym property and a separable dual. An $\mathscr{E}$-valued amart of class* (B) *converges weakly* a.e.

We also need a Lemma.

Lemma 1 ([4]; *a proof is also given in* [7]). *Let* $\mathscr{E}$ *be an arbitrary Banach space,* $(X_n, \mathscr{F}_n)$ *an* $\mathscr{E}$*-valued amart, m a fixed integer. Then*

$$\lim_n E_A X_n = \mu_m(A) \tag{5}$$

exists uniformly in $A \in \mathscr{F}_n$.

Now by the Vitali-Hahn-Saks theorem (cf. [6], p. 321) μ_m is a (countably additive) measure on $\mathscr{F}_m$. We observe that (4) implies that

$$\text{variation } \mu_m \leqq \liminf E|X_n| = M < \infty. \tag{6}$$

To see this, given $\varepsilon > 0$ choose disjoint sets A_i, $i = 1, 2, \ldots, k$ so that

$$\text{variation } \mu_m - \sum_1^k |\mu_m(A_i)| < \varepsilon.$$

Next find N so large that $n > N$ implies for all i

$$|\mu(A_i) - E_{A_i} X_n| < \frac{\varepsilon}{k}.$$

Now

$$\liminf \sum_1^k |E_{A_i} X_n| \leqq \liminf \sum_{i=1}^k E_{A_i} |X_n| \leqq \liminf E|X_n| = M$$

implies that variation $\mu_m \leqq M + 2\varepsilon$. Since ε is arbitrary, (6) follows.

Thus the Radon-Nikodym derivative $\dfrac{d\mu_m}{dP} = Y_m$ exists. $(Y_m, \mathscr{F}_m)$ is a martingale, since for each $A \in \mathscr{F}_m$

$$E_A Y_{m+1} = \mu_{m+1}(A) = \mu_m(A) = E_A Y_m.$$

Let $Z_m = X_m - Y_m$; then for $n \geqq m$

$$E_A Z_n = E_A X_n - E_A Y_m,$$

hence

$$\lim_n E_A Z_n = 0, \qquad A \in \mathscr{F}_m. \tag{7}$$

Now let $\varepsilon_i \downarrow 0$. For each m choose $A_m \in \mathscr{F}_m$ so that

$$\sup_{A \in \mathscr{F}_m} |E_A Z_m| - |E_{A_m} Z_m| < \varepsilon_m. \tag{8}$$

Because of (7) we can find an integer $n_m > m$ such that

$$|E_{A_m^c} Z_{n_m}| < \varepsilon_m.$$

Define a stopping time τ_m by

$$\tau_m = m \quad \text{on} \quad A_m, \qquad \tau_m = n_m \quad \text{on} \quad A_m^c.$$

Then for each m

$$|E_{A_m} Z_m - E Z_{\tau_m}| = |E_{A_m^c} Z_{n_m}| < \varepsilon_m.$$

Since Z_n is an amart,

$$\lim_m E Z_{\tau_m} = \lim_m E Z_m = 0.$$

It follows that $E_{A_m} Z_m \to 0$; hence, by (8),

$$\sup_{A \in \mathscr{F}_m} |E_A Z_m| \to 0$$

(3) now implies that $\|Z_m\| \to 0$.

For each increasing sequence of bounded stopping times τ_n, $(Z_{\tau_n})_{n \in \mathbb{N}}$ is an amart with respect to $(\mathscr{F}_{\tau_m})_{m \in \mathbb{N}}$ ([7], Proposition 1.6; the proof is valid also in the Banach-valued case). Therefore $\|Z_{\tau_m}\| \to 0$; it follows that $\lim_T \|Z_\tau\| = 0$.

Proof of (ii).

Lemma 2. *Let $\mathscr{E}$ be an arbitrary Banach space, m a constant $\in \mathbb{N}$, and $(X_n, \mathscr{F}_m)_{n \in \mathbb{N}}$ any sequence (not necessarily an amart) of class* (B). *Then $E \sup |X_n| < \infty$.*

Proof. Suppose that $E \sup |X_n| = \infty$. Then for each integer $M > 0$ there is N_M such that $E \max_{n \leq N_M} |X_n| > M$. Let

$$\tau_M = \inf \{k \leq N_M : |X_k| = \max_{n \leq N_M} |X_n|\},$$

then $E|X_{\tau_M}| \geq M$. This contradicts (B).

Observe that if $\mathscr{E} = \mathbb{R}$ this lemma is valid assuming only $\sup_T |EX_\tau| < \infty$ instead of (B) (cf. the proof of Proposition 2.4, [7]).

Now assume that $(X_n, \mathscr{F}_n)$ is an amart of class (B); then $(E^{\mathscr{F}_m} X_n, \mathscr{F}_m)_{n \in \mathbb{N}}$ is an amart for each m; hence by Lemma 2

$$\sup_n |E^{\mathscr{F}_m} X_n| \in L^1. \tag{9}$$

Assume $\mathscr{E}'$ separable; by Theorem 2, $\lim_n E^{\mathscr{F}_m} X_n = Y'_m$ exists a.e. in the weak topology.

Integrating over sets $A \in \mathscr{F}_m$, as we may by (9), we obtain

$$E_A Y'_m = \lim_n E_A X_n = E_A Y_m.$$

Thus $Y_m = \lim_n E^{\mathscr{F}_m} X_n$. Since $E^{\mathscr{F}_m}$ is an L^1 contraction and X_n is of class (B), it follows that $(Y_m)_{m \in \mathbb{N}}$ is L^1 bounded. Now $(|Y_n|)_{n \in \mathbb{N}}$ is a (numerical) submartingale, hence $E|Y_\tau|$ is an increasing function of $\tau \in T$. Therefore L^1-boundedness of Y_n implies that Y_n is of class (B); thus Z_n is also of class (B). By Theorem 2, weak limit a.e. of Z_n exists; this limit is necessarily zero, since $\|Z_n\| \to 0$. □

Remark 1. Let $(X_n, \mathscr{F}_n)$ be an amart taking values in an arbitrary Banach space. Then by Lemma 1 $\lim_n E_A X_n = \mu_m(A)$ exists for each m and is a measure on $\mathscr{F}_m$.

The necessary and sufficient condition that X_n have the Riesz decomposition is that the Radon-Nikodym derivative $\frac{d\mu_m}{dP}$ exist for each m. Indeed, if $\frac{d\mu_m}{dP}$ exists, the proof given above applies; conversely, if $X_m = Y_m + Z_m$ is the Riesz decomposition of X_m, Y_m is easily seen to be $\frac{d\mu_m}{dP}$. In particular, if all the σ-algebras $\mathscr{F}_m$ are atomic, then the Riesz decomposition holds whether or not $\mathscr{E}$ has the Radon-Nikodym property and whether or not (4) holds.

In general, the assumption that $\mathscr{E}$ have the R-N property cannot be omitted. This is already known for Theorem 2, even if all the σ-algebras $\mathscr{F}_m$ agree ([7], Section 5, example 3), and in fact the same example applies to Theorem 1.

Let $\{e_n^i, n \in \mathbb{N}\ 1 \leqq i \leqq 2^n\}$ be the standard basis for the Banach space c_0 (in any order). Let

$$A_n^i \cap A_n^j = \emptyset \quad \text{if } i \neq j,$$

and $P(A_n^i) = 2^{-n}$. Let

$$X_n = \sum_{k=1}^{n} \sum_{i=1}^{2k} e_k^i 1_{A_k^i}.$$

Let $\{e_n^i, n \in \mathbb{N}\ 1 \leqq i \leqq 2^n\}$ be the standard basis for the Banach space c_0 (in any order). Let

$$\frac{d\mu_m}{dP} = \sum_{k=1}^{\infty} \sum_{i=1}^{2k} e_k^i 1_{A_k^i}$$

exists as an $\mathscr{E}''$-valued random variable ($\mathscr{E}'' = l_\infty$), but not as an $\mathscr{E}$-valued random variable. Therefore the Riesz decomposition fails (cf. Remark 1 above).

The first example of [4] (repeated in [7]) shows that there is a bounded potential that fails to converge strongly a.e. or in L_E^1 norm.

The second example in [4], appearing also in [7], contributed by *W.J.* Davis, shows that in part (ii) of Theorem 1 the assumption that X_n is of class (B) cannot be replaced by the assumption that X_n is L^1-bounded, i.e., $\sup_n E|X_n| < \infty$.

The following question is still open: Is $\mathscr{E}'$ necessarily separable if $\mathscr{E}$ is separable and has the Radon-Nikodym property and each $\mathscr{E}$-valued amart of class (B) converges weakly a.e.? Theorem 1 shows that to prove that $\mathscr{E}$ is a counterexample to this conjecture, it suffices to prove that each $\mathscr{E}$-valued potential of class (B) converges weakly a.e. This follows because the separability of the dual is not needed for the convergence a.e. of martingales (theorem of Chatterji; a simple proof is sketched in [4]).

We now show that in part (i) of the theorem the assumption $\liminf E|X_n| < \infty$ cannot be omitted.

Example 1. *An amart in l^p, p fixed, $1 \leqq p \leqq 2$, with no Riesz decomposition. If $\mathscr{E} = l^p$, then $\mathscr{E}$ has the Radon-Nikodym property and, except in case $p=1$, separable dual.*

Given p, choose numbers $a_n > 0$ with $\sum_n a_n^p = \infty$, $\sum_n a_n^2 < \infty$. Let Z_n be independent real random variables with $P(Z_n = 1) = P(Z_n = -1) = \frac{1}{2}$. Set

$$X_n = \sum_{k=1}^{n} a_k e_k Z_k$$

where (e_n) is the usual unit basis for l^p. Then

$$|X_n(\omega)| = \left(\sum_{k=1}^{n} a_k^p\right)^{1/p}$$

for all ω, so that $E|X_n| \to \infty$. Thus the boundedness condition (4) is violated.

We claim that X_n is an amart for the constant sequence of σ-algebras $\mathcal{F}_n = \mathcal{F}$. Given $\varepsilon > 0$, choose $N \in \mathbb{N}$ so large that

$$4 \sum_{k=N}^{\infty} a_k^2 < \varepsilon^2.$$

Let $\tau \in T$, $\tau \geqq N$. Then

$$X_\tau = \sum_{k=1}^{\infty} a_k e_k Z_k 1_{B_k}$$

where $B_k = \{\tau \geqq k\}$. Thus

$$EX_\tau = \sum_{k=1}^{\infty} a_k e_k E(Z_k 1_{B_k}) = \sum_{k=N}^{\infty} a_k e_k E(Z_k 1_{B_k}),$$

since $EZ_k = 0$, and $B_k = \Omega$ for $k < N$. Now, with $\alpha_k = \pm a_k$, one has

$$\begin{aligned} |EX_\tau|^p &= \sum_{k=N}^{\infty} |a_k E(Z_k 1_{B_k})|^p \leqq \left(\sum_{k=N}^{\infty} |a_k E(Z_k 1_{B_k})|\right)^p \\ &= \left[\sum_{k=N}^{\infty} \alpha_k E(Z_k 1_{B_k})\right]^p \leqq \left[E\left(\sup_{n \geqq N}\left|\sum_{k=N}^{n} \alpha_k Z_k\right|\right)\right]^p. \end{aligned}$$

Observe that $Y_n = \sum_N^n \alpha_k Z_k$ is a martingale. Doob's (or Kolmogorov's) dominated estimates give (cf. [5], p. 317)

$$E(\sup_{n \geqq N} |Y_n|) \leqq [E(\sup_{n \geqq N} |Y_n|)^2]^{1/2} \leqq 2\left(\sum_{k=N}^{\infty} a_k^2\right)^{1/2} < \varepsilon,$$

hence $|EX_\tau| < \varepsilon$. This shows that $EX_\tau \to 0$; thus $(X_n, \mathcal{F}_n)$ is an amart.

Therefore for all $A \in \mathcal{F}_1 = \mathcal{F}$, $\mu(A) = \lim_n E_A X_n$ exists and defines a measure on l^p. We now claim that this measure does not have a σ-finite variation. Let $F \in \mathcal{F}$, $P(F) > 0$. We will show that μ has infinite variation on F. Fix n, let $\{A_1, \ldots, A_{2^n}\}$ be the atoms of the algebra generated by $Z_1, Z_2, \ldots, Z_n$. Thus $P(A_i) = 2^{-n}$ for $i = 1, 2, \ldots, 2^n$. Let $F_i = F \cap A_i$. For $k \leqq n$, Z_k is constant on F_i, hence $|E_{F_i} Z_k| = P(F_i)$. Thus for $m \geqq n$

$$|E(1_{F_i} X_m)|_p^p = \sum_{k=1}^{m} a_k^p |E_{F_i} Z_k|^p \geqq \sum_{k=1}^{n} a_k^p P(F_i)^p.$$

Hence

$$|\mu(F_i)| \geqq \left(\sum_{k=1}^{n} a_k^p\right)^{1/p} P(F_i).$$

Thus the variation of μ on F is at least

$$\sum_{i=1}^{2^n} |\mu(F_i)| \geqq \left(\sum_{k=1}^{n} a_k^p\right)^{1/p} \sum_{i=1}^{2^n} P(F_i) = \left(\sum_{k=1}^{n} a_k^p\right)^{1/p} P(F).$$

Since $\sum_{k=1}^{\infty} a_k^p = \infty$, this shows that μ has infinite variation on F.

Therefore μ does not have σ-finite variation. This shows that the Radon-Nikodym derivative $\left(\frac{d\mu}{dP}\right)_{\mathscr{F} = \mathscr{F}_i}$ does not exist, since if it did, μ would have variation at most λ on the set $\left\{\left|\frac{d\mu}{dP}\right| < \lambda\right\}$. Remark 1 above now implies that (X_n) does not have the Riesz decomposition. □

Our final result may be considered as a consequence of the *real* Riesz decomposition theorem stated in the beginning of the paper. The proof also uses the first important result obtained about the (then still nameless) amarts: the amart convergence theorem which asserts that an L^1-bounded real amart converges a.e. [1]. To see the analogy of what follows with martingales, recall that $(X_n, \mathscr{F}_n)$ is a martingale if and only if for each n, each bounded stopping time $\tau_n \geqq n$, one has $E^{\mathscr{F}_n} X_{\tau_n} - X_n = 0$.

Theorem 3. *A sequence of real-valued adapted random variables* $(X_n, \mathscr{F}_n)$ *is an amart if and only if for each increasing sequence of bounded stopping times* $\tau_n \geqq n$, *one has*

$$E^{\mathscr{F}_n} X_{\tau_n} - X_n \to 0 \quad \text{a.e. } \textit{and in } L^1. \tag{10}$$

Proof. The "if" part is immediate and only requires in (10) convergence in L^1: On integrating (10) one obtains $E(X_{\tau_n} - X_n) \to 0$, hence the net EX_τ is Cauchy and converges. Conversely, suppose that X_n is an amart. If $X_n = Y_n + Z_n$ is the Riesz decomposition of X_n, then the martingale Y_n clearly satisfies (10); it remains to show that the potential Z_n does. Since $(Z_n, \mathscr{F}_n)$ is an amart, $(Z_{\tau_n}, \mathscr{F}_{\tau_n})$ is, where $\mathscr{F}_{\tau_n} = \{A: A \cap \{\tau_n = k\} \in \mathscr{F}_k \ \forall k\}$ ([7], Proposition 1.6). Since $n \leqq \tau_n$, one has $\mathscr{F}_n \subseteq \mathscr{F}_{\tau_n}$ which implies that $(E^{\mathscr{F}_n} Z_{\tau_n}, \mathscr{F}_n)$ is an amart. The conditional expectation is an L^1-contraction; therefore $E|Z_{\tau_n}| \to 0$ implies that $E|E^{\mathscr{F}_n} Z_{\tau_n}| \to 0$; by the amart convergence theorem, $E^{\mathscr{F}_n} Z_{\tau_n} \to 0$ a.e. Hence also $E^{\mathscr{F}_n} Z_{\tau_n} - Z_n \to 0$ in L^1 and a.e. □

An adapted family $(X_n, \mathscr{F}_n)$ is called a *game fairer with time* [10] iff for every $\varepsilon > 0$

$$P[|E^{\mathscr{F}_n} X_p - X_n| > \varepsilon] \to 0$$

as $n, p \to \infty$ with $p \geqq n$. Theorem 3 implies that an amart is a game fairer with time. So clearly is every adapted sequence converging in L^1 (and vice-versa, if X_n is uniformly integrable, as shown in [10]). Therefore there are games fairer with time which are not amarts.

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[1] The authors of [3] wish to use the present occasion to make the following slight correction. On page 1014 the parenthetical expressions "(ou stationnaire)" and "(ou le théorème ergodique)" should be deleted