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Some Inequalities for Randomly Stopped Variables with Applications to Pointwise Convergence

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In this paper, first, we prove some inequalities for randomly stopped variables, which arise naturally in the gambling theory, then we show that a theorem of Chacon and some pointwise convergence theorems, which imply the submartingale convergence theorem, are immediate consequences of these inequalities.

1. Introduction

Throughout this paper $(\Omega, \mathscr{F}, P)$ is a probability space, $\{X_n\}$ a sequence of random variables, $\{\mathscr{F}_n\}$ an increasing sequence of sub σ-fields of $\mathscr{F}$ to which $\{X_n\}$ is adapted, $X^* = \limsup_{n\to\infty} X_n$, and $X_* = \liminf_{n\to\infty} X_n$. We recall that a mapping $\tau\colon \Omega \to N^* \cup \{\infty\} = \{1, 2, \ldots, \infty\}$ is called a stopping time (with respect to $\{\mathscr{F}_n\}$) if $P([\tau < \infty]) = 1$ and $[\tau \leqq n] \in \mathscr{F}_n$ for each $n \in N^*$. We denote by T_1 the set of all stopping times (with respect to $\{\mathscr{F}_n\}$) and by T_0 the set of all bounded stopping times in T_1. For any two stopping times τ and t in T_1, we write $\tau \leqq t$ if $\tau(w) \leqq t(w)$ for all $w \in \Omega$. With this natural partial ordering, we write $\limsup_{\tau \in T_j} E(X_\tau)$ for "$\inf_{s \in T_j} \{\sup_{s \leqq t, t \in T_j} E(X_t)\}$" and $\liminf_{\tau \in T_j} E(X_\tau)$ for "$\sup_{s \in T_j} \{\inf_{s \leqq t, t \in T_j} E(X_t)\}$", $j = 0, 1$.

In Section 2 the following inequalities for randomly stopped variables, which arise naturally in the gambling theory, are proved: (1) if $\{X_n^-\}$ is uniformly integrable then $\limsup_{\tau \in T_0} E(X_\tau) \geqq \limsup_{\tau \in T_1} E(X_\tau) \geqq E(X^*)$ and (2) if $\{X_\tau^+ \mid \tau \in T_0\}$ is uniformly integrable then $\limsup_{\tau \in T_0} E(X_\tau) \leqq \limsup_{\tau \in T_1} E(X_\tau) = E(X^*)$. It is easy to see that Inequality (2) strengthens and generalizes the usual Fatou lemma and the "Fatou equation" for randomly stopped variables obtained by Sudderth [5]. In Section 3 a new and simple proof of a theorem of Chacon [3] will be given as an application of these inequalities. In Section 4 some pointwise convergence theorems, which imply the submartingale convergence theorem, will be shown as immediate consequences of these inequalities. These pointwise convergence theorems are parallel to that studied by Austin, Edgar, and Tulcea in [1], by Baxter in [2].

2. Some Inequalities

We start with the following lemma which is known to Sudderth.

(2.1) **Lemma.** *Let $\{X_n\}$, T_1, X^*, and X_* be as defined above then we have the following inequalities whenever all the expectations occuring in them are well-defined.*

(1) $\limsup_{\tau \in T_1} E(X_\tau) \geqq E(X^*)$.

(2) $\liminf_{\tau \in T_1} E(X_\tau) \leqq E(X_*)$.

Proof. Sudderth [5] proved this lemma by an application of Lévy's martingale convergence theorem, but a constructive proof is possible and we sketch it as follows:

Step 1. Assume that X^* is integrable and construct a sequence $\{Y_n\}$ of integrable random variables with the following properties: (a) $Y_n < X^*$ a.s. for all $n \geqq 1$, (b) $Y_n \to X^*$ a.s. and $E(Y_n) \to E(X^*)$ as $n \to \infty$, and (c) for each $n \geqq 1$, $Y_n = \sum_{i=1}^{\infty} a_i \chi_{A_i}$ where $a_1 > a_2 > \cdots$ and $\{A_j\}$ is a measurable partition of Ω.

Step 2. For each Y in the sequence $\{Y_n\}$ (constructed in Step 1), each $\varepsilon > 0$, and each stopping time τ in T_1, construct a stopping time t in T_1 such that $t \geqq \tau$ and $E(X_t) \geqq E(Y) - \varepsilon$. (For example, for each $k \geqq 1$, let $\varepsilon_k = \varepsilon/[2^k(a_1 - a_{k+1})]$, $\tau_k(w) = \inf\{n \mid n \geqq \max\{\tau(w), N_{k-1}\}$ and $X_n(w) \geqq a_k\}$, $\tau_k(w) = \infty$ if no such n exists, $w \in \Omega$, and let N_k be the smallest positive integer such that $P([N_k \leqq \tau_k < \infty]) < \varepsilon_k$, $N_0 = 1$. If we let $t = \min\{\tau_k \mid k \geqq 1\}$, then $t \geqq \tau$ and $E(X_t) \geqq E(Y) - \varepsilon$.)

Step 3. If $E(X^*) = \infty$, then, for each constant c, let $X_n^c = \min\{X_n, c\}$ for all $n \geqq 1$. Then $\limsup_{\tau \in T_1} E(X_\tau) \geqq \limsup_{\tau \in T_1} E(X_\tau^c) \geqq E\{\limsup_{n \to \infty} X_n^c\}$. Letting $c \to \infty$, we get $\limsup_{\tau \in T_1} E(X_\tau) \geqq E(X^*) = \infty$.

The proof of Lemma (2.1) now is complete since Inequality (1) is obvious if $E(X^*) = -\infty$ and Inequality (2) can be proved by the same argument.

(2.1) **Theorem.** *Let $\{X_n\}$, T_0, T_1, X^* and X_* be as defined above and suppose that $E(X_\tau)$ is well-defined for all $\tau \in T_1$, then*

(3) *If $\{X_n^-\}$ is uniformly integrable then* $\limsup_{\tau \in T_0} E(X_\tau) \geqq \limsup_{\tau \in T_1} E(X_\tau) \geqq E(X^*)$.

(4) *If $\{X_n^+\}$ is uniformly integrable then* $\liminf_{\tau \in T_0} E(X_\tau) \leqq \liminf_{\tau \in T_1} E(X_\tau) \leqq E(X_*)$.

Proof. It suffices to prove (3). Since $\{X_n^-\}$ is uniformly integrable, $E(X^*)$ is well-defined and $E(X^*) > -\infty$. By Lemma 2.1, $\limsup_{\tau \in T_1} E(X_\tau) \geqq E(X^*)$. Hence it is enough to show that $\limsup_{\tau \in T_0} E(X_\tau) \geqq \limsup_{\tau \in T_1} E(X_\tau)$.

First, we assume that $\limsup_{\tau \in T_1} E(X_\tau) = v$ is finite and show that $\limsup_{\tau \in T_0} E(X_\tau) \geqq v$. Since $\sup_{t \leqq \tau,\, \tau \in T_1} E(X_\tau) \geqq \limsup_{\tau \in T_1} E(X_\tau)$ for all stopping times t in T_1, for every $\varepsilon > 0$, we can and do choose a strictly increasing sequence $\{\tau_n\}$ from T_1 such that $E(X_{\tau_n})>v-\varepsilon$ for all $n\geqq 1$. For each positive integer n, let $X_{t_{n,j}}=X_{\tau_n}\chi_{[\tau_n\leqq j]}$ for all $j\geqq 1$. Then $X^+_{t_{n,j}}\uparrow X^+_{\tau_n}$ a.s., $X^-_{t_{n,j}}\uparrow X^-_{\tau_n}$ a.s. as $j\to\infty$, and by the monotone convergence theorem, $E(X^+_{t_{n,j}})\to E(X^+_{\tau_n})$ and $E(X^-_{t_{n,j}})\to E(X^-_{\tau_n})$ as $j\to\infty$. Hence there is a positive integer J_1 such that $E(X_{t_{n,j}})>v-2\varepsilon$ if $j\geqq J_1$. Since $\{X_n^-\}$ is uniformly integrable and $P([\tau_n<j])\to 1$ as $j\to\infty$, there is a positive integer J_2 such that $\sup_k \int_{[\tau_n>j]} X_k^-\,dP<\varepsilon$ for all $j\geqq J_2$. Let $J=\max(J_1,J_2,n)$ and let $t_n=\min(\tau_n,J)$, then $t_n\in T_0$ and

$$E(X_{t_n})=E(X_{t_{n,J}})+E(X_J\chi_{[\tau_n>J]})>v-2\varepsilon-E(X_J^-\chi_{[\tau_n>J]})>v-3\varepsilon.$$

Since $\{\tau_n\}$ is strictly increasing, $t_n\geqq n$ for all $n\geqq 1$ and

$$\limsup_{\tau\in T_0} E(X_\tau)\geqq \liminf_{n\to\infty} E(X_{t_n})\geqq v-3\varepsilon.$$

Since ε is arbitrary, $\limsup_{\tau\in T_0} E(X_\tau)\geqq v$.

Now suppose that $\limsup_{\tau\in T_1} E(X_\tau)=\infty$. Then it is enough to show that $\limsup_{\tau\in T_0} E(X_\tau)\geqq N$ for all positive integers N. Since $\limsup_{\tau\in T_1} E(X_\tau)=\infty$, for each positive integer N, there is a strictly increasing sequence $\{\tau_n\}$ of stopping times in T_1 such that $E(X_{\tau_n})>N+1$ for all $n\geqq 1$. By the same technique used above, we can and do construct a sequence $\{t_n\}$ of bounded stopping times in T_0 such that $t_n\geqq n$ and $E(X_{t_n})>N$ for all $n\geqq 1$. Hence $\limsup_{\tau\in T_0} E(X_\tau)\geqq \liminf_{n\to\infty} E(X_{t_n})\geqq N$. Letting $N\to\infty$, we get $\limsup_{\tau\in T_0} E(X_\tau)=\infty$ and the proof of Theorem 2.1 now is complete.

For Theorem (2.2), we state a lemma which does not seem to have appeared before.

(2.2) **Lemma.** *Let $\{X_n\}$, T_0, and T_1 be as defined above, then the following two assertions are equivalent.*

(5) *$\{X_\tau \mid \tau\in T_0\}$ is uniformly integrable.*

(6) *$\{X_\tau \mid \tau\in T_1\}$ is uniformly integrable.*

Proof. The implication "(6)⇒(5)" is obvious, we only prove the implication "(5)⇒(6)". To prove it, it suffices to show that (a) $\sup_{\tau\in T_1} E(|X_\tau|)<\infty$ and (b) for each $\varepsilon>0$, there is a $\delta>0$ such that $\sup_{\tau\in T_1}\int_A |X_\tau|<\varepsilon$ for all $A\in\mathscr{F}$ and $P(A)<\delta$. Since, for each τ in T_1, there exists a sequence $\{t_k\}$ in T_0 such that $X_{t_k}\to X_\tau$ a.s. as $k\to\infty$ and since $\{X_\tau \mid \tau\in T_0\}$ is uniformly integrable, (a) and (b) hold. So $\{X_\tau \mid \tau\in T_1\}$ is uniformly integrable.

Now we state and prove our second inequality which strengthens and generalizes the usual Fatou lemma and Sudderth's Fatou equation for randomly stopped variables [5]. We replace dominace by an integrable random variable (the assumption for Sudderth's Theorem 2, [5]) with uniform integrability and we also consider the class T_0 of bounded stopping times, which is not considered by Sudderth.

(2.2) **Theorem.** *Let $\{X_n\}$, T_0, T_1, X^*, and X_* be as defined above, then*

(7) *If $\{X_\tau^+ \mid \tau \in T_0\}$ is uniformly integrable, then*
$$\limsup_{\tau \in T_0} E(X_\tau) \leqq \limsup_{\tau \in T_1} E(X_\tau) = E(X^*).$$

(8) *If $\{X_\tau^- \mid \tau \in T_0\}$ is uniformly integrable, then*
$$\liminf_{\tau \in T_0} E(X_\tau) \geqq \liminf_{\tau \in T_1} E(X_\tau) = E(X_*).$$

Proof. It suffices to prove (7).

By Lemma (2.2), the uniform integrability of $\{X_\tau^+ \mid \tau \in T_0\}$ implies that $\{X_\tau^+ \mid \tau \in T_1\}$ is uniformly integrable. Hence all expectations occurring in Inequality (7) exist and are uniformly bounded above. By the uniform integrability of $\{X_\tau^+ \mid \tau \in T_1\}$, for every $\varepsilon > 0$, there is a constant $c > 0$ such that $\int_{[X_\tau^+ > c]} X_\tau^+ \, dP < \varepsilon$ for every τ in T_1. Hence $E(X_\tau) \leqq \int_{[X_\tau^+ \leqq c]} X_\tau \, dP + \varepsilon$ for every τ in T_1. For each $n \geqq 1$, let $X_n^c = \min(X_n, c)$ then $\{X_n^c\}$ is adapted to $\{\mathscr{F}_n\}$ and is uniformly bounded above by c. For each $n \geqq 1$, let $Z_n = \sup_{k \geqq n} X_k^c$, then $Z_n \downarrow \min(X^*, c) \leqq X^*$, and by the monotone convergence theorem $\lim_{n \to \infty} E(Z_n) = E\{\min(X^*, c)\} \leqq E(X^*)$. Also $X_t^c \leqq Z_n$ for all $t \geqq n$, hence $E(X_t) \leqq E(X_t^c) + \varepsilon \leqq E(Z_n) + \varepsilon$ for all $t \geqq n$. Therefore $\limsup_{\tau \in T_0} E(X_\tau) \leqq \lim_{n \to \infty} \{E(Z_n) + \varepsilon\} \leqq E(X^*) + \varepsilon$ and $\limsup_{\tau \in T_1} E(X_\tau) \leqq \lim_{n \to \infty} \{E(Z_n) + \varepsilon\} \leqq E(X^*) + \varepsilon$. Since ε is arbitrary, $\limsup_{\tau \in T_0} E(X_\tau) \leqq E(X^*)$ and $\limsup_{\tau \in T_1} E(X_\tau) \leqq E(X^*)$. By Lemma (2.1), $\limsup_{\tau \in T_1} E(X_\tau) \geqq E(X^*)$. Hence $\limsup_{\tau \in T_0} E(X_\tau) \leqq \limsup_{\tau \in T_1} E(X_\tau) = E(X^*)$ and the proof of Theorem (2.2) now is complete.

Remarks. 1. Since $\limsup_{n \to \infty} E(X_n) \leqq \limsup_{\tau \in T_0} E(X_\tau)$ and $\liminf_{n \to \infty} E(X_n) \geqq \liminf_{\tau \in T_0} E(X_\tau)$ whenever all the expectations occurring in them are well-defined, Theorem (2.2) implies the usual Fatou lemma.

2. It is worth noting that the sufficient condition for Theorem (2.2) is much stronger than that for Theorem (2.1). Example 1 below shows that the inequality of Theorem (2.2) is false even if $\{X_n\}$ is uniformly integrable. Moreover, in general, there is no inequality related to "$\limsup_{\tau \in T_0} E(X_\tau)$" ($\liminf_{\tau \in T_0} E(X_\tau)$) and "$\limsup_{\tau \in T_1} E(X_\tau)$" ($\liminf_{\tau \in T_1} E(X_\tau)$) (see Examples 1 and 2 below).

Example 1. Modified double or nothing. Let $V_1, V_2, \ldots$ be i.i.d. random variables such that $P(V_1 = 0) = P(V_1 = 1) = \frac{1}{2}$, $Y_n = 2^n V_1 V_2 \ldots V_n$ for all $n \geqq 1$, and let $Y_0 = 1$. For each $n \geqq 1$, let $j(n)$ be the largest integer k such that $2^k \leqq n$ and let $Z_n = Y_{j(n)}$. Let $\{U_n\}$ be a sequence of random variables such that $P(U_n = 1) = 2^{-k}$, $P(U_n = 0) = 1 - 2^{-k}$ for $2^k \leqq n < 2^{k+1}$, $\sum_{2k \leqq n < 2k+1} U_n = 1$, and $\{U_n\}$ is independent of $\{Z_n\}$. Now let $X_n = U_n Z_n$ for all $n \geqq 1$, then $\{X_n\}$ is uniformly integrable and $\{X_\tau^+ \mid \tau \in T_0\}$ is not. Moreover, for each $k \geqq 1$, if we let $t = n$ on the set $[X_n > 0]$ for $2^k \leqq n < 2^{k+1}$ and let $t = 2^{k+1} - 1$ otherwise. Then it is easy to see that t is a bounded stopping time in T_0 such that $E(X_t) = 1$. Hence $\limsup_{\tau \in T_0} E(X_\tau) \geqq 1$. Now let τ be the stopping time defined by $\tau(w) = 2^m - 1$ if $m = \inf\{k \mid X_n(w) = 0$ for all

n such that $2^{k-1} \leqq n < 2^k\}$ and $\tau(w) = \infty$ if no such m exists. It is easy to see that τ is in T_1 and for any s in T_1, $\tau \leqq s$, then $E(X_s) = 0$. Therefore $\limsup_{\tau \in T_0} E(X_\tau) \geqq 1 > 0 = \limsup_{\tau \in T_1} E(X_\tau)$.

Example 2. Negative double or nothing. Let $\{V_n\}$ be as defined in Example 1 and let $X_n = -2^n V_1 V_2 \ldots V_n$ for all $n \geqq 1$. It is obvious that $\{X_\tau^+ | \tau \in T_0\}$ is uniformly integrable, hence, by Theorem (2.2), $\limsup_{\tau \in T_1} E(X_\tau) = E(\limsup_{n \to \infty} X_n) = 0$. But, by the optional sampling theorem, $E(X_\tau) = E(X_1) = -1$ for all bounded stopping times τ in T_0. Hence $\limsup_{\tau \in T_0} E(X_\tau) = -1 < \limsup_{\tau \in T_1} E(X_\tau) = E(\limsup_{n \to \infty} X_n) = 0$.

Combining Theorems (2.1) and (2.2), we have the following equalities for randomly stopped variables.

(2.3) **Theorem.** *Let $\{X_n\}$, T_0, T_1, X^*, and X_* be as defined above, then*

(9) *If both of $\{X_n^-\}$ and $\{X_\tau^+ | \tau \in T_0\}$ are uniformly integrable then*

$$\limsup_{\tau \in T_0} E(X_\tau) = \limsup_{\tau \in T_1} E(X_\tau) = E(X^*).$$

(10) *If both of $\{X_n^+\}$ and $\{X_\tau^- | \tau \in T_0\}$ are uniformly integrable then*

$$\liminf_{\tau \in T_0} E(X_\tau) = \liminf_{\tau \in T_1} E(X_\tau) = E(X_*).$$

3. A Simple Proof of a Theorem of Chacon

In this section, a simple proof of the result stated as Theorem (3.1), due to Chacon [3], will be given as an application of the inequalities for the class T_0 in Section 2. A pointwise convergence theorem and the submartingale convergence theorem follow immediately from this theorem

(3.1) **Theorem** (Chacon). *Let $\{X_n\}$, T_0, X^*, and X_* be as defined above. Suppose that $E(|X_n|) < \infty$ for all $n \geqq 1$ and $\liminf_{n \to \infty} E(|X_n|) < \infty$. Then*

(11) $$\limsup_{\tau, t \in T_0} E(X_\tau - X_t) \geqq E(X^* - X_*).$$

Furthermore, if there is a constant M such that $\sup_{\tau \in T_0} E(|X_\tau|) \leqq M$ then X^* and X_* are integrable.

Proof. By Lemma 1 of [1] and the Borel-Cantelli lemma, we can choose two strictly increasing sequences $\{\tau_k\}$ and $\{t_k\}$ of bounded stopping times such that $X_{\tau_k} \to X^*$ a.s. and $X_{t_k} \to X_*$ a.s. as $k \to \infty$. Hence the second assertion follows immediately from Fatou's lemma and we need only prove (11). To prove (11), it suffices to show that:

(12) $$\sup_{\tau, t \in T_0} E(X_\tau - X_t) \geqq E(X^* - X_*).$$

It is also easy to see that, under the assumption of the theorem, if $\sup_{\tau \in T_0} E(|X_\tau|) = \infty$ then $\sup_{\tau, t \in T_0} E(X_\tau - X_t) = \infty$. Hence we can and do assume that $\sup_{\tau \in T_0} E(|X_\tau|) < \infty$ and by the previous argument, X^* and X_* are integrable.

Let λ be a positive constant, define an incomplete stopping time σ as follows: $\sigma(w)=\inf\{n \mid |X_n(w)| \geqq a\}$, $\sigma(w)=\infty$ if no such n exists, $w \in \Omega$. Let $A=[\sigma<\infty]$, then $X_{n\wedge\sigma}\chi_A \to X_\sigma \chi_A$ as $n\to\infty$. Hence, by the usual Fatou lemma and the fact that $\tau_n = n\wedge\sigma$ is in T_0 for each $n \geqq 1$, $E(|X_\sigma \chi_A|) \leqq \liminf_{n\to\infty} E(|X_{\tau_n}\chi_A|) \leqq \sup_{\tau\in T_0} E(|X_\tau|) < \infty$. Now, let $Y=|X_\sigma\chi_A| + \lambda\chi_{A^c}$ and $Y_n = X_{n\wedge\sigma}$ for all $n\geqq 1$, then $E(Y)<\infty$ and $|Y_n| \leqq Y$ for all $n\geqq 1$. By Theorem (2.3), $\limsup_{\tau\in T_0} E(Y_\tau) = E(Y^*)$ and $\liminf_{\tau\in T_0} E(Y_\tau) = E(Y_*)$, where $Y^* = \limsup_{n\to\infty} Y_n$ and $Y_* = \liminf_{n\to\infty} Y_n$. So $\sup_{\tau, t\in T_0} E(Y_\tau - Y_t) \geqq E(Y^* - Y_*)$. Since the set $\{X_{\tau\wedge\sigma} \mid \tau\in T_0\} = \{Y_\tau \mid \tau \in T_0\}$ is a subset of $\{X_\tau \mid \tau\in T_0\}$, $\sup_{\tau, t\in T_0} E(X_\tau - X_t) \geqq E(Y^* - Y_*)$. Letting $\lambda\to\infty$, we get $\sup_{\tau, t\in T_0} E(X_\tau - X_t) \geqq E(X^* - X_*)$ and the proof of Theorem (3.1) now is complete.

(3.1) **Corollary** (Theorem 2 of [1]). *Suppose that* $\{X_n\}$, T_0 *are as defined above and suppose that* $\liminf_{n\to\infty} E(|X_n|) < \infty$. *Consider the following two assertions.*

(i) *The generalized sequence* $\{E(X_\tau) \mid \tau \in T_0\}$ *converges.*

(ii) X_n *converges pointwise almost surely on* Ω.

Then (i) $\Rightarrow$ (ii).

(3.2) **Corollary** (the Submartingale Convergence Theorem). *Suppose that* $\{X_n\}$ *is a sequence of* L_1*-bounded random variables adapted to the increasing sequence* $\{\mathscr{F}_n\}$ *of* σ*-fields. Suppose that* $E(X_{n+1} \mid \mathscr{F}_n) \geqq X_n$ *a.s. for all* $n \geqq 1$. *Then* X_n *converges almost surely to a finite limit.*

Remark. Theorem (3.1) and Corollary (3.1) also hold under any one of the following two conditions.

(i) $\sup E(X_n^+) < \infty$.

(ii) $\sup E(X_n^-) < \infty$.

4. Pointwise Convergence

The connection between almost sure convergence of a sequence of random variables and convergence of certain related expectations has been studied in [1, 2] and [4]. In this section we will give three criteria for almost sure convergence of a sequence of random variables. Finally, we will extend the main theorem of [2] as an application of the inequalities in Section 2.

(4.1) **Theorem.** *Let* $\{X_n\}$, T_0, T_1, X^*, *and* X_* *be as defined above. Suppose that* Y *is a non-negative integrable random variable such that* $|X_n| \leqq Y$ *for all* $n \geqq 1$. *Then the following three statements are equivalent*

(i) X_n *converges a.s. to a finite limit.*

(ii) *The generalized sequence* $\{E(X_\tau) \mid \tau\in T_0\}$ *is convergent.*

(iii) *The generalized sequence* $\{E(X_\tau) \mid \tau\in T_1\}$ *is convergent.*

Proof. By Theorem (2.3), $\limsup_{\tau\in T_0} E(X_\tau) = \limsup_{\tau\in T_1} E(X_\tau) = E(X^*)$ and $\liminf_{\tau\in T_0} E(X_\tau) = \liminf_{\tau\in T_1} E(X_\tau) = E(X_*)$. So the theorem now is obvious.

Remark. Theorem (4.1) is an extension of Corollary 1 in [1] and Sudderth's Example [5] is related to Theorem (4.1).

For each positive constant c and each random variable Y, let $Y^c = c$ if $Y > c$, $Y^c = -c$ if $Y < -c$, and $Y^c = Y$ if $-c \leqq Y \leqq c$.

(4.2) **Theorem.** *Let* $\{X_n\}$, T_0, T_1, X^*, *and* X_* *be as defined above. Suppose that* $\liminf_{n\to\infty} E(|X_n|) < \infty$. *Then the following three statements are equivalent.*

(i) X_n *converges almost surely to a finite limit.*

(ii) $\lim_{c\to\infty}\{\lim_{\tau\to\infty,\,\tau\in T_0}\{\sup_{t,\sigma\geqq\tau;\,t,\sigma\in T_0} E(X_t^c - X_\sigma^c)\}\} = 0.$

(iii) $\lim_{c\to\infty}\{\lim_{\tau\to\infty,\,\tau\in T_1}\{\sup_{t,\sigma\geqq\tau;\,t,\sigma\in T_1} E(X_t^c - X_\sigma^c)\}\} = 0.$

Proof. Since, for each positive constant c, $\limsup_{\tau\in T_0} E(X_\tau^c) = \limsup_{\tau\in T_1} E(X_\tau^c) = E(\limsup_{n\to\infty} X_n^c)$ and $\liminf_{\tau\in T_0} E(X_\tau^c) = \liminf_{\tau\in T_1} E(X_\tau^c) = E(\liminf_{n\to\infty} X_n^c)$. Hence

$$\lim_{\tau\to\infty,\,\tau\in T_0}\{\sup_{t,\sigma\geqq\tau;\,t,\sigma\in T_0} E(X_t^c - X_\sigma^c)\} = \lim_{\tau\to\infty,\,\tau\in T_1}\{\sup_{t,\sigma\geqq\tau;\,t,\sigma\in T_1} E(X_t^c - X_\sigma^c)\}$$
$$= E\{\limsup_{n\to\infty} X_n^c - \liminf_{n\to\infty} X_n^c\}$$

for all $c > 0$. But it is easy to see that $\{\limsup_{n\to\infty} X_n^c - \liminf_{n\to\infty} X_n^c\} \uparrow (X^* - X_*)$ as $c\to\infty$. By the monotone convergence theorem,

$$\lim_{c\to\infty}\{\lim_{\tau\to\infty,\,\tau\in T_0}\{\sup_{t,\sigma\geqq\tau;\,t,\sigma\in T_0} E(X_t^c - X_\sigma^c)\}\}$$
$$= \lim_{c\to\infty}\{\lim_{\tau\to\infty,\,\tau\in T_1}\{\sup_{t,\sigma\geqq\tau;\,t,\sigma\in T_1} E(X_t^c - X_\sigma^c)\}\} = E(X^* - X_*).$$

The implications (i)$\Rightarrow$(ii) and (ii)$\Rightarrow$(iii) now are abvious. But if we have (iii), then $E(X^* - X_*) = 0$, i.e., $P(X^* \neq X_*) = 0$. By the usual Fatou lemma, we have $E(\liminf_{n\to\infty} |X_n|) \leqq \liminf_{n\to\infty} E(|X_n|) < \infty$. So X_n converges almost surely to a finite limit and the proof of Theorem (4.2) now is complete.

Remarks. 1. Theorem (4.2) is conceptually interesting because, almost sure convergence is shown to be equivalent to a certain kind of first moment convergence, with only the mild regularity condition $\liminf_{n\to\infty} E(|X_n|) < \infty$ assured.

2. Theorem (4.2) is similar to Theorem 2 of [1] since both of them are concerned the pointwise convergence in terms of expectations. But we consider the classes T_0 and T_1, and truncation moments in Theorem (4.2) and Austin, Edgar, and Tulcea only consider the class T_0 and first moments in Theorem 2 of [1].

Let η_1 be the class of all functions ϕ such that $\phi(x) = c$ if $x \leqq a$, $\phi(x) = d$ if $x \geqq b$, and $\phi(x) = c + [(x-a)(d-c)/(b-a)]$ if $a < x < b$, where a, b, c, d are finite constants and $a < b$, $c < d$.

(4.3) **Theorem.** *Let* $\{X_n\}$, T_0, T_1, X^*, *and* X_* *be as defined above and suppose that* $\liminf_{n\to\infty} E(|X_n|) < \infty$, *then the following three statements are equivalent.*

(i) X_n *converges a.s. to a finite limit.*

(ii) *For all* ϕ *in* η_1 *the generalized sequence* $\{E(\phi(X_\tau)) \mid \tau \in T_0\}$ *is convergent.*

(iii) *For all* ϕ *in* η_1 *the generalized sequence* $\{E(\phi(X_\tau)) \mid \tau \in T_1\}$ *is convergent.*

Proof. Since each ϕ in η_1 is a non-decreasing function, $\limsup_{n\to\infty} \phi(X_n) = \phi(\limsup_{n\to\infty} X_n)$ and $\liminf_{n\to\infty} \phi(X_n) = \phi(\liminf_{n\to\infty} X_n)$. Since, for all ϕ in η_1, $\{\phi(X_n)\}$ is uniformly bounded, by Theorem (2.3), $\limsup_{\tau\in T_0} E\{\phi(X_\tau)\} = \limsup_{\tau\in T_1} E\{\phi(X_\tau)\} = E\{\limsup_{n\to\infty} \phi(X_n)\}$ and $\liminf_{\tau\in T_0} E\{\phi(X_\tau)\} = \liminf_{\tau\in T_1} E\{\phi(X_\tau)\} = E\{\liminf_{n\to\infty} \phi(X_n)\}$ for all ϕ in η_1. The implications (i)$\Rightarrow$(ii) and (ii)$\Rightarrow$(iii) now are obvious. Now we show that (iii)$\Rightarrow$(i). By the proceeding argument, we get $E\{\phi(X^*)\} = E\{\limsup_{n\to\infty} \phi(X_n)\} = \limsup_{\tau\in T_1} E\{\phi(X_\tau)\}$ and $E\{\phi(X_*)\} = E\{\liminf_{n\to\infty} \phi(X_n)\} = \liminf_{\tau\in T_1} E\{\phi(X_\tau)\}$ for all ϕ in η_1. By (iii), we get $E\{\phi(X^*)\} = E\{\phi(X_*)\}$ for all ϕ in η_1. Since η_1 is a separating class, we get $P\{X^* \neq X_*\} = 0$. Now, by the usual Fatou lemma, X_n converges a.s. to a finite limit and the proof of Theorem (4.3) now is complete.

In the remainder of this section, let $(\Omega, \mathscr{F}, P)$ be a probability space, $\{\mathscr{F}_n\}$ an increasing sequence of sub σ-fields of $\mathscr{F}$, S a compact metric space, and $\{X_n\}$ a sequence of S-valued random variables adapted to $\{\mathscr{F}_n\}$. Let δ be the metric on S and $C(S)$ be the set of all real-valued continuous functions defined on S. The following is another application of the inequalities in Section 2 to the pointwise convergence which is a generalization of the main theorem of [2].

(4.4) **Theorem.** *The following three statements are equivalent:*

(i) *X_n converges pointwise almost surely on Ω.*

(ii) *For all φ in $C(S)$, the generalized sequence $\{E(\varphi(X_\tau)) \mid \tau\in T_0\}$ is convergent.*

(iii) *For all φ in $C(S)$, the generalized sequence $\{E(\varphi(X_\tau)) \mid \tau\in T_1\}$ is convergent.*

Remark. The equivalence of (i) *and* (ii) *is Baxter's main result and we extend it to the class T_1.*

Proof. Since S is compact, for each φ in $C(S)$, $\{\varphi(X_n)\}$ is uniformly bounded. By Theorem (2.3), for each φ in $C(S)$, $\limsup_{\tau\in T_0} E\{\varphi(X_\tau)\} = \limsup_{\tau\in T_1} E\{\varphi(X_\tau)\} = E\{\limsup_{n\to\infty} \varphi(X_n)\}$ and $\liminf_{\tau\in T_0} E\{\varphi(X_\tau)\} = \liminf_{\tau\in T_1} E\{\varphi(X_\tau)\} = E\{\liminf_{n\to\infty} \varphi(X_n)\}$. Hence the implications (i)$\Rightarrow$(ii) and (ii)$\Rightarrow$(iii) are obvious. Now we show that (iii)$\Rightarrow$(i). By the proceeding argument and (iii), for each φ in $C(S)$, $\varphi(X_n)$ converges pointwise almost surely on Ω. Now, choose a dense set of points $\{z_j\}$ in S and define φ_j in $C(S)$ by $\varphi_j(x) = \delta(x, z_j)$ for each $j \geqq 1$. Then there is a set $\Omega_1 \subseteq \Omega$ such that $P(\Omega_1) = 1$ and $\varphi_j(X_n(w))$ converges as $n\to\infty$ for each $w\in\Omega_1$ and for all $j \geqq 1$. By the compactness of S, it follows easily that $X_n(w)$ converges for each $w\in\Omega_1$ and the proof of Theorem (4.4) now is complete.

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