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# Differentiability Preserving Properties of a Class of Semigroups*

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It is well known that for a large class of Markov process the associated semigroup $T(t)f(x)=\int f(y)\,P(t,x;dy)$ satisfies the Kolmogorov backward differential equation, that is, if $U(t,x)=T(t)f(x)$ then $\frac{\partial U}{\partial t}=\frac{1}{2}a(x)\frac{\partial^2 U}{\partial x^2}+b(x)\frac{\partial U}{\partial x}$ and $\lim_{t\downarrow 0} U(t,x)=U(0,x)=f(x)$.

In this paper we are considering the opposite problem: given the diffusion and drift coefficients we study the differentiability preserving properties of the semigroup $T(t)$ having as infinitesimal generator $A=\frac{1}{2}a(x)\frac{\partial^2}{\partial x^2}+b(x)\frac{\partial}{\partial x}$. More specifically, for a large class of functions $a(x)$ and $b(x)$, we will prove for $k=0,\ldots,3$ the existence of $T(t)$ such that $T(t)\colon C^k(I)\to C^k(I)$ and the existence of a constant $\mu_k$ such that $|T(t)f|_k\leqq|f|_k\exp(\mu_k t)$ for $f\in C^k(I)$. Moreover an explicit expression of $\mu_k$ in terms of the coefficients $a(x)$ and $b(x)$ is obtained. As a side result we obtain the necessity of the boundary conditions imposed.

## 0. Introduction

Given functions $a(x)$ and $b(x)$ the problem of constructing a process $X(t)$ having $a(x)$ and $b(x)$ as diffusion and drift coefficients respectively has attracted much attention in recent years. The question of existence has been completely studied by Feller ([6, 7]), Dynkin [5] and Mandl [9]. The more delicate question of their importance to the study of diffusion approximations in genetics and numerical analysis, and in a general manner the question of their independent mathematical interest have been raised by Borovkov [1], Norman [10] and

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Skorohod [11]. Specifically one wants to know to what extent the semigroup of operators $T(t)$ associated with the process $X(t)$ preserves the differentiability of the initial data. Special results were obtained by Brezis-Rosenkrantz-Singer [2], general results were obtained by Norman [10], who also raised the question of obtaining his theorems via semigroup method. By using a semigroup approach and under different conditions than those of Norman's (however including most of the examples of application), we obtain a slightly stronger result. Let $I$ be a closed and bounded interval, $|f|=\sup_{x\in I}|f(x)|$, $|f|_k=\sum_{i=0}^{k}|f^{(i)}|$ and $C^k(I)=\{f: f^{(i)}$ continuous in $I$ for $i=0,\dots,k\}$. For $k=0,1,2,3$ we show the existence of $T(t)$ and of a constant $\mu_k$ such that $T(t)$: $C^k(I)\to C^k(I)$ and $|T(t)f|_k \leqq |f|_k \exp(\mu_k t)$ for $f\in C^k(I)$. Moreover an explicit expression of $\mu_k$ in terms of the coefficients $a(x)$ and $b(x)$ will be given, and as a side result we obtain the necessity of the boundary conditions imposed. For further applications of estimates such as $|T(t)f|_3 \leqq |f|_3 \exp(\mu_3 t)$ we refer the reader to Brezis-Rosenkrantz-Singer [3], Lax-Richtmyer [8] and Trotter [12].

We now present a brief sketch of the approach used. Let the linear operator $A$ be defined by the relation $Af(x)=\lim_{t\downarrow 0} t^{-1}(T(t)f(x)-f(x))$ provided the limit on the right-hand side exists, $A$ is called the *infinitesimal generator* of the semigroup $T(t)$. Comparing the definition of $A$ with the Kolmogorov backward equation $\frac{\partial U}{\partial t}=\frac{1}{2}a(x)\frac{\partial^2 U}{\partial x^2}+b(x)\frac{\partial U}{\partial x}$ where $U(t,x)=T(t)f(x)$ and $f\in C^2(I)$, we conclude that $A=\frac{a(x)}{2}\frac{d^2}{dx^2}+b(x)\frac{d}{dx}$. As examples of generators $A$ that occur in application we have

$$A=cx(1-x)\frac{d^2}{dx^2}+m(\bar{x}-x)\frac{d^7}{dx} \quad \text{on } [0,1]$$

with $c>0$, $m\geqq 0$, $0<\bar{x}<1$.

$$A=Vx^2(1-x)^2\frac{d^2}{dx^2}+x(1-x)\frac{d}{dx} \quad \text{on } [0,1]$$

with $V>0$.

The idea is to show that for $f\in C^k(I)$ and $\lambda>\mu_k$ (where $\mu_k$ is a prefixed constant) there exists a unique $F(\lambda,x)\in\mathscr{D}(A)$ satisfying $\lambda F(\lambda,x)-AF(\lambda,x)=f(x)$ and $(\lambda-\mu_k)|F(\lambda)|_k\leqq|f|_k$. Where $\mathscr{D}(A)$, the domain of operator $A$, will be determined by the boundary conditions (to be properly defined in Section 1 and the necessity of these conditions will be proved in Section 3). In addition one must show that $\mathscr{D}(A)$ is dense in $C^k(I)$ with respect to $|\cdot|_k$. These facts, once established, yield, as a consequence of the Hille-Yosida theorem, the existence of the semigroup $T(t)$ with infinitesimal generator $A$, moreover, we obtain the desired estimate $|T(t)f|_k\leqq|f|_k\exp(\mu_k t)$. We remark here that the basic ideas of our proofs were taken from [2].

In Section 1 preliminary results on the boundary behavior of $F(\lambda)$ is obtained. In Section 2 the differentiability preserving properties of the resolvent $(\lambda-A)^{-1}$

is studied ($(\lambda - A)^{-1}$ is such that $F(\lambda) = (\lambda - A)^{-1} f$). The maximum principle will be heavily used and growth functions will be introduced as a tool to obtain the results. In Section 3 the main theorems are stated and proved.

## 1. Preliminary Results on the Boundary Behavior

As a general reference for this section see Mandl [9].

Let $I = [r_0, r_1]$ with $-\infty < r_0 < 0 < r_1 < +\infty$, let $a(x)$ and $b(x)$ be continuous functions defined on $I$ with $a(x) > 0$ on $(r_0, r_1)$. For $A = \frac{a(x)}{2}\frac{d^2}{dx^2} + b(x)\frac{d}{dx}$ we are concerned with the differentiability properties of the semigroup $T(t)$, this can be done by studying the resolvent $(\lambda - A)^{-1}$ which in turn reduces to the analysis of the solution $F(\lambda)$ of the differential equation (1) with initial data $f$ and $\lambda > 0$

$$\lambda F(\lambda) - A F(\lambda) = f. \tag{1}$$

A detailed analysis of Equation (1) is carried out in Mandl, in order to use his results we will express $A$ as "Feller differential operator" $D_m D_p^+$.

$$A = \frac{a(x)}{2}\frac{d^2}{dx^2} + b(x)\frac{d}{dx} = \frac{a(x)}{2} e^{-B(x)} \frac{d}{dx}\left(e^{B(x)} \frac{d}{dx}\right) = D_m D_p^+$$

where for $x \in (r_0, r_1)$ we have:

$$B(x) = \int_0^x 2 b(y)\, a(y)^{-1}\, dy$$

$$m(x) = \int_0^x 2 a(y)^{-1} e^{B(y)}\, dy; \qquad p(x) = \int_0^x e^{-B(y)}\, dy \tag{2}$$

$$D_p^+ f(x) = \lim_{y \to x^+} \frac{f(y) - f(x)}{p(y) - p(x)}$$

$$D_m f(x) = \lim_{y \to x} \frac{f(y) - f(x)}{m(y) - m(x)}$$

provided the limit exists.

*Remark 1.* There is no loss of generality in assuming $0 \in (r_0, r_1)$, if not take $x_0 \in (r_0, r_1)$ and let $x_0$ play the role of 0. Whenever defined we can write $D_p^+ f(x) = e^{B(x)} f^{(1)}(x)$.

Now Equation (1) can be written as:

$$\lambda F(\lambda) - D_m D_p^+ F(\lambda) = f. \tag{3}$$

The general solution of the non-homogeneous Equation (3) is given by (for details see Mandl)

$$F(\lambda, x) = F_0(x) + c_+ u_+(x) + c_- u_-(x) \tag{4}$$

where $F_0(x)$ constitutes a solution of (3), $u_+(x)$ and $u_-(x)$ constitute a fundamental system of solutions of $\lambda w - D_m D_p^+ w = 0$ and $c_+$ and $c_-$ are constants.

*Remark 2.* (a) To simplify the notation we sometimes omit the dependence on $\lambda$ in our symbolisms by writting $F = F(\lambda)$, $u_+(x) = u_+(\lambda, x)$, etc.

(b) Let $u^1(x) = \int_0^x m(s)\,dp(s)$ and $v^1(x) = \int_0^x p(s)\,dm(s)$. It can be shown that if $u^1(r_0) = \infty$ then $u_+(x) \notin C(I) = C^0(I)$, hence the general solution of (3) is of the form $F = F_0 + c_- u_-$ with $c_-$ possibly zero and no boundary conditions can be imposed at $r_0$. Similarly if $u^1(r_1) = \infty$ then $u_-(x) \notin C(I)$ and $F = F_0 + c_+ u_+$ with no boundary conditions at $r_1$. Finally if $u^1(r_i) = \infty$ for $i = 0, 1$ then we have $F = F_0$. This leads us to a classification of the boundary points $r_i$ as regular, entrance, exit and natural (Mandl pp. 24–25).

Boundary conditions will be imposed in case of regular or exit boundary to ensure uniqueness of solution $F(\lambda)$. Our conditions will be

$$\begin{aligned} D_p^+ F(r_i) &= 0 \quad \text{for } r_i \text{ regular} \\ D_m D_p^+ F(r_i) &= 0 \quad \text{for } r_i \text{ exit.} \end{aligned} \tag{5}$$

*Remark 3.* (a) We shall denote by B.C. the boundary conditions (5). The condition $D_m D_p^+ F(r_i) = 0$ means adhesive boundary, that is, the process remains in the point $r_i$ after having reached it. The condition $D_p^+ F(r_i) = 0$ means reflecting boundary.

We now prove a few results on the boundary behavior of $F(\lambda)$. The proofs of Proposition 1 and 2 will be omitted, they are straight forward (for details see [4]).

**Proposition 1.** *Let $a, b \in C^k(I)$ and $a(x) > 0$ on $(r_0, r_1)$ then*

(a) *$B, m, p \in C^{k+1}(r_0, r_1)$.*
(b) *$u_+, u_- \in C^{k+2}(r_0, r_1)$.*
(c) *if $F$ is a solution of $\lambda F - D_m D_p^+ F = f$ and $f \in C^k(I)$ then $F \in C^{k+2}(r_0, r_1)$.*

**Proposition 2.** *Let $a, b, f \in C(I)$ with $a(x) > 0$ on $(r_0, r_1)$ and suppose $F$ satisfies B.C. with $\lambda F - D_m D_p^+ F = f$. Then*

(a) *$D_m D_p^+ F(r_i) = 0$ if $r_i$ natural or exit.*
(b) *$D_p^+ F(r_i) = 0$ if $r_i$ entrance or regular or natural with $m(r_i)$ finite.*

*Let Condition $k$ $(k \geqq 1)$ be:*

(a 0) *$I = [r_0, r_1]$, $-\infty < r_0 < r_1 < +\infty$.*
(a 1) *$a, b \in C^k(I)$.*
(a 2) *$a(x) > 0$ on $(r_0, r_1)$ and $a(r_i) = 0$ for $i = 0, 1$.*
(a 3) *$(-1)^i\, b(r_i) \geqq 0$ for $i = 0, 1$.*
(a 4) *$B(x)$ converges as $x \to r_i$. The limit, $B(r_i)$, may be finite or infinite.*
(a 5) *$\liminf_{x \to r_i} b(x)\, a^{(1)}(x)/a(x) > -\infty$ for $i = 0, 1$.*

*Remark 4.* (a) Conditions (a 4) and (a 5) are satisfied if $a^{(1)}(r_i) \equiv 0$. For example, if $i = 1$, $b(x)/a(x) \to k_1 < \infty$ as $x \to r_i$, and $a^{(1)}(x) \to a^{(1)}(r_1) < 0$. Condition (a 4) holds if $b(x) \geqq 0$ near $r_i$ (or $b(x) \leqq 0$ near $r_i$), since $B(x)$ is then monotonic near $r_i$.

(b) Condition (a 5) holds if $a(x) = \sum_{n=k}^{\infty} a_n (x - r_i)^n$ $(k \geqq 1, a_k \neq 0)$ is analytic at $r_i$.

For $a(x) \sim a_k(x-r_i)^k$ and $a^{(1)}(x) \sim k a_k (x-r_i)^{k-1}$, so $a^{(1)}(x)/a(x) \sim k/(x-r_i)$ as $x \to r_i$. If $(-1)^i b(r_i) > 0$ then $b(x)\, a^{(1)}(x)/a(x) \to \infty$, and if $b(r_i)=0$, $b(x)\, a^{(1)}(x)/a(x) \to k b^{(1)}(r_i)$ as $x \to r_i$.

(c) The examples mentioned in the introduction satisfy the *Condition k*.

**Proposition 3.** *If Condition 1 holds then:*

(a) *If $r_i$ is regular or entrance or natural with $m(r_i)$ finite then* $\lim_{x \to r_i} e^{B(x)} = 0$.

(b) *If $r_i$ is exit or natural with $|m(r_i)| = \infty$ then $b(r_i)=0$.*

*Proof.* Let's assume $r_i = r_1$.

(a) Since $a(r_1)=0$, in a neighborhood of $r_1$ we have (say $x \geqq \xi$):

$$a(x) = a^{(1)}(r_1)(x-r_1) + o(x-r_1)$$

and $0 < a(x) \leqq |a^{(1)}|(r_1 - x)$. Since $|a^{(1)}| > 0$ we have:

$$\int_0^x \frac{2}{a(y)}\, dy \geqq \int_0^\xi \frac{2}{a(y)}\, dy + \int_\xi^x \frac{2}{|a^{(1)}|(r_1 - y)}\, dy \to \infty \qquad \text{as } x \to r_1$$

hence $\int_0^x \frac{2}{a(y)}\, dy \to \infty$ as $x \to r_1$ and we must have $\lim_{x \to r_1} e^{B(x)} = 0$ since

$$m(r_1) = \int_0^{r_1} \frac{2}{a(y)}\, e^{B(y)}\, dy$$

is finite ($m(r_1) < \infty$ by assumption when $r_1$ is natural and $m(r_1) < \infty$ when $r_1$ is entrance or regular (Mandl, p. 25)).

(b) By assumption $m(r_1) = \infty$ when $r_1$ is natural and $m(r_1) = \infty$ when $r_1$ is exit (Mandl, p. 25), now since $b(r_1) \leqq 0$ enough to show if $b(r_1) < 0$ then $m(r_1)$ is finite. Let's assume $b(r_1) < 0$, then there exists $\xi < r_1$ and $\alpha < 0$ such that $b(x) \leqq \alpha$ if $x \geqq \xi$. Now $e^{B(x)}$ is decreasing for $x \geqq \xi$ since $B^{(1)}(x) = 2b(x)a(x)^{-1} < 0$ for $x \geqq \xi$, hence $\frac{d e^{B(x)}}{dx} = \frac{2b(x)}{a(x)}\, e^{B(x)}$ is integrable over $[\xi, r_1)$. Therefore $m^{(1)}(x) = \frac{1}{b(x)} \frac{d}{dx} e^{B(x)}$ is integrable over $[\xi, r_1)$ since $b(x) < 0$ on $[\xi, r_1]$. So that

$$m(r_1) = \int_0^\xi m^{(1)}(x)\, dx + \int_\xi^{r_1} \frac{1}{b(x)} \frac{d}{dx} e^{B(x)}\, dx < \infty. \quad \square$$

**Proposition 4.** *Let $f \in C^1(I)$, $r_i$ be an exit or natural boundary with $|m(r_i)| = \infty$, $a^{(1)}(r_i) \neq 0$ and Condition 1 holds then $\lim_{x \to r_i} F^{(2)}(x)$ exists and finite provided $F$ is such that $F \in C^1(I)$, satisfies the B.C. and $\lambda F - D_m D_p^+ F = f$.*

*Proof.* By hypothesis $a(x) > 0$ on $(r_0, r_1)$ and $\lambda F - \frac{a}{2} F^{(2)} - b F^{(1)} = f$ on $(r_0, r_1)$, so we have on $(r_0, r_1)$

$$F^{(2)}(x) = \frac{2}{a(x)} (\lambda F(x) - f(x)) - \frac{2b(x)}{a(x)} F^{(1)}(x).$$

Now $\lim_{x\to r_i} \frac{2b(x)}{a(x)} F^{(1)}(x)$ exists and finite since $F\in C^1(I)$ and $\lim_{x\to r_i} b(x)/a(x)$ is finite for Proposition 3 shows that $b(r_i)=0$ hence $\lim_{x\to r_i} b(x)/a(x)=b^{(1)}(r_i)/a^{(1)}(r_i)$ is finite. It remains to show $\lim_{x\to r_i} 2(\lambda F(x)-f(x))/a(x)$ is finite. By hypothesis $a(r_i)=0$ and by Proposition 2 $D_m D_p^+ F(r_i)=\lambda F(r_i)-f(r_i)=0$, applying l'Hospital rule we have

$$\lim_{x\to r_i} \frac{2}{a(x)}(\lambda F(x)-f(x))=2\lim_{x\to r_i}(\lambda F^{(1)}(x)-f^{(1)}(x))/a^{(1)}(x)$$

finite since $a^{(1)}(r_i)\neq 0$. □

## 2. Study of the Resolvent $(\lambda-A)^{-1}$

Let the domain of the operator $A$ be defined by $\mathscr{D}(A)=\mathscr{D}_0\cap\mathscr{D}_1$ where

$\mathscr{D}_i=\mathscr{D}=\{F\colon F\in C(I)$ and $AF\in C(I)\}$ if $r_i$ is inaccessible,

$\mathscr{D}_i=\{F\colon F\in\mathscr{D}$ and $AF(r_i)=0\}$ if $r_i$ is exit,

$\mathscr{D}_i=\{F\colon F\in\mathscr{D}$ and $D_p^+ F(r_i)=0\}$ if $r_i$ is regular.

For $k=0,1,\dots$ let $\mathscr{D}_k(A)=\mathscr{D}(A)\cap C^k(I)\cap A^{-1}C^k(I)$. Let

$$\mu_0=0,$$
$$\mu_1=|b^{(1)}|,$$
$$\mu_2=\left|\frac{a^{(2)}}{2}\right|+2|b^{(1)}|+|b^{(2)}|+N_2,$$
$$\mu_3=\tfrac{1}{2}|a^{(3)}|+\tfrac{3}{2}|a^{(2)}|+|b^{(3)}|+4|b^{(2)}|+3|b^{(1)}|+N_3$$

where $N_2=\max\{\zeta_0,\zeta_1\}$ and $N_3=\max\{-m_0,-m_1,0\}$ with $m_i=\inf_{x\in I_i} b(x)a^{(1)}(x)/a(x)$, $I_0=(r_0,0)$, $I_1=(0,r_1)$, and

$$\zeta_i=\begin{cases}0 & \text{if } B(r_i)<\infty \text{ or } m_i\geqq 0\\ -m_i & \text{if } B(r_i)=\infty \text{ and } m_i<0,\end{cases}$$

Let *Condition 0* be: $a,b\in C(I)$ with $a(x)>0$ on $(r_0,r_1)$.

**Theorem k** ($k=0,1,2,3$). "*If Condition k holds and $f\in C^k(I)$ then the equation $\lambda F-AF=f$ has a unique solution $F=(\lambda-A)^{-1}f\in\mathscr{D}_k(A)$ provided $\lambda>\mu_k$, moreover, we have:*

$$|F|_k=|(\lambda-A)^{-1}f|_k\leqq|f|_k(\lambda-\mu_k)^{-1}\text{"}$$

*Remark* 5. (a) *If Condition k* ($k=0,1,2,3$) *holds then $\mu_k$ is well defined.* (*Note that $m_i>-\infty$, so $0\leqq\zeta_i<\infty$*).

(b) $C^0(I)=C(I)$ *and* $|\cdot|_0=|\cdot|$.

(c) *Theorem* 0 *is a direct application of Theorem* 2 (*Mandl*, p. 39) *and Hille-Yosida theorem.*

## 2.1. Proof of Theorem 1

Theorem 0 ensures the existence and uniqueness of the solution $F$. Will show $|F^{(1)}| \leqq K_1 = |f^{(1)}|(\lambda-\mu_1)^{-1}$ since then by Theorem 0 we have:

$$\begin{aligned}|F|_1 &= |(\lambda-A)^{-1}f|_1 = |F| + |F^{(1)}| \leqq |f|\lambda^{-1} + K_1 \\ &\leqq |f|\lambda^{-1} + |f^{(1)}|(\lambda-\mu_1)^{-1} \leqq |f|_1(\lambda-\mu_1)^{-1}.\end{aligned}$$

Since $a, b, f \in C^1(I)$ and $a(x)>0$ on $(r_0, r_1)$, by Proposition 1, we have $F \in C^3(r_0, r_1)$. To show $|F^{(1)}| \leqq K_1$ enough to show $|F^{(1)}(x)| \leqq K_1$ on $(r_0, r_1)$ and $\lim_{x\to r_j} F^{(1)}(x)$ exists for $j=0, 1$ and $\lambda > \mu_1$. The second part follows from the following lemma which will be proved at the end of this section.

**Lemma 1.** *For $\alpha, \beta, \gamma, f \in C(I)$ with $\gamma(x)>0$ on $(r_0, r_1)$, let $\psi \in C^2(r_0, r_1)$ and satisfying*

$$(\lambda-\alpha)\psi - \beta\psi^{(1)} - \gamma\psi^{(2)} = f.$$

*Then for $\lambda > |\alpha|$ and $i=0, 1$ we have $\lim_{x\to r_i} \psi(x)$ exists finite or infinite.*

Now $F \in C^3(r_0, r_1)$ and $\lambda F - AF = f$ so on $(r_0, r_1)$ we have

$$(\lambda - b^{(1)})F^{(1)} - \left(\frac{a^{(1)}}{2} + b\right)F^{(2)} - \frac{a}{2}F^{(3)} = f^{(1)}. \tag{6}$$

Lemma 1 applied to equation (6) gives us the existence of $\lim_{x\to r_j} F^{(1)}(x)$ for $j=0, 1$. To show $|F^{(1)}(x)| \leqq K_1$ maximum principle can be applied to (6) provided the maximum occurs at some interior point. For example, if maximum of $F^{(1)}$ occurs at $x_0 \in (r_0, r_1)$ then (6) holds with $F^{(2)}(x_0)=0$ and $F^{(3)}(x_0) \leqq 0$ therefore

$$-\frac{a(x_0)}{2}F^{(3)}(x_0) \geqq 0$$

and for $\lambda > \mu_1$:

$$\begin{aligned}&(\lambda - b^{(1)}(x_0))F^{(1)}(x_0) \leqq f^{(1)}(x_0), \\ &F^{(1)}(x) \leqq F^{(1)}(x_0) \leqq |f^{(1)}|(\lambda - |b^{(1)}|)^{-1} = K_1.\end{aligned}$$

Similarly if maximum of $-F^{(1)}$ occurs at $\tilde{x}_0 \in (r_0, r_1)$ then applying maximum principle we get $F^{(1)}(x) \geqq F^{(1)}(\tilde{x}_0) \geqq -K_1$ which yields $|F^{(1)}(x)| \leqq K_1$ on $(r_0, r_1)$. But maximum principle cannot be directly applied when maximum does not occur at an interior point. We now introduce auxiliary function $\phi_{\varepsilon,1}$ which depends on certain growth function $g$ defined according to the boundary behavior of $F^{(1)}$. The proof of Theorem 1 will be devided into two parts: first we study the properties of the auxiliary function $\phi_{\varepsilon,1}$ and then the maximum principle applied to $\phi_{\varepsilon,1}$ completes the proof.

### 2.1.1. The Auxiliary Function $\phi_{\varepsilon,1}$

Under the hypothesis of Theorem 1 and for $\varepsilon>0$ we define

$$\phi_{\varepsilon,1}(x)=\begin{cases} F^{(1)}(x)/(1+\varepsilon g_0(x)e^{-B(x)}) & r_0<x<0 \\ F^{(1)}(x)/(1+\varepsilon g_1(x)e^{-B(x)}) & 0<x<r_1 \\ \max\limits_{i=0,1} F^{(1)}(0)/(1+\varepsilon g_i(0)) & x=0 \end{cases} \tag{7}$$

where

$$g_i(x)=\begin{cases} 1 & \text{if } r_i \text{ is entrance or regular} \\ |m(x)| & \text{if } r_i \text{ is natural or exit} \end{cases}$$

notice $e^{-B(0)}=1$ and we can write $\phi_{\varepsilon,1}=F^{(1)}/1+\varepsilon g e^{-B}$ where $g=1$ or $|m|$. Now $\phi_{\varepsilon,1}$ as above defined has the following properties:

(a) $\phi_{\varepsilon,1}\in C^2((r_0,r_1)\setminus\{0\})$.

(b) $\lim\limits_{x\to r_i}\phi_{\varepsilon,1}(x)=0$ for $i=0,1$.

(c) $\phi_{\varepsilon,1}$ attains a maximum on $[r_0,r_1]$ (even though $\phi_{\varepsilon,1}$ may be not continuous at 0).

(d) $\phi_{\varepsilon,1}$ satisfies on $\{(r_0,r_1)\setminus\{0\}\}$ the differential equation

$$\alpha\phi_{\varepsilon,1}+\beta\phi_{\varepsilon,1}^{(1)}+\gamma\phi_{\varepsilon,1}^{(2)}=f^{(1)} \tag{8}$$

with $\alpha=\lambda-b^{(1)}+\varepsilon\lambda e^{-B}g$ and $\gamma=-\dfrac{a}{2}(1+\varepsilon g e^{-B})$.

To prove this:

(a) We have $F\in C^3(r_0,r_1)$, $m$ and $B\in C^2(r_0,r_1)$ by Proposition 1, hence $g\in C^2((r_0,r_1)\setminus\{0\})$ therefore $\phi_{\varepsilon,1}\in C^2((r_0,r_1)\setminus\{0\})$ since $ge^{-B}>0$ for $x\neq 0$.

(b) Let's assume $r_i=r_1$.

If $r_1$ is entrace or regular then we have $g(x)=1$ for $0<x<r_1$ and

$$|\phi_{\varepsilon,1}(x)|=\left|\frac{F^{(1)}(x)}{1+\varepsilon e^{-B(x)}}\right|\leqq\left|\frac{D_p^+F(x)}{\varepsilon}\right|\to 0 \quad \text{as } x\to r_1$$

since $D_p^+F(r_1)=0$ by Proposition 2.

If $r_1$ is natural or exit we have $g(x)=m(x)$ for $0<x<r_1$

$$|\phi_{\varepsilon,1}(x)|=\left|\frac{F^{(1)}(x)}{1+\varepsilon m(x)e^{-B(x)}}\right|\leqq\left|\frac{D_p^+F(x)}{\varepsilon m(x)}\right|$$

so it is enough to show $\lim\limits_{x\to r_1}D_p^+F(x)/m(x)=0$. If $r_1$ is natural with $m(r_1)$ finite then there is nothing to prove since $D_p^+F(r_1)=0$ by Proposition 2. So let's assume $m(r_1)=\infty$. Let $0<x_1<x_2<r_1$, since $D_p^+F(x)$ and $m(x)$ are continuous and differentiable on $[x_1,x_2]$ we have by the Cauchy generalized mean value theorem

$$\frac{D_p^+F(x_2)-D_p^+F(x_1)}{m(x_2)-m(x_1)}=\frac{(D_p^+F(x))^{(1)}}{(m(x))^{(1)}}=D_mD_p^+F(x)$$

for some $x \in (x_1, x_2)$. By Proposition 2, $\lim_{x \to r_1} D_m D_p^+ F(x) = D_m D_p^+ F(r_1) = 0$. So given $\delta > 0$ there exists $\xi$ such that for $\xi < x_1 < x_2 < r_1$ we have

$$-\frac{\delta}{3} < (D_p^+ F(x_2) - D_p^+ F(x_1))/(m(x_2) - m(x_1)) < \frac{\delta}{3}$$

now, for fixed $x_1$ let $x_2 \to r_1$, then we must have $m(x_2) \to \infty$, since we are assuming $m(r_1) = \infty$. So for $x_2$ large enough we have $|D_p^+ F(x_1)/(m(x_2) - m(x_1))| < \delta/3$. This implies

$$\frac{2\delta}{3} > |D_p^+ F(x_2)/(m(x_2) - m(x_1))| > |D_p^+ F(x_2)/m(x_2)|_{x_2 \to r_1} \to 0.$$

(c) By (a) and (b) we conclude $\phi_{\varepsilon, 1}$ continuous on $\{[r_0, r_1] \setminus \{0\}\}$ although $\phi_{\varepsilon, 1}$ may be not continuous at 0 it will attain a maximum on $[r_0, r_1]$ since at 0 it is defined by the largest value and $\phi_{\varepsilon, 1}(0^+)$ and $\phi_{\varepsilon, 1}(0^-)$ exist.

(d) A routine calculation shows (d). □

Next the maximum principle will be applied to $\phi_{\varepsilon, 1}$ in order to complete the proof, but if the maximum is at $x = 0$ the principle cannot be applied. We will then make use of the following lemma which will be proved at the end of this section.

**Lemma 2.** *Let's assume* $[l_0, l_1] \subset (r_0, r_1)$.

(i) $\psi \in C(r_0, r_1)$,

(ii) $g(x) \geqq 0$ *on* $[l_0, l_1]$, *and* $g(x) > 0$ *on* $\{(r_0, r_1) - [l_0, l_1]\}$.

For $\varepsilon > 0$ let $\phi_\varepsilon(x) = \psi(x)/(1 + \varepsilon g(x))$. If a maximum of $\psi$ does not occur in $[l_0, l_1]$, then there is an $\varepsilon_1 > 0$ such that, for all $\varepsilon < \varepsilon_1$, $\phi_\varepsilon$ does not have a positive maximum on $[l_0, l_1]$.

### 2.1.2. The Maximum Principle

We will show

$$F^{(1)}(x) \leqq K_1 \quad \text{for } x \in (r_0, r_1) \text{ and } \lambda > \mu_1 \tag{9}$$

then exactly the same reasoning applied to $-F^{(1)}$ (that is, replace $F^{(1)}$ by $-F^{(1)}$ in the definition of $\phi_{\varepsilon, 1}$) completes the proof. To prove (9) if a maximum of $F^{(1)}$ occurs at some interior point the maximum principle applied to (6) gives us (9). If not we use the function $\phi_{\varepsilon, 1}$. It suffices to show for $\varepsilon$ small enough we have $\phi_{\varepsilon, 1}(x) \leqq K_1$ on $[r_0, r_1]$ for $\lambda > \mu_1$. Then for $x \in (r_0, r_1)$ we have

$$\lim_{\varepsilon \to 0} \phi_{\varepsilon, 1}(x) = \lim_{\varepsilon \to 0} \frac{F^{(1)}(x)}{1 + \varepsilon g(x) e^{-B(x)}} = F^{(1)}(x) \leqq K_1.$$

Now assuming $\dot{F}^{(1)}$ does not have an interior maximum, in particular there exists an interval $0 \in [l_0, l_1] \subset (r_0, r_1)$ such that $F^{(1)}$ does not have maximum on $[l_0, l_1]$, now by Lemma 2 there exists $\varepsilon_1 > 0$ such that for all $\varepsilon < \varepsilon_1$ the function $\phi_{\varepsilon, 1}$ will not have a positive maximum on $[l_0, l_1]$, in particular $\phi_{\varepsilon, 1}$ will not have a positive maximum at $x=0$ for $\varepsilon<\varepsilon_1$. Now let $0<\varepsilon<\varepsilon_1$ we will show $\phi_{\varepsilon,1}(x)\leqq K_1$ and this completes the proof. There are three cases to study:

If $\phi_{\varepsilon,1}$ has a maximum at $x=0$ then $\phi_{\varepsilon,1}(x)\leqq\phi_{\varepsilon,1}(0)\leqq 0\leqq K_1$.

If $\phi_{\varepsilon,1}$ has a maximum at one of the boundaries then $\phi_{\varepsilon,1}(x)=\lim_{x\to r_i}\phi_{\varepsilon,1}(x)=0\leqq K_1$.

If $\phi_{\varepsilon,1}$ has a maximum at some $x_{0,\varepsilon}\in\{(r_0,r_1)\setminus\{0\}\}$ then the maximum principle can be applied since at $x_{0,\varepsilon}$ (8) holds with

$$\phi^{(1)}_{\varepsilon,1}(x_{0,\varepsilon})=0, \qquad \phi^{(2)}_{\varepsilon,1}(x_{0,\varepsilon})\leqq 0,$$
$$g(x_{0,\varepsilon})>0, \qquad a(x_{0,\varepsilon})>0,$$

hence

$$\phi_{\varepsilon,1}(x)\leqq\phi_{\varepsilon,1}(x_{0,\varepsilon})\leqq\frac{|f^{(1)}|}{\alpha(x_{0,\varepsilon})}\leqq\frac{|f^{(1)}|}{\lambda-\mu_1}=K_1 \qquad \text{for } \lambda>\mu_1. \quad \Box$$

## 2.2. Proof of Theorems 2 and 3

The proofs are similar to Theorem 1. Hwere we present a quick sketch (for details see [4]).

*Proof of Theorem 2.* It suffices to show for $\lambda>\mu_2$ and $\delta_2=f^{(2)}+b^{(2)}F^{(1)}$

$$|F^{(2)}|\leqq K_2=|\delta_2|(\lambda-\mu_2)^{-1}.$$

Equation (6) becomes

$$\left(\lambda-\frac{a^{(2)}}{2}-2b^{(1)}\right)F^{(2)}-(a^{(1)}+b)F^{(3)}-\frac{a}{2}F^{(4)}=\delta_2.$$

$\phi_{\varepsilon,2}$ is defined as in (7) with $F^{(2)}$ in place of $F^{(1)}$ and

$$g_i(x)=\begin{cases}2e^{B(x)}/a(x) & \text{if } \lim_{x\to r_i}e^{-B(x)}=0\\ 2/a(x) & \text{otherwise.}\end{cases}$$

We will show that the $\lim_{x\to r_i}\phi_{\varepsilon,2}(x)=0$ for $i=0,1$. Let's assume $r_i=r_1$

$$|\phi_{\varepsilon,2}(x)|\leqq|e^{B(x)}F^{(2)}(x)/\varepsilon g_1(x)|$$

there are two cases to study:

*Case 1.* If $r_1$ is natural with $m(r_1)<\infty$ or entrance of regular then $\lim_{x\to r_1}e^{-B(x)}=\infty$ (by Proposition 3), also $\frac{a}{2}F^{(2)}=\lambda F-bF^{(1)}-f$ therefore $\lim_{x\to r_1}a(x)F^{(2)}(x)$ is finite so that we have $\lim_{x\to r_1}\phi_{\varepsilon,2}(x)=0$.

*Case 2.* If $r_1$ is natural with $m(r_1)=\infty$ or exit then $b(r_1)=0$ (by Proposition 3) and $AF(r_1)=0$ (by Proposition 2). Now $\frac{a}{2}F^{(2)}=AF-bF^{(1)}$ so that $\lim_{x\to r_1}\frac{a(x)}{2}F^{(2)}(x)=0$ (notice $F\in C^1(I)$ by Theorem 1). Therefore $\lim_{x\to r_1}\phi_{\varepsilon,2}(x)=0$.

We have an equivalent equation to (8) where

$$\alpha=\lambda-\frac{a^{(2)}}{2}-2b^{(1)}+\varepsilon g e^{-B}(\lambda-b^{(1)}) \qquad \text{if } g=2/a,$$

$$\alpha=\lambda-\frac{a^{(2)}}{2}-2b^{(1)}+\varepsilon g e^{-B}\left(\lambda-2b^{(1)}+\frac{b a^{(1)}}{a}\right) \qquad \text{if } g=\frac{2}{a}e^{B}.$$

*Proof of Theorem 3.* It is enough to show for $\lambda>\mu_3$ that

$$|F^{(3)}|\leqq K_3=|\delta_3|(\lambda-\mu_3)^{-1}$$

where

$$\delta_3=f^{(3)}+\left(\frac{a^{(3)}}{2}+3b^{(2)}\right)F^{(2)}+b^{(3)}F^{(1)}.$$

Equation (6) becomes

$$(\lambda-\tfrac{3}{2}a^{(2)}-3b^{(1)})F^{(3)}-(\tfrac{3}{2}a^{(1)}+b)F^{(4)}-\frac{a}{2}F^{(5)}=\delta_3.$$

$\phi_{\varepsilon,3}$ is defined as in (7) with $F^{(3)}$ in place of $F^{(1)}$ and

$$g_i(x)=\begin{cases}2/a(x) & \text{if } \lim_{x\to r_i} e^{B(x)}=0\\ 2|m(x)|/a(x) & \text{otherwise.}\end{cases}$$

We will show $\lim_{x\to r_i}\phi_{\varepsilon,3}(x)=0$ for $i=0,1$. Let's assume $r_i=r_1$. First notice that for $\lambda>\mu_2$ by Theorem 2 $F\in C^2(I)$ hence $\lim_{x\to r_1}\frac{a(x)}{2}F^{(3)}(x)$ exists and is finite. Now

$$|\phi_{\varepsilon,3}(x)|\leqq|F^{(3)}(x)/\varepsilon g_1(x)e^{-B(x)}|.$$

*Case 1.* If $g_1=2/a$ then the result follows since $\lim_{x\to r_1}e^{-B(x)}=\infty$.

*Case 2.* If $g_1=2m/a$ it suffices to prove $m(x)e^{-B(x)}\to\infty$ as $x\to r_1$. If $\lim_{x\to r_1}e^{-B(x)}\neq 0$ then there is nothing to prove since $m(r_1)=\infty$ (by Proposition 3); otherwise applying l'Hospital rule we have

$$\lim_{x\to r_1}|m(x)e^{-B(x)}|=\lim_{x\to r_1}\left|\frac{m^{(1)}(x)}{(e^{B(x)})^{(1)}}\right|=\lim_{x\to r_1}\left|\frac{1}{b(x)}\right|=\infty$$

since $b(r_1)=0$ by Proposition 3.

We have an equivalent equation to (8) where

$$\alpha=\lambda-\tfrac{3}{2}a^{(2)}-3b^{(1)}+\varepsilon e^{-B}g\rho$$

and $\rho=\lambda-a^{(2)}-2b^{(1)}+\dfrac{(a^{(1)})^2}{2a}+\dfrac{a^{(1)}b}{a}$.

### 2.3. Proof of Lemma 1 and Lemma 2

*Proof of Lemma 1.* Suppose $r_i = r_1$ (the case when $r_i = r_0$ is similar). If $\psi$ is monotonic in a neighborhood of $r_1$ then the $\lim_{x \to r_1} \psi(x)$ exists and is finite or infinite. If not, then there exists $\xi < r_1$ such that $\psi$ has infinitely many local maximums and local minimums on $[\xi, r_1)$. Let $\{x_i\}_{i=1}^{\infty}$ be such that $\lim_{i \to \infty} x_i = r_1$. For $\lambda > |\alpha|$, given $\varepsilon > 0$ choose $\tilde{x}_0$ such that $\xi < \tilde{x}_0 < r_1$ and

$$\left| \frac{f(r_1)}{\lambda - \alpha(r_1)} - \frac{f(x)}{\lambda - \alpha(x)} \right| < \varepsilon \quad \text{for } x \in [\tilde{x}_0, r_1].$$

Let

$$A = \{x_M : x_M \in [\tilde{x}_0, r_1) \text{ and } x_M \text{ is a local maximum of } \psi\},$$
$$B = \{x_m : x_m \in [\tilde{x}_0, r_1) \text{ and } x_m \text{ is a local minimum of } \psi\}.$$

Then we have:

$$\psi(x_M) \leqq \frac{f(x_M)}{\lambda - \alpha(x_M)} \leqq \frac{f(r_1)}{\lambda - \alpha(r_1)} + \varepsilon,$$
$$\psi(x_m) \geqq \frac{f(x_m)}{\lambda - \alpha(x_m)} \geqq \frac{f(r_1)}{\lambda - \alpha(r_1)} - \varepsilon$$

moreover,

$$\limsup_{x_i \to r_1} \psi(x_i) \leqq \sup_{x_M \in A} \psi(x_M) \leqq \frac{f(r_1)}{\lambda - \alpha(r_1)} + \varepsilon,$$
$$\liminf_{x_i \to r_1} \psi(x_i) \leqq \inf_{x_m \in B} \psi(x_m) \leqq \frac{f(r_1)}{\lambda - \alpha(r_1)} - \varepsilon$$

and this completes the proof. □

*Proof of Lemma 2.* First notice that $\phi_\varepsilon(x)$ and $\psi(x)$ will always have the same sign since $g(x) \geqq 0$. By hypothesis maximum of $\psi$ not on $[l_0, l_1]$ so there exists $x_0 \notin [l_0, l_1]$ and $\delta > 0$ such that $\psi(x_0) > \psi(x) + \delta$ for $x \in [l_0, l_1]$. Let $K = \sup_{x \in [l_0, l_1]} \psi(x)$ then $K$ is finite. If $K \leqq 0$ then $\psi(x) \leqq 0$ on $[l_0, l_1]$ so that $\phi_\varepsilon(x) \leqq 0$ on $[l_0, l_1]$ hence $\phi_\varepsilon$ cannot have a positive maximum on $[l_0, l_1]$. If $K > 0$ let $\varepsilon_1 = \{\delta/2K g(x_0)\}$. Suppose for some $\varepsilon < \varepsilon_1$, $\phi_\varepsilon$ has a positive maximum at some $x_{1,\varepsilon} \in [l_0, l_1]$ then $\phi_\varepsilon(x_0) \leqq \phi_\varepsilon(x_{1,\varepsilon})$ and $\psi(x_0) \leqq \psi(x_{1,\varepsilon}) + \varepsilon g(x_0) \psi(x_{1,\varepsilon})$, but for $\varepsilon < \varepsilon_1$ we have $\varepsilon g(x_0) \psi(x_{1,\varepsilon}) \leqq \frac{\delta}{2}$, this implies $\psi(x_0) < \psi(x_{1,\varepsilon}) + \frac{\delta}{2}$ (contradiction). □

## 3. Main Theorem. The Converse

*Main Theorem.* If *condition 3* holds then there exists a strongly continuous semigroup $T(t)$ in $C^3(I)$ whose infinitesimal generator is $A$ with domain $\mathscr{D}_3(A)$, $T(t)$: $C^3(I) \to C^3(I)$ and there exists a constant $\mu_3$ independent of $t$ and $f$ such that for $f \in C^3(I)$ we have $|T(t)f|_3 \leqq |f|_3 \exp(\mu_3 t)$.

*Proof.* Let $G=A-\mu_3$, $\lambda'=\lambda-\mu_3$, $\mathscr{D}(G)=\mathscr{D}_3(A)$, we will show that $G$ satisfies the conditions of Hille-Yosida theorem, then it follows that there exists a strongly continuous contraction semigroup $\{S(t), t\geqq 0\}$ in $C^3(I)$ whose infinitesimal generator is $G$, moreover, $|S(t)f|_3\leqq |f|_3$ for $f\in C^3(I)$. Now let $T(t)=S(t)\exp(\mu_3 t)$, then $\{T(t), t\geqq 0\}$ is a strongly continuous semigroup in $C^3(I)$ and the infinitesimal generator of $T(t)$ is $G+\mu_3=A$, also, since $T(t)=S(t)\exp(\mu_3 t)$ and $|S(t)f|_3\leqq |f|_3$ we have $T(t)$: $C^3(I)\to C^3(I)$ and $|T(t)f|_3\leqq |f|_3\exp(\mu_3 t)$ for $f\in C^3(I)$.

(a) $\mathscr{D}(G)$ is dense in $C^3(I)$ with respect to $|\cdot|_3$. Enough to show $\mathscr{D}(A)\supset C^2(I)$. Now $\mathscr{D}(A)=\mathscr{D}_0\cap\mathscr{D}_1$ will show $\mathscr{D}_i\supset C^2(I)$ for $i=0, 1$. There are three cases:

(i) if $r_i$ is inaccessible we have $\mathscr{D}_i=\mathscr{D}\supset C^2(I)$.

(ii) if $r_i$ is an exit we need to show that $Af(r_i)=0$ for $f\in C^2(I)$. Now $b(r_i)=0$ by Proposition 3 and $a(r_i)=0$, so

$$Af(r_i)=\lim_{x\to r_i}\left\{\frac{a(x)}{2}f^{(2)}(x)+b(x)f^{(1)}(x)\right\}=0.$$

(iii) if $r_i$ is regular we need to show $D_p^+ f(r_i)=0$ for $f\in C^2(I)$. Now $\lim_{x\to r_i} e^{B(x)}=0$ by Proposition 3, hence $D_p^+ f(r_i)=0$.

(b) For $\lambda'>0$ and $f\in C^3(I)$ the equation $\lambda' F-GF=f$ has a unique solution $F\in\mathscr{D}(G)$.

It follows from the fact that $(\lambda-A)F=f$ has a unique solution $F\in\mathscr{D}_3(A)=\mathscr{D}(G)$ provided $\lambda>\mu_3$ that is $\lambda'>0$, and $\lambda' F-GF=(\lambda-A)F$.

(c) $|F|_3\leqq |f|_3\lambda^{-1}$ for $f\in C^3(I)$.

From Theorem 3 we have for $\lambda'>0$

$$|F|_3=|(\lambda-A)^{-1}f|_3\leqq |f|_3/(\lambda-\mu_3)=|f|_3/\lambda'.$$

And this completes the proof. □

*Remark 6.* The preceding theorem can be generalized: "For $k=0, 1, 2, 3$, if *Condition k* holds then there exists a strongly continuous semigroup $T(t)$ in $C^k(I)$ whose infinitesimal generator is $A$ with domain $\mathscr{D}_k(A)$. Moreover, $T(t)$: $C^k(I)\to C^k(I)$ and there exists $\mu_k$ independent of $t$ and $f$ such that for $f\in C^k(I)$ we have

$$|T(t)f|_k\leqq |f|_k\exp(\mu_k t)."$$

Now we will prove the necessity of the boundary conditions $D_p^+ F(r_i)=0$ and $AF(r_i)=0$.

**The Converse Theorem.** *If the Main Theorem is true then:*

(a) $D_p^+ F(r_i)=0$ *if* $r_i$ *is regular.*

(b) $AF(r_i)=0$ *if* $r_i$ *is an exit.*

*Proof.* (a) if $r_i$ is regular then $\lim_{x\to r_i} e^{B(x)}=0$ by Proposition 3, also $F\in C^2(I)$ by the Main Theorem, so $D_p^+ F(r_i)=0$.

(b) if $r_i$ is an exit then $b(r_i)=0$ by Proposition 3, $a(r_i)=0$ by hypothesis and $F\in C^2(I)$, so $AF(r_i)=0$. □

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