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On the Tail Events of a Markov Chain

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1. Introduction and Summary

Let $(\Omega, \mathscr{F}, P)$ be a probability space, $\{X_n\colon n \geqq 0\}$ a Markov chain defined on this space, $E=\{a_1, a_2, \ldots\}$ the set of its states, $\mathscr{F}_m^n$ the σ-algebra generated by the random variables $X_m, \ldots, X_n$ $(m, n=0, 1, \ldots, m \leqq n)$, $\mathscr{F}_n^n = \mathscr{F}_n$ and $\mathscr{F}_n^\infty$ the σ-algebra generated by $X_n, X_{n+1}, \ldots$. Set $P_j^{(n)} = P\{X_n = a_j\}$ and $P_{i,j}^{*(k,n)} = P\{X_k = a_i | X_n = a_j\}$ for $n=2, 3, \ldots, n>k$ and $a_i, a_j \in E$.

The σ-algebra $\mathscr{T} = \bigcap_{n=1}^{\infty} \mathscr{F}_n^\infty$ is called the *tail* σ-algebra of the considered chain.

Let $\mathscr{F}_0$ be an arbitrary sub σ-algebra of $\mathscr{F}$. A set A in $\mathscr{F}_0$ is called *atomic* if $P(A)>0$ and A does not contain two disjoints subsets of positive probability belonging to $\mathscr{F}_0$. A set A in $\mathscr{F}_0$ is called *completely nonatomic* if $P(A)>0$ and A does not contain any atomic subset belonging to $\mathscr{F}_0$.

According to a well known property (see Loève [9], p. 100) the sample space Ω can be represented by means of disjoint events belonging to $\mathscr{F}_0$ as

$$\Omega = \bigcup_{n=0}^{\infty} A_n$$

where A_0 and some of the A_n, $n \geqq 1$ may be absent and if present A_0 is a completely nonatomic set and $A_1, A_2, \ldots$ are atomic sets. This decomposition is unique modulo null probability sets of $\mathscr{F}_0$.

A σ-algebra $\mathscr{F}_0$ will be said to be *trivial* if it contains only Ω and ϕ, modulo null probability sets.

In the case when the representation corresponding to $\mathscr{F}_0$ is of the form $\Omega = \bigcup_{i=1}^{s} A_i$ with s a natural number, $\mathscr{F}_0$ will be said to be *finite*.

When the representation corresponding to $\mathscr{F}_0$ is of the form $\Omega = \bigcup_{i=1}^{\infty} A_i$, $\mathscr{F}_0$ will be called *atomic*.

There are several papers investigating the structure of the tail σ-algebra of a Markov chain. Blackwell and Freedman [3] have proved that $\mathscr{T}$ is finite in the case of a homogeneous, irreducible, recurrent and denumerable Markov chain, the atomic sets being $\{X_0 \in E_c\}$ modulo null probability sets, where $\{E_c, c \in C\}$ is the partition of E into its cyclically moving subclasses. Using the space time harmonic functions Jamison and Orey [8] extended this result to Markov chains recurrent in the sense of Harris.

Some conditions ensuring the atomic structure of the tail σ-algebra which corresponds to a homogeneous, denumerable Markov chain, by means of its connections with the invariant σ-field of the chain, have been obtained by Abrahamse [1].

Other papers consider some kind of mixing conditions under which $\mathcal{T}$ is trivial or finite. In [4] such a property has been studied by means of the random variables

$$\sup_{A\in\mathcal{F}_k^\infty}|P(A|\mathcal{F}_1^n)-P(A)| \qquad (k>n,\ k,n=1,2,\ldots). \tag{1}$$

Bartfay and Révész [2] have proved that in the case of a Markov chain (1) remains unchanged if $\mathcal{F}_1^n$ is replaced by $\mathcal{F}_n$ and $\mathcal{F}_k^\infty$ by $\mathcal{F}_{k-1}$ and have got some necessary and sufficient conditions for the finiteness of $\mathcal{T}$. Subsequently in [5] it has been proved that any finite nonhomogeneous Markov chain has a finite tail σ-algebra. Iosifescu [7] gave further extensions in this direction and has proved, among other results, that the finiteness of the $\mathcal{T}$ holds true also for a continuous parameter Markov chain with a finite set of states.

The aim of the paper is to give a characterization of the tail σ-algebra of an arbitrary denumerable nonhomogeneous Markov chain by means of a certain convergent sequence of random variables derived from the chain. The random variables used are functions of the transition probabilities of the reversed chain and the proof of our main theorem will be seen to utilize as its main tools the time reversibility of the Markov property and the martingale convergence theorem. Some criteria insuring the triviality, finiteness and atomicity of $\mathcal{T}$ are obtained.

The probability distribution of the atomic sets in $\mathcal{T}$ is established and some special cases of nonhomogeneous Markov chains are considered. In particular when the chain assumes only s values, its tail σ-algebra will be proved to contain at most s atomic sets. In the final section an extension of the main theorem to a continuous parameter Markov process is given.

2. Some Basic Results

Let us define for any k and n with $n>k$ the random variable

$$\gamma_{k,n}(\omega)=\tfrac{1}{2}\sum_{i=1}^{\infty}|P_i^{(k)}-P_{i,j}^{*(k,n)}| \qquad \text{for } \omega\in\{X_n=a_j\}. \tag{2}$$

Let us notice that

$$\gamma_{k,n}(\omega)=\sup_{A\in\mathcal{F}_k}(P\{A\}-P\{A|\mathcal{F}_n\}). \tag{3}$$

Indeed, we have

$$\sup_{A\in\mathcal{F}_k}(P\{A\}-P\{A|\mathcal{F}_n\})=\sum_{i=1}^{\infty}(P_i^{(k)}-P_{i,j}^{*(k,n)})^+$$

for $\omega\in\{X_n=a_j\}$, where $a^+=\max(a,0)$. On the other hand, if we denote by $\bar{A}$ the complementary set of A, for any A and B with $P(B)>0$, one has

$$P\{A\}-P\{A|B\}=-(P\{\bar{A}\}-P\{\bar{A}|B\})$$

and (3) follows.

Now, we shall see that a stronger result than (3) holds. Namely, we have

Lemma 1. *If* $\{X_n\colon n \geqq 1\}$ *is a denumerable Markov chain and* $\{\gamma_{k,n}(\omega)\}$ *is defined by* (2), *then*

$$\gamma_{k,n}(\omega) = \sup_{A \in \mathscr{F}_1^k} (P\{A\} - P\{A | \mathscr{F}_n^\infty\}) \quad a.s. \tag{4}$$

The proof of this lemma can be achieved using the same technique as in the proof of Lemma 3 of [6] by taking into consideration that the Markov property is reversible.

Lemma 2. *If* $\{X_n\colon n \geqq 1\}$ *is a denumerable Markov chain and* $\{\gamma_{k,n}(\omega)\}$ *is defined by* (2), *then*

$$\gamma_{k,n}(\omega) \leqq \gamma_{k',n}(\omega) \quad a.s. \tag{5}$$

for $k' > k$.

The proof follows directly from Lemma 1.

Lemma 3. *If* B *is an event of positive probability and* $\mathscr{F}_0$ *a sub σ-algebra of* $\mathscr{F}$ *then*

$$\sup_{A \in \mathscr{F}_0} (P\{A\} - P\{A | B\}) \leqq 1 - P\{B\}. \tag{6}$$

Proof. We have

$$P(A) - P(A|B) = P(A \cap \bar{B}) + P(A \cap B) - P(A \cap B | B) \leqq P(A \cap \bar{B}) \leqq 1 - P(B).$$

We notice easily that in (6) equality holds if and only if $B \in \mathscr{F}_0$.

Let us denote by T_0 the completely nonatomic set and by $T_1, T_2, \ldots,$ the atomic sets occurring in the representation of Ω corresponding to $\mathscr{T}$.

Theorem 1. (i) *There exist the limits*

$$\lim_{n \to \infty} \gamma_{k,n}(\omega) = \gamma_k(\omega) \quad a.s., \tag{7}$$

$$\lim_{k \to \infty} \gamma_k(\omega) = \gamma(\omega) \quad a.s. \tag{8}$$

(ii) $\gamma(\omega) = 1$ *for almost all* $\omega \in T_0$

and

$\gamma(\omega) = 1 - P(T_i)$ *for almost all* $\omega \in T_i$ $(i = 1, 2, \ldots)$.

Proof. Let us consider first the following decomposition of $\gamma_{k,n}(\omega)$:

$$\gamma_{k,n}(\omega) = \sum_{i \in \Gamma_N} (P_i^{(k)} - P_{i,j}^{*(k,n)})^+ + \sum_{i \in \bar{\Gamma}_N} (P_i^{(k)} - P_{i,j}^{*(k,n)})^+ \tag{9}$$

where $\Gamma_N = \{1, 2, \ldots, N\}$. Denote the first sum of the right hand side of (9) by $\lambda_{k,n}^N(\omega)$ and the second one by $\lambda_{k,n}^{N'}(\omega)$.

Now, for a given $\varepsilon > 0$ we can choose a number N such that for any n and ω: $\lambda_{k,n}^{N'}(\omega) < \varepsilon$. Indeed, it is enough to notice that

$$\lambda_{k,n}^{N'}(\omega) \leqq \sum_{i \in \bar{\Gamma}_N} P_i^{(k)}$$

which obviously could be made as small as we like by choosing N sufficiently large.

Let us deal now with the first sum of (9). By a well-known property of martingales due to Doob (see e.g. [9], p. 409) we have

$$P\{X_k(\omega) = a_i | \mathscr{F}_n^\infty\} \to P\{X_k(\omega) = a_i | \mathscr{T}\} \quad \text{a.s.} \tag{10}$$

as $n\to\infty$. Since $\lambda_{k,n}^N(\omega)$ contains a finite number of summands, we get from (10) and by the time reversibility of the Markov property

$$\lim_{n\to\infty}\lambda_{k,n}^N(\omega)=\sum_{i\in I_N}\left(P_i^{(k)}-P\{X_k(\omega)=a_i|\mathscr{T}\}\right)^+ \quad \text{a.s.}$$

If we notice that

$$|\gamma_k(\omega)-\lim_{n\to\infty}\lambda_{k,n}^N(\omega)|<\varepsilon \quad \text{a.s.}$$

we conclude that

$$\lim_{n\to\infty}\gamma_{k,n}(\omega)=\gamma_k(\omega)=\sup_{A\in\mathscr{F}_1^k}(P\{A\}-P\{A|\mathscr{T}\}) \quad \text{a.s.}$$

Further, taking into account (5) we deduce that the following limit also exists:

$$\lim_{k\to\infty}\gamma_k(\omega)=\gamma(\omega) \qquad \text{a.s.},$$

and the first part of the theorem is proved.

Set now $\phi(A,\omega)=P\{A\}-P\{A|\mathscr{T}\}$ and $\mathscr{G}=\{A\colon \phi(A,\omega)\leqq\gamma(\omega) \text{ a.s.}\}$. It is easy to see that $\mathscr{G}$ is a monotone class and contains $\bigcup_{n=1}^{\infty}\mathscr{F}_1^n$. Therefore

$$P\{A\}-P\{A|\mathscr{T}\}\leqq\gamma(\omega) \qquad \text{a.s.} \tag{11}$$

for any $A\in\mathscr{F}_1^\infty$.

Consider further an atomic set T_i belonging to $\mathscr{T}$. If we denote $A_k=\{j\colon P_j^{(k)}-P\{X_k=j|T_i\}>0\}$, then we have $P\{X_k\in A_k\}-P\{X_k\in A_k|\mathscr{T}\}=\gamma_k(\omega)$ for almost all $\omega\in T_i$. This fact, together with Lemma 3 and (11) yields

$$\gamma_k(\omega)\leqq P\{\overline{T}_i\}-P\{\overline{T}_i|T_i\}=1-P\{T_i\}\leqq\gamma(\omega)$$

for almost all $\omega\in T_i$. Hence

$$\gamma(\omega)=1-P\{T_i\} \tag{12}$$

for almost all $\omega\in T_i$.

To complete the proof let us notice that it follows from (11) that for any $T\in\mathscr{T}$ we have

$$P\{T\}-P\{T|\mathscr{T}\}\leqq\gamma(\omega) \qquad \text{a.s.} \tag{13}$$

Let us suppose that the event $T=\{\omega\colon\gamma(\omega)<\delta\}$ has positive probability for a given δ with $\delta<1$. Consider an arbitrary subevent of T, say T', which belongs to $\mathscr{T}$. Writing (13) for $\overline{T}'$ and integrating over T' we get $P(T')P(\overline{T}')\leqq\delta P(T')$. If $P(T')>0$ we deduce easily that $P(T')\geqq 1-\delta$. Therefore T is a union of atomic sets and contains at most

$$\left[\frac{P(T)}{1-\delta}\right]$$

atomic sets. This fact and (12) prove the remaining part of the Theorem.

Corollary 1. *The tail σ-algebra of a denumerable Markov chain $\{X_n\colon n\geqq 1\}$ has the same structure as the tail σ-algebra of any of its subsequence of random variables $\{X_{n_k}\colon k\geqq 1\}$.*

Corollary 2. (i) $\gamma(\omega)=0$ *a.s. iff* $\mathcal{T}$ *is trivial.*
(ii) $\gamma(\omega)<\delta$ *a.s. with* $\frac{1}{2}\leqq\delta<1$ *iff* $\mathcal{T}$ *is finite.*
(iii) $\gamma(\omega)<1$ *a.s. and* (i) *and* (ii) *do not hold iff* $\mathcal{T}$ *is atomic.*

Proof. By Theorem 1 we need only show that $\gamma(\omega)<\frac{1}{2}$ a.s. implies $\gamma(\omega)=0$ a.s. We notice easily that we have either $P\{\gamma(\omega)=0\}=1$ or $P\{\gamma(\omega)=0\}=0$. If the second case were possible, we would have according to Theorem 1 $\frac{1}{2}<P(T)<1$ for any atomic set T of $\mathcal{T}$. But since there cannot be more than one set with such a property, $\bar{T}$ must also be atomic and $\gamma(\omega)=P(T)<\frac{1}{2}$ for almost all $\omega\in\bar{T}$, which is a contradiction.

Corollary 3. *If* $P\{\gamma(\omega)<1\}>0$ *and we denote the probability distribution of* γ *by*

$$\gamma\colon \begin{pmatrix} 1 & x_1\ldots x_n\ldots \\ P_0 & P_1\ldots P_n\ldots \end{pmatrix},$$

then $\mathcal{T}$ *contains* $P_i/(1-x_i)$ *atomic sets having probability* $1-x_i$, $i\geqq 1$.

Proof. Let us denote by $\{T_k^i\colon k=1,\ldots,k_i\}$ the atomic sets of $\mathcal{T}$ which have the same probability (for $i=1,2,\ldots$). According to (12) we have

$$1-P(T_k^i)=x_i, \qquad k=1,\ldots,k_i.$$

Therefore $P\{T_k^i\}=1-x_i$ for $k=1,\ldots,k_i$ and the number k_i is $P_i/(1-x_i)$. The proof is complete.

Remark. While the expression of $\gamma(\omega)$ is in general difficult to obtain explicitly, we can nevertheless get effectively the probability distribution of the atomic sets of $\mathcal{T}$ by using Corollary 3. Indeed, we know that the distribution function of $\gamma(\omega)$ can be obtained by taking the limit of the distribution functions of $\{\gamma_{k,n}(\omega)\}$. According to Theorem 1 $\gamma(\omega)$ is a discrete variable, $\{x_i\}$ are the points of discontinuity and $\{P_i\}$ the jumps of its distribution function.

3. Applications

In this section we shall give some theoretical applications of the preceding results to nonhomogeneous Markov chains.

Theorem 2. *Let* $\{X_n\colon n\geqq 1\}$ *be a denumerable Markov chain assuming a state* a_i *such that* $P\{X_n=a_i \text{ i.o.}\}=1$ *and* $\liminf_{n\to\infty} P_i^{(n)}\geqq a>0$. *Then the tail* σ*-algebra* $\mathcal{T}$ *is finite and contains at most* $[1/a]$ *atoms.*

Proof. Without any loss of generality we may take $\Omega=N^{*\infty}$, i.e. the space of all sequences $\omega=(i_1,i_2,\ldots)$ with $i_1,i_2,\ldots\in N^*$, $X_n(\omega)=i_n$, the smallest σ-algebra with respect to which the $\{X_n\colon n\geqq 1\}$ are measurable, and P the probability constructed on $\mathcal{F}$ in the standard way.

Now, there exists a set $\Lambda\in E^\infty$ with $P(\Lambda)=1$ such that in any $\omega\in\Lambda$ the coordinate i occurs infinitely often. Further applying Lemma 3 to the sequence $\{\gamma_{k,n}(\omega)\}$ and using Theorem 1 we get $\gamma(\omega)\leqq 1-a$ for $\omega\in\Lambda$ and the theorem follows.

Let us define for an arbitrary $\varepsilon>0$ and for a positive integer m the set

$$N_\varepsilon^m=\{i\colon\ P_i^{(m)}>\varepsilon\}. \tag{14}$$

Theorem 3. *Let $\{X_n\colon n \geqq 1\}$ be a denumerable Markov chain and $\{N_\varepsilon^m\}$ the family of sets defined by* (14). *If* $\lim_{\varepsilon\to\infty} \limsup_{m\to\infty} P\{X_m \in N_\varepsilon^m\} = 1$, *then the tail σ-field $\mathscr{T}$ is at most atomic.*

Proof. According to Theorem 1 we need to prove that $P\{\omega\colon \gamma(\omega) < 1\} = 1$. By the assumption made, for an arbitrary η we may find a number ε and a subsequence $\{m_n\}$ such that

$$\lim_{n\to\infty} P\{X_{m_n} \in N_\varepsilon^{m_n}\} > 1 - \eta. \tag{15}$$

From Lemma 3 we deduce easily that $\gamma_{k,m_n}(\omega) \leqq 1 - \varepsilon$ for $\omega \in \{X_n(\omega) = a_j\}$ with $j \in N_\varepsilon^{m_n}$ and $k < m_n$. Now, by (15) we get $P\{\gamma_k(\omega) \leqq 1 - \varepsilon\} \geqq 1 - \eta$ for any k and the same relationship will hold true if we replace $\gamma_k(\omega)$ by $\gamma(\omega)$. The proof is complete.

Corollary 1. *Any Markov chain $\{X_n\colon n \geqq 1\}$ with a stationary absolute probability distribution admits at most an atomic tail σ-algebra.*

We note that in particular when the chain is homogeneous the same result is a consequence of the Blackwell and Freedman paper [3], because such a chain is recurrent.

Corollary 2. *Any Markov chain $\{X_n\colon n \geqq 1\}$ assuming s states has a finite tail σ-algebra which consists of at most s atomic sets.*

Proof. First, note that since $\gamma_{k,n}(\omega)$ assumes at most s values, the same will hold true for $\gamma(\omega)$. We can see this most easily by a double application of the following

Lemma.[1] *Let $X_1, X_2, \ldots, X$ be random variables having distributions $\pi_1, \pi_2, \ldots, \pi$, and taking values in a separable metric space, and let X_n converge to X "in law"; i.e. let π_n converge weakly to π. Then if π contains $s+1$ distinct points in its support, the same must be true for π_n for all sufficiently large n.*

To prove the lemma (noting that the support of a probability measure π is the set of all points y such that $\pi(G) > 0$ whenever the open set G contains y) notice that under the stated hypothesis there will exist $s+1$ disjoint open sets all having positive π-measure. We can take these open sets to be spheres, and since there are continuum-many choices possible for the radii of the spheres, we can suppose that each sphere has a boundary of π-measure zero, that is, is a π-continuity-set. Calling the spheres so chosen $S_1, S_2, \ldots, S_{s+1}$, we shall have $\pi_n(S_i) > 0$ for $n > N_i$ (all $i = 1, 2, \ldots, s+1$), and so $\pi_n(S_i) > 0$ for $i = 1, 2, \ldots, s+1$ provided that $n > \max(N_i) = N$. Thus the support of π_n (for $n > N$) contains at least $s+1$ points (it is here that we use the separability of the metric space).

Turning now to the corollary, if the distribution of γ contains $s+1$ distinct points in its support, then so does that of γ_k for $k > K$. Choose such a k, and apply the lemma again; we see that the distribution of $\gamma_{k,n}$ must have at least $s+1$ points in its support for all $n > N$, yet $\gamma_{k,n}$ is a real-valued random variable assuming at most s distinct values.

Next, we shall show that the conditions of Theorem 3 are satisfied. For that, suppose the contrary, i.e. there exists a positive number p such that

$$P\{X_m \in \overline{N_\varepsilon^m}\} > p$$

[1] It was Prof. D.G. Kendall who noticed that this result holds true in such general conditions and provided the proof.

for m sufficiently large and arbitrary positive ε. But if we choose $\varepsilon < p/s$, we deduce that there must be more than s states which is a contradiction. Therefore $\mathscr{T}$ is finite, and if we denote by N the number of the atomic sets in $\mathscr{T}$, then by Corollary 3 after Theorem 1 we get

$$N = \sum_{i=1}^{t} \frac{P_i}{1-x_i}, \tag{16}$$

$x_1, \ldots, x_t$ being the distinct values assumed by γ and $P_1, \ldots, P_t$ their probabilities.

Now, given two arbitrary positive ε and η we may find two positive integers k and n such that

$$P\{|\gamma_{k,n} - \gamma| < \varepsilon\} > 1 - \eta.$$

Let us denote by $x'_1, \ldots, x'_s$ the values assumed by $\gamma_{k,n}$ on the sets

$$\{X_n = a_1\}, \ldots, \{X_n = a_s\}$$

respectively and by $E_1, \ldots, E_t$ the subsets of E with the property that for any $a_j \in E_i$, $|x'_j - x_i| < \varepsilon$, $i = 1, \ldots, t$. Further, let x_i^* be the value of $\gamma_{k,n}$ taken on the set $\{X_n = a_{i'}\}$ where $P_{i'}^{(n)} = \max_{a_j \in E_i} P_j^{(n)}$. According to Lemma 3, $x_i^* \leqq 1 - P_{i'}^{(n)}$. Then if we denote by N'_i the number of states in E_i and set $P'_i = \sum_{j \in E_i} P_j^{(n)}$, we get

$$\frac{P'_i}{1-x_i} \leqq N'_i, \quad i = 1, \ldots, t. \tag{17}$$

Further, we may choose ε, η, k and n such that $E_1, \ldots, E_t$ be disjoint and

$$\sum_{i=1}^{t} \frac{P_i}{1-x_i} < \sum_{i=1}^{t} \frac{P'_i}{1-x_i^*} + \lambda \tag{18}$$

with $\lambda < 1$.

Therefore by (16), (17) and (18) we get

$$N < N'_1 + \cdots + N'_t + \lambda < s + 1.$$

As N is an integer we deduce that $N \leqq s$ and the proof is complete.

4. The Continuous Parameter Case

The above results can be extended to the case of a continuous parameter Markov process with a denumerable set of states $\{X(t), t \in [0, \infty)\}$. For such a process the tail σ-algebra $\mathscr{T}$ is defined as

$$\mathscr{T} = \bigcap_{t \in [0, \infty)} \mathscr{F}_t^{\infty},$$

$\mathscr{F}_{t_1}^{t_2}$ being the σ-algebra generated by the random variables $\{X(s), t_1 \leqq s \leqq t_2\}$, $t_1, t_2 \in [0, \infty)$, $t_1 < t_2$. We have

Theorem 4. *Let $\{X(t), t \in [0, \infty)\}$ be a Markov process having a denumerable set of states, $\{t_n, n \geqq 1\}$ an arbitrary increasing sequence of positive numbers with $\lim_{n \to \infty} t_n = \infty$ and $\{\gamma_{t_k, t_n}(\omega)\}$ a sequence of random variables defined in the manner* *of* (2). *Then*

(i) *There exists the limits*

$$\lim_{n\to\infty} \gamma_{t_k, t_n}(\omega) = \gamma_{t_k}(\omega) \quad a.s.,$$

$$\lim_{k\to\infty} \gamma_{t_k}(\omega) = \gamma(\omega) \quad a.s.;$$

(ii) $\gamma(\omega)=1$ *for almost all* $\omega \in T_0$

and

$\gamma(\omega)=1-P(T_i)$ *for almost all* $\omega \in T_i$, $(i=1,2,\ldots)$.

Proof. (a) is Theorem 1 applied to the Markov chain $\{X_{t_n}, n \geqq 1\}$.

To prove (b) we may notice that as in the proof of Theorem 8 of [8] we deduce that $\{X_{t_n}, n \geqq 1\}$ provides all the information about $\mathscr{T}$. Indeed an obvious extension of Lemma 1 to Markov processes yields

$$\gamma_{t_k, t_n} = \sup_{A \in \mathscr{F}_0^k} (P\{A\} - P\{A | \mathscr{F}_{t_n}^{\infty}\}).$$

Now, if we remark that in (10) $\mathscr{T}$ may be replaced by $\mathscr{T}' = \bigcap_{n=1}^{\infty} \mathscr{F}_{t_n}^{\infty}$, we deduce that $\mathscr{T}'$ is the tail σ-algebra of $\{X_{t_n}, n \geqq 1\}$. But $\mathscr{T}' = \mathscr{T}$ and the proof is complete.

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