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Extensions of Billingsley's Theorems on Weak Convergence of Empirical Processes

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0. Summary

In [4], Sen proved Theorems 22.1 and 22.2 of Billingsley [1] on the weak convergence of empirical distribution functions for sequences of ϕ-mixing random variables to appropriate Gaussian random functions under less stringent regularity conditions. In this note, we shall prove the same theorems under weaker conditions than Sen's ones.

1. The Main Result

Let $\{x_j, -\infty<j<\infty\}$ be a strictly stationary ϕ-mixing sequence of random variables defined on a probability space $(\Omega, \mathscr{B}, P)$. Thus, $\{x_j\}$ satisfies the condition that

$$\sup_{A\in\mathfrak{M}_{-\infty}^{k},\, B\in\mathfrak{M}_{k+n}^{\infty}} \frac{1}{P(A)}|P(A\cap B)-P(A)P(B)|=\phi(n)\downarrow 0 \quad (n\to\infty) \tag{1}$$

where $\mathfrak{M}_a^b$ denotes the σ-algebra generated by events of the type

$$\{(x_{i_1},\dots,x_{i_k})\in E\}, \quad a\leqq i_1<\dots<i_k\leqq b$$

and E is a k-dimensional Borel set. Let

$$c(u)=\begin{cases}1 & \text{if } u\geqq 0,\\ 0 & \text{if } u<0.\end{cases} \tag{2}$$

Assuming x_i has a continuous distribution function $F(u)$ and working with $x_i^*=F(x_i)$ for any i, we define the empirical distribution function by

$$F_n(t)=n^{-1}\sum_{i=1}^{n} c(t-x_i^*), \quad 0\leqq t\leqq 1. \tag{3}$$

In [1], Billingsley proved that the sequence $\{Y_n\}$ of random elements in $D[0,1]$ defined by

$$Y_n(t)=n^{\frac{1}{2}}[F_n(t)-t], \quad 0\leqq t\leqq 1, \tag{4}$$

converges weakly to a Gaussian random function under the condition $\sum n^2\phi^{\frac{1}{2}}(n)<\infty$. Recently, in [4], Sen proved that the same result holds even if the condition $\sum n^2\phi^{\frac{1}{2}}(n)<\infty$ is replaced by the condition $\sum n\phi^{\frac{1}{2}}(n)<\infty$. We show here that the latter condition $\sum n\phi^{\frac{1}{2}}(n)<\infty$ can be replaced by the condition

$\phi(n)=O(n^{-2})$. We use the same notations and definitions in § 22, [1]. Let

$$g_t(x_i^*)=c(t-x_i^*)-t, \quad 0\leqq t\leqq 1, \ i\geqq 0. \tag{5}$$

Theorem. *Suppose that $\{x_n\}$ is ϕ-mixing with $\phi(n)=O(n^{-2})$, and suppose x_0 has a continuous distribution function F on $[0,1]$. Then*

$$Y_n \xrightarrow{\mathscr{D}} Y \tag{6}$$

where Y_n is defined by (4) and Y is the Gaussian random function specified by

$$E\{Y(t)\}=0 \tag{7}$$

and

$$E\{Y(s)\,Y(t)\}=E\{g_s(x_0^*)\,g_t(x_0^*)\}+\sum_{k=1}^{\infty}E\{g_s(x_0^*)\,g_t(x_k^*)\}+\sum_{k=1}^{\infty}E\{g_s(x_k^*)\,g_t(x_0^*)\}. \tag{8}$$

These series converges absolutely and $P(Y\in C)=1$ (cf. Theorem 22.1 in [1] and Theorem 3.1 in [4]).

2. Basic Lemmas

Let $\{z_i\}$ be stationary and ϕ-mixing with $Ez_i=0$, $Ez_i^2=\tau$, $P(|z_i|>1)=0$ and

$$E|z_i|\leqq c\tau \quad (c<\infty). \tag{9}$$

It is clear that (9) holds when the z_i are Bernoullian variables, centered at expectation. Let $S_0=0$ and

$$S_m=z_1+\cdots+z_m, \quad m\geqq 1.$$

In what follows, by the letter K, we shall denote any quantity (not always the same) which is bounded in absolute value.

Lemma 1. *If $\phi(j)=O(j^{-2})$ and (9) holds, then for all m sufficiently large*

$$ES_m^4\leqq Km^2(\log m)^2\,\tau. \tag{10}$$

Proof. We follow the proof of Lemma 2.1 in [4]. We denote by $\sum_m$ the summation over all $i, j, k\geqq 0$ for which $i+j+k\leqq m$, and let $\sum_m^{(1)}$, $\sum_m^{(2)}$ and $\sum_m^{(3)}$ be respectively the components of $\sum_m$ for which $i\geqq(j,k)$, $j\geqq(i,k)$ and $k\geqq(i,j)$. Then, we have

$$ES_m^4\leqq 24m\{\textstyle\sum_m^{(1)}+\sum_m^{(2)}+\sum_m^{(3)}\}\,|Ez_0\,z_i\,z_{i+j}\,z_{i+j+k}|. \tag{11}$$

Since $\phi(j)=O(j^{-2})$,

$$\sum_{j=1}^{m}(j+1)^2\,\phi(j)\leqq Km \quad\text{and}\quad \sum_{j=1}^{m}\phi^{\frac{1}{2}}(j)\leqq K\log m. \tag{12}$$

So, using (9), (12) and the assumption that $P(|z_i|>1)=0$, we have the following inequalities:

$$\begin{aligned}\textstyle\sum_m^{(1)}|Ez_0\,z_i\,z_{i+j}\,z_{i+j+k}| &\leqq 2\textstyle\sum_m^{(1)}\phi(i)\,E|z_0| \\ &\leqq K\tau\sum_{i=1}^{m}(i+1)^2\,\phi(i)\leqq K\tau m,\end{aligned} \tag{13}$$

$$\begin{aligned}
&\textstyle\sum_m^{(2)} |E z_0 z_i z_{i+j} z_{i+j+k}| \\
&\quad \leqq \textstyle\sum_m^{(2)} |E z_0 z_i E z_0 z_k| + 2 \sum_m^{(2)} \phi(j) |E z_0 z_i| \\
&\quad \leqq K \tau^2 \textstyle\sum_m^{(2)} \phi^{\frac{1}{2}}(i) \phi^{\frac{1}{2}}(k) + K \tau \sum_m^{(2)} \phi(j) \\
&\quad \leqq K m \tau^2 \left(\sum_{i=1}^{m} \phi^{\frac{1}{2}}(i) \right)^2 + K \tau \left(\sum_{j=1}^{m} (j+1)^2 \phi(i) \right) \\
&\quad \leqq K m (\log m)^2 \tau, \\
&\textstyle\sum_m^{(3)} |E z_0 z_i z_{i+j} z_{i+j+k}| \\
&\quad \leqq 2 \textstyle\sum_m^{(3)} \phi(k) E|z_0| \leqq K \tau \sum_m^{(3)} \phi(k) \\
&\quad \leqq K \tau \sum_{k=1}^{m} (k+1)^2 \phi(k) \leqq K m \tau.
\end{aligned} \tag{13}$$

Thus, (10) follows from (11) through (13), and the proof is completed.

Let $q = [n^{\frac{5}{6}}]$, and $h = [n/2q]$. Further, let

$$\begin{aligned}
v_i &= \sum_{j=1}^{q} z_{2iq+j} \qquad (i = 0, \ldots, h-1), \\
\bar{v}_i &= \sum_{j=1}^{q} z_{(2i+1)q+j} \quad (i = 0, \ldots, h-1), \\
v_h &= S_n - \sum_{i=0}^{h-1} v_i - \sum_{i=0}^{h-1} \bar{v}_i.
\end{aligned}$$

Lemma 2. *There exists a number* $\gamma > 0$ *such that for all* n *sufficiently large*

$$\max \left(E \left(\sum_{i=0}^{h} v_i \right)^4, E \left(\sum_{i=0}^{h-1} \bar{v}_i \right)^4 \right) \leqq K (n^{2-\gamma} \tau + n^2 \tau^2). \tag{14}$$

Proof. We shall evaluate the following quantity:

$$\begin{aligned}
E \left(\sum_{i=0}^{h} v_i \right)^4 &= E \left(\sum_{i=0}^{h-1} v_i \right)^4 + 4 E \left(\left(\sum_{i=0}^{h-1} v_i \right)^3 v_h \right) \\
&\quad + 6 E \left(\left(\sum_{i=0}^{h-1} v_i \right)^2 v_h^2 \right) + 4 \left(\left(\sum_{i=0}^{h-1} v_i \right)^3 v_h \right) + E v_h^4.
\end{aligned} \tag{15}$$

Since $E v_h^4 \leqq K E v_0^4$ for all n sufficiently large, so, by Lemma 1 (p. 170) in [1],

$$\begin{aligned}
\left| E \left(\left(\sum_{i=0}^{h-1} v_i \right)^3 v_h \right) \right| &\leqq \sum_{j=2hq+1}^{n} \left| E \left(\left(\sum_{i=0}^{h-1} v_i \right)^3 z_j \right) \right| \\
&\leqq \sum_{j=1}^{\infty} 2 \phi(j) E \left| \sum_{i=0}^{h-1} v_i \right|^3 \leqq K \left\{ E \left(\sum_{i=0}^{h-1} v_i \right)^4 \right\}^{\frac{3}{4}}, \\
\left| E \left(\left(\sum_{i=0}^{h-1} v_i \right) v_h^3 \right) \right| &\leqq \sum_{i=0}^{h-1} |E v_i v_h^3| \\
&\leqq 2 \sum_{i=1}^{h} \phi^{\frac{1}{4}}((2i-1) q) (E v_0^4)^{\frac{1}{4}} (E v_h^4)^{\frac{3}{4}} \leqq K h q^{-\frac{1}{2}} E v_0^4,
\end{aligned} \tag{16}$$

$$E\left(\left(\sum_{i=0}^{h-1} v_i\right)^2 v_h^2\right) \leqq 2\phi^{\frac{1}{2}}(q)\left(E\left(\sum_{i=0}^{h-1} v_i\right)^4\right)^{\frac{1}{2}} (E v_h^4)^{\frac{1}{2}}. \tag{16}$$

Now, we consider $E\left(\sum_{i=0}^{h-1} v_i\right)^4$. As in the proof of Lemma 1,

$$E\left(\sum_{i=0}^{h-1} v_i\right)^4 \leqq 24h \textstyle\sum_{h-1} |E v_0 v_i v_{i+j} v_{i+j+k}|. \tag{17}$$

It follows from the fact $\phi(m)=O(m^{-2})$ that if $(i, j, k) \geqq 1$ then

$$\begin{aligned} &|E v_0 v_i v_{i+j} v_{i+j+k}| \\ &\quad \leqq |E v_0 v_i E v_0 v_k| + 2\phi^{\frac{1}{2}}((2j-1)q)\{E(v_0 v_i)^2 E(v_0 v_k)^2\}^{\frac{1}{2}} \\ &\quad \leqq 4(E v_0^2)^2 \phi^{\frac{1}{2}}((2i-1)q)\phi^{\frac{1}{2}}((2k-1)q) \\ &\qquad + 2\phi^{\frac{1}{2}}((2j-1)q)\{(E v_0^2)^2 + 2\phi^{\frac{1}{2}}((2i-1)q) E v_0^4\}^{\frac{1}{2}} \\ &\qquad \cdot \{(E v_0^2)^2 + 2\phi^{\frac{1}{2}}((2k-1)q) E v_0^4\}^{\frac{1}{2}} \\ &\quad \leqq K E v_0^4 \{\phi^{\frac{1}{2}}((2i-1)q)\phi^{\frac{1}{2}}((2k-1)q) + \phi^{\frac{1}{2}}((2j-1)q)\} \\ &\quad \leqq K q^{-1} E v_0^4 \left\{\frac{1}{(2i-1)(2k-1)} + \frac{1}{2j-1}\right\}. \end{aligned} \tag{18}$$

Similarly, for $i \geqq 1$

$$\begin{aligned} &|E v_0^3 v_i| \leqq 2\phi^{\frac{1}{4}}((2i-1)q) E v_0^4 \leqq K q^{-\frac{1}{2}} E v_0^4, \\ &E v_0^2 v_i^2 \leqq (E v_0^2)^2 + 2\phi^{\frac{1}{2}}((2i-1)q) E v_0^4 \leqq (E v_0^2)^2 + K q^{-1} E v_0^4, \\ &|E v_0^2 v_i v_{i+j}| \leqq 2\phi^{\frac{1}{2}}((2j-1)q)(E v_0^2)^2 + 2\phi^{\frac{1}{2}}((2i-1)q) E v_0^4 \leqq K q^{-1} E v_0^4 \quad (j \geqq 1), \\ &\max(|E v_0 v_i^2 v_{i+j}|, |E v_0 v_i v_{i+j}^2|) \leqq 2\phi^{\frac{3}{4}}((2i-1)q) E v_0^4 \leqq K q^{-\frac{3}{2}} E v_0^4 \quad (j \geqq 0). \end{aligned} \tag{19}$$

So from (18), (19) and the definitions of q and h, we have

$$\begin{aligned} \textstyle\sum_h |E v_0 v_i v_{i+j} v_{i+j+k}| &\leqq E v_0^4 + \sum_{i=1}^{h-1} \{|E v_0^3 v_i| + |E v_0 v_i^3| + E v_0^2 v_i^2\} \\ &\quad + \sum_{j=1}^{h-1} \sum_{i=1}^{h-j-1} \{|E v_0^2 v_i v_{i+j}| + |E v_0 v_i^2 v_{i+j}| + |E v_0 v_i v_{i+j}^2|\} \\ &\quad + \sum_{(i,j,k) \geqq 1} |E v_0 v_i v_{i+j} v_{i+j+k}| \\ &\leqq K\{(1 + q^{-\frac{1}{2}} h + q^{-1} h^2 \log h) E v_0^4 + h(E v_0^2)^2\} \\ &\leqq K\{E v_0^4 + h(E v_0^2)^2\}. \end{aligned} \tag{20}$$

As $E v_0^2 = q\tau(1+o(1))$ (cf. Theorem 1.6 in [2]), so from (17), (20) and Lemma 1, we have

$$E\left(\sum_{i=0}^{h-1} v_i\right)^4 \leqq K\{h q^2 (\log q)^2 \tau + h^2 q^2 \tau^2\} \leqq K\{n^{\frac{13}{7}} \tau + n^2 \tau^2\}. \tag{21}$$

On the other hand, $E\left(\sum_{i=0}^{h-1} \bar{v}_i\right)^4 = E\left(\sum_{i=0}^{h-1} v_i\right)^4$. Thus, (14) with $\gamma = \frac{1}{7}$ is obtained from (15), (16) and (21).

3. Proof of Theorem

Let

$$Y_n'(t)=\frac{1}{\sqrt{n}}\left\{\sum_{i=0}^{h-1}\sum_{j=1}^{q} g_t(x^*_{2iq+j})+\sum_{j=2iq+j}^{n} g_t(x^*_j)\right\}$$

and

$$Y_n''(t)=\frac{1}{\sqrt{n}}\sum_{i=0}^{h-1}\sum_{j=1}^{q} g_t(x^*_{(2i+1)q+j}).$$

Let $z_j=g_t(x^*_j)-g_s(x^*_j)$ $(0\leqq s\leqq t\leqq 1)$ and put

$$v_i=\sum_{j=1}^{q} z_{2iq+j} \qquad (i=0,1,\ldots,h),$$

$$\bar{v}_{i'}=\sum_{j=1}^{q} z_{(2i'+1)q+j} \quad (i'=0,1,\ldots,h-1).$$

Since z_j is a Bernoullian variable, (9) holds with $c=2$ and $Ez_j=0$,

$$Ez_j^2=(t-s)(1-t+s)\leqq t-s.$$

Hence, it follows from Lemma 2 that for all n sufficiently large

$$\max\left(E\left(\sum_{i=0}^{h} v_i\right)^4, E\left(\sum_{i=0}^{h-1}\bar{v}_i\right)^4\right)\leqq K\{n^{2-\gamma}(t-s)+(t-s)^2\}$$

where $0\leqq s\leqq t\leqq 1$ and $0<\gamma\leqq\frac{1}{7}$. Therefore, if ε $(0<\varepsilon<1)$ is a fixed number such that

$$\frac{\varepsilon}{n}\leqq t-s,$$

we have

$$\begin{aligned}\max(E|Y_n'(t)-Y_n'(s)|^4, E|Y_n''(t)-Y_n''(s)|^4)\\ \leqq K\left\{\frac{(t-s)^{1+\gamma}}{\varepsilon^\gamma}+(t-s)^2\right\}\\ \leqq K\left(\frac{1}{\varepsilon^\gamma}+1\right)(t-s)^{1+\gamma}.\end{aligned} \tag{22}$$

Assume now that p is a number satisfying $\varepsilon/n\leqq p$. Since

$$\begin{aligned}&Y_n(s+ip)-Y_n(s)\\ &=\{Y_n'(s+ip)-Y_n'(s)\}+\{Y_n''(s+ip)-Y_n''(s)\}\\ &=\sum_{j=1}^{i}\{Y_n'(s+jp)-Y_n'(s+(j-1)p)\}+\sum_{j=1}^{i}\{Y_n''(s+jp)-Y_n''(s+(j-1)p)\}\\ &\qquad (i=1,\ldots,m)\end{aligned}$$

where m is a positive integer, so by (22) and Theorem 12.2 in [1],

$$\begin{aligned}&P(\max_{i\leqq m}|Y_n(s+ip)-Y_n(s)|\geqq\lambda)\\ &\leqq P\left(\max_{i\leqq m}|Y_n'(s+ip)-Y_n'(s)|\geqq\frac{\lambda}{2}\right)+P\left(\max_{i\leqq m}|Y_n''(s+ip)-Y_n''(s)|\geqq\frac{\lambda}{2}\right)\\ &\leqq\frac{32K}{\lambda^4}\left(\frac{1}{\varepsilon^\gamma}+1\right)(mp)^{1+\gamma}.\end{aligned}$$

Thus, as (22.20) in [1], we have

$$P\big(\sup_{s\leqq t\leqq s+mp}|Y_n(t)-Y_n(s)|\geqq 4\varepsilon\big)\leqq\frac{K_0}{\varepsilon^4}\left(\frac{1}{\varepsilon^\gamma}+1\right)(mp)^{1+\gamma} \tag{23}$$

if

$$\frac{\varepsilon}{n}\leqq p<\frac{\varepsilon}{\sqrt{n}}. \tag{24}$$

Now, we choose δ so that $K_0\,\delta^\gamma\,\varepsilon^{-4}(1+\varepsilon^{-\gamma})<\eta$, where $\eta>0$ is an arbitrarily given number. From (23) it will follow that

$$P\big(\sup_{s\leqq t\leqq s+\delta}|Y_n(t)-Y_n(s)|\geqq 4\varepsilon\big)<\eta\,\delta, \tag{25}$$

provided there exist a p and an integer m such that (24) holds and $mp=\delta$. But this is equivalent to the existence of an integer m with

$$\frac{\delta}{\varepsilon}\sqrt{n}<m\leqq\frac{\delta}{\varepsilon}n,$$

which is true for all sufficiently large n. The rest of the proof is identical to the proof of Theorem 22.1 in [1] and hence, is omitted.

Remark. By Lemma 2, we can prove that the conclusion of Theorem 22.2 in [1] holds, even if the condition $\sum n^2\,\phi^{\frac{1}{2}}(n)$ is replaced by the condition $\phi(n)=O(n^{-2})$. This is an extension of Sen's result in [4].

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