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Combinatorial Solution of the Buffon Sylvester Problem

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1. Introduction

In the present note we consider integrals with respect to a certain measure μ, introduced on the set $\mathfrak{M}$ of geodesic lines (g-lines) which belong to a "simple domain" $\mathscr{S}$ on a smooth surface in $\mathbb{R}^3$. A domain $\mathscr{S}$ is called simple if there is a single g-path inside $\mathscr{S}$ connecting every pair of points from the interior of $\mathscr{S}$.

In the case where $\mathscr{S}$ has constant curvature the measure μ coincides with the usual invariant measure. In the general case many properties of the latter are shared by μ. In particular [1]

$$d\mu = \sin\psi \, dl \, d\psi \tag{1}$$

where ψ is the angle of intersection of $g \in \mathfrak{M}$ with a fixed line $\mathscr{K}$ and l is the longitudinal coordinate of the point of intersection on $\mathscr{K}$.

We show (§ 2) that the correctness of such an introduction of μ follows from a certain symmetry principal in the theory of g-lines. The Buffon-Sylvester problem for a simple $\mathscr{S}$ is posed as follows (see [2, 3] for planar and spacial cases):

Let a set of needles $\{\Delta_i\}_1^n$ (a needle Δ_i is an open finite segment of $g \in \mathfrak{M}$) be fixed on $\mathscr{S}$. Let

$$A_i = \{g \in \mathfrak{M};\ g \cap \Delta_i \neq \emptyset\},$$

$$E = \bigcap_1^n A_i, \qquad F = \bigcup_1^n A_i.$$

Find $\mu(E)$ and $\mu(F)$.

The method of "invariant imbedding" [3] gave the answer in the case of a planar $\mathscr{S}$:

$$\mu(E) = 2\sum_{T'} \rho\, I_{n-1}(g) + \sum_{W^-} \rho\, I_{n-2}(g) - \sum_{W^+} \rho\, I_{n-2}(g). \tag{2}$$

(2) is valid under the additional assumption that the set of endpoints of the needles $\{\Delta_i\}$ is nondegenerate in the sense that no three points from the set $\{\mathscr{P}_i\}$ of the needles' endpoints lie on the same g-line. In (2) T', W^- and W^+ are disjoint subsets of the set $\{g \in \mathfrak{M};$ a pair of points from $\{\mathscr{P}_i\}$ lies on $g\}$, ρ is always the distance between the points in the pair, $T' = \{g \in \mathfrak{M};$ a needle lies on $g\}$, $W^- = \{g \in \mathfrak{M};$ the two needles with endpoints on g lie entirely in different halfplanes with respect to $g\}$, $W^+ = \{g \in \mathfrak{M};$ the two needles with endpoints on g lie in the same halfplane with respect to $g\}$, $I_k(g) = 1$ if g intersects k needles and 0 otherwise.

Although the method of invariant imbedding is no more useful in the case of general $\mathscr{S}$, the result itself holds, and this is shown in § 3. In the general case, in (2) ρ stands for geodesic distance, and due to the simplicity of $\mathscr{S}$, W^+ and W^- are also well defined.

A process of integration of (2), analogous to that used in [3] reveals some interesting properties of g-convex domains on $\mathscr{S}$. In particular we give an isoperimetric inequality which reduces to the classical one when $\mathscr{S}$ has constant curvature. We demonstrate some extremal properties of the halfsphere in the set of simple domains on surfaces in $\mathbb{R}^3$.

2. The Symmetry Principal

Consider the problem of defining the measure μ on M.

Let Δ be an infinitesimal segment of $g \in \mathfrak{M}$ on $\mathscr{S}$, dl the length of Δ. Consider the set $\{g \in \mathfrak{M};\, g \cap \Delta \neq \emptyset$, the angle of intersection at $g \cap \Delta$ belongs to $(\psi, \psi + d\psi)\}$. Assume that the μ-measure of the above set may be presented in the form $d\mu = F(g, l, \psi)\, dl\, d\psi$. It is natural to ask whether there exists a μ for which

$$F(g, l, \psi) = F(\psi), \qquad \text{independently of } g \text{ and } l. \tag{3}$$

Let us show that the condition (3) defines F up to a constant factor.

Let Δ_1 and Δ_2 be two infinitesimal g-segments on $\mathscr{S}$ of length dl_1 and dl_2, $\mathscr{P}_1$ and $\mathscr{P}_2$ be the midpoints of Δ_1 and Δ_2 respectively. The μ-measure of the set $\{g \in \mathfrak{M};\, \Delta_1 \cap g \neq \emptyset,\, \Delta_2 \cap g \neq \emptyset\}$ under (3) may be written in both forms

$$d\mu = F(\psi_1)\, dl_1\, d\psi_1 = F(\psi_2)\, dl_2\, d\psi_2 \tag{4}$$

where ψ_i is the polar angle (measured from Δ_i) of $\mathscr{P}_j$, $d\psi_i$ is the difference between the polar angles of the endpoints of Δ_j in the system of geodesic polar coordinates on $\mathscr{S}$ with $\mathscr{P}_i$ as center, $i = 1, 2$, $j = 2, 1$. In general the relation between the length $d\lambda_i$ of the infinitesimal arc through $\mathscr{P}_j$ of a geodesic circle centered at $\mathscr{P}_i$ and the corresponding polar angle $d\psi_i$ has the form $d\lambda_i = h(\mathscr{P}_i, \mathscr{P}_j)\, d\psi_i$, $i = 1, 2$ with some function h. The geodesic radius is always orthogonal to the geodesic circle. Therefore in (4) we have to insert

$$d\psi_i = \frac{\sin \psi_j\, dl_j}{h(\mathscr{P}_i, \mathscr{P}_j)}, \qquad i = 1, 2.$$

This yields

$$d\mu = \frac{F(\psi_1) \sin \psi_2}{h(\mathscr{P}_1, \mathscr{P}_2)}\, dl_1\, dl_2 = \frac{F(\psi_2) \sin \psi_1}{h(\mathscr{P}_2, \mathscr{P}_1)}\, dl_1\, dl_2. \tag{5}$$

Since ψ_1 and ψ_2 are independent variables we easily conclude from (5) that

$$F(\psi) = C \sin \psi, \qquad h(\mathscr{P}_1, \mathscr{P}_2) = h(\mathscr{P}_2, \mathscr{P}_1). \tag{6}$$

Obviously the symmetry condition (6) is necessary and sufficient for the existence of a μ with the property (1). The existence of such a μ has been established for simple $\mathscr{S}$ (see [1]), and this proves (6) as well as

$$d\mu = \frac{\sin \psi_1 \sin \psi_2}{h}\, dl_1\, dl_2. \tag{7}$$

Note that for $\mathscr{S}$ of constant curvature κ, h depends only on the distance r between $\mathscr{P}_i$ and $\mathscr{P}_j$. For instance

$$h = r \quad \text{for } \kappa = 0, \qquad h = \sin r \quad \text{for } \kappa = 1 \quad \text{and} \quad h = \operatorname{sh} r \quad \text{for } \kappa = -1.$$

It follows by integration of (1) that for every g-convex domain $\mathscr{D}\subseteq\mathscr{S}$ the μ-measure of the set $\{g\in\mathfrak{M};\, g\cap\mathscr{D}\neq\emptyset\}$ is equal to the length of the perimeter of $\mathscr{D}$. We use this fact in § 3.

3. Convex Polygons

To prove (2) we start with a somewhat more general problem. Namely let $\mathscr{D}_1,\dots,\mathscr{D}_n$ be a set of g-convex open polygons on $\mathscr{S}$. Let

$$B_i=\{g\in\mathfrak{M};\, g\cap\mathscr{D}_i\neq\emptyset\},\qquad E=\bigcap_1^n B_i,\qquad F=\bigcup_1^n B_i.$$

$$I_k(g)=1 \quad \text{if } g \text{ intersects } k \text{ polygons, 0 otherwise.}$$

The results of [3] again suggest the form of $\mu(E)$ and $\mu(F)$ in the case of non-degenerate $\{\mathscr{P}_i\}$, $\{\mathscr{P}_i\}=$the set of all vertices of $\{\mathscr{D}_i\}_1^n$:

$$\mu(E)=\sum_T \rho\, I_{n-1}(g)+\sum_{W^-}\rho\, I_{n-2}(g)-\sum_{W^+}\rho\, I_{n-2}(g), \tag{8}$$

$$\mu(F)=\sum_T \rho\, I_0(g)+\sum_{W^+}\rho\, I_0(g)-\sum_{W^-}\rho\, I_0(g). \tag{9}$$

Here T, W^- and W^+ are disjoint subsets of the set $\mathfrak{M}_0=\{g\in\mathfrak{M};$ a pair of points from $\{\mathscr{P}_i\}$ lies on $g\}$, $\rho=\rho(g)$ is the g-distance between the points of the pair, $T=\{g\in\mathfrak{M}_0;$ a side of a polygon from $\{\mathscr{D}_i\}$ lies on $g\}$. To define W^- and W^+ note that each $g\in\mathfrak{M}$ divides $\mathscr{S}$ into two disjoint regions which we call g-parts. $W^-=\{g\in\mathfrak{M}_0;$ the two polygons which have vertices on g lie entirely in different g-parts$\}$, $W^+=\{g\in\mathfrak{M}_0;$ the two polygons which have vertices on g lie entirely on the same g-part$\}$. Denote by J a subset of the $\{1,\dots,n\}$. We add J to the symbols introduced above when referring to the set $\{\mathscr{D}_i\}_{i\in J}$.

By the universal theorem of logic

$$\mu(E)=\sum_{i=1}^n(-1)^{i+1}\Pi_i,\qquad \Pi_i=\sum_{\operatorname{card} J=i}^n \mu(F_J), \tag{10}$$

$$\mu(F)=\sum_{i=1}^n(-1)^{i+1}\pi_i,\qquad \pi_i=\sum_{\operatorname{card} J=i}\mu(E_J). \tag{11}$$

Let us verify that the right side of (8) and (9) satisfy (10) and (11).

Upon setting $I_{T(J)}=1$ if $g\in T(J)$, 0 otherwise, changing the order of summation yields

$$\sum_{\operatorname{card} J=i}\ \sum_{T(J)}\rho\, I_0(g,J)=\sum_T\rho\sum_{\operatorname{card} J=i} I_{T(J)}\, I_0(g,J)=\sum_T\rho\, C_{n-r-1}^{i-1},$$

$r=r(g)$ is the number of polygons from $\{\mathscr{D}_i\}_1^n$ intersected by g.

Quite analogously

$$\sum_{\operatorname{card} J=i}\ \sum_{W^\pm(J)}\rho\, I_0(g,J)=\sum_{W^\pm}\rho\, C_{n-r-2}^{i-2}.$$

Furthermore

$$\sum_{i=1}^n(-1)^{i+1}\sum_T\rho\, C_{n-r-1}^{i-1}=\sum_T\rho\sum_{i=1}^n(-1)^{i+1}C_{n-r-1}^{i-1}=\sum_T\rho\, I_{n-1}(g),$$

$$\sum_{i=1}^n(-1)^{i+1}\sum_{W^\pm}\rho\, C_{n-r-2}^{i-2}=\sum_{W^\pm}\rho\sum_{i=1}^n(-1)^{i+1}C_{n-r-2}^{i-2}=-\sum_{W^\pm}\rho\, I_{n-2}(g).$$

We see that substitution of the values for $\mu(F, J)$ from (9) into (10) yields the right hand side of (8). In the same way we can obtain the right hand side of (9) by substituting (8) into (11).

Clearly this by no means proves the Eqs. (8) and (9). For instance, the quantities $\sum_T \rho\, I_{n-1}(g)$ and $\sum_T \rho\, I_0(g)$ also satisfy (10) and (11).

We will prove (8) and (9) by induction with respect to n in the case of non-overlapping polygons $\{\mathscr{D}_i\}$.

According to the remark at the end of § 2, (8) and (9) are true for a single polygon ($n=1$). Assume that (8) and (9) are true for sets of $n-1$ nonoverlapping polygons with nondegenerate $\{\mathscr{P}_i\}$. Let us show that (8) and (9) hold for $\{\mathscr{D}_i\}_1^n$.

Consider the possibilities:

a) The set E is empty.

b) The set E is not empty.

If a) is the case, $\pi_n = \mu(E) = 0$. For $\mu(E_J)$, card $J < n$, (8) is true by assumption, and (9) is verified by using (11) as above. At the same time, the indicators on the right hand side of (8) vanish, which proves that (8) is also correct.

Now let us look at case b). Choose $g_0 \in E$ and assume that the numbering of $\{\mathscr{D}_i\}_1^n$ is by order of their intersection with g_0.

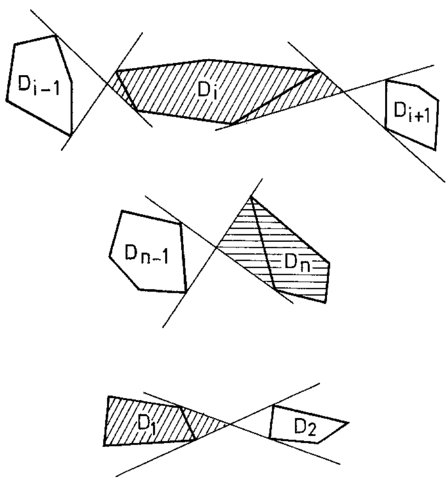

Fig. 1. $\tilde{\mathscr{D}}_i$ are the shaded polygons, $i = 1, \ldots, n$

Using the lines from W^- construct for each $\mathscr{D}_i$ the polygon $\tilde{\mathscr{D}}_i$ by the method shown in Fig. 1. We mark the symbols introduced above with tildas when referring to the set $\{\tilde{\mathscr{D}}_i\}_1^n$.

Lemma. $E = \tilde{E}$.

The proof is obtained by induction with respect to n, and draws on the uniqueness of the point of intersection of two g-lines on $\mathscr{S}$.

Another important property of the set $\{\tilde{\mathscr{D}}_i\}_1^n$ is that $\mu(\tilde{F}) =$ the length of the perimeter of the minimal convex hull of the set $\{\tilde{\mathscr{D}}_i\}$.

In general the sets $\{\tilde{\mathscr{D}}_i\}_{i\in J}$ do not possess nondegenerate $\{\tilde{\mathscr{P}}_i\}$. For this reason we have to derive the limiting form of (9) for $\{\tilde{\mathscr{D}}_i\}_{i\in J}$, card $J<n$. This happens to be

$$\mu(\tilde{F}_J)=\sum_{\tilde{T}(J)}\rho\,\tilde{I}_0(g,J)+\sum_{\tilde{W}(J)}\rho\,\tilde{I}_0(g,J)-\sum_{\tilde{W}^-(J)}\rho\,\tilde{I}_0(g,J). \tag{12}$$

The following symbols need special defining:

$\tilde{T}(J)$ is the set of all pairs (a,g), where a belongs to the set of all sides of the polygons $\{\tilde{\mathscr{D}}_i\}_{i\in J}$. g is the geodesic through a. $\rho=\rho(a)$ is the length of a.

Also when $g\in W^-(J)$ it may occur that three points from $\{\tilde{P}_i\}$ lie on g; in this case ρ equals the distance between the outermost points on g. We emphasise that (12) is true for card $J<n$ because of our assumption. But in fact (12) remains true for card $J=n$ also, since its right hand side in this case reduces to the perimeter of the minimal convex hull of $\{\tilde{\mathscr{D}}_i\}_1^n$.

By the lemma above

$$\mu(E)=\sum_1^n(-1)^{i+1}\tilde{\Pi}_i,\qquad \tilde{\Pi}_i=\sum_{\text{card }J=i}\mu(\tilde{F}_J). \tag{13}$$

Substitution of (12) into (13) yields

$$\mu(E)=\sum_{\tilde{T}}\rho\,\tilde{I}_{n-1}(g)+\sum_{\tilde{W}^-}\rho\,\tilde{I}_{n-2}(g)-\sum_{\tilde{W}^+}\rho\,\tilde{I}_{n-2}(g). \tag{14}$$

But the right hand side of (14) does not change when the tildas are removed. Hence (8) is proved. Now we are in a position to check (9) by substituting (8) into (11). With this the proof by induction is completed.

In the next section we apply Eq. (2) only to needles. The latter is obtained from (8) by the following procedure.

Let $I_i^{(1)},\ldots,I_i^{(s_i)}$ be the parts into which Δ_i is partitioned by other needles of the set $\{\Delta_i\}_1^n$. Choose integers $k_1,\ldots,k_n$, $1\leqq k_i\leqq s_i$. Denote

$$E_{k_1,\ldots,k_n}=\{g;\,g\cap I_i^{(k_i)}\neq\emptyset,\ i=1,\ldots,n\}.$$

Obviously

$$\mu(E)=\sum\mu(E_{k_1,\ldots,k_n}).$$

Each $I_i^{(k_i)}$ may be considered as an infinitely narrow polygon and (8) may be used to express $\mu(E_{k_1,\ldots,k_n})$. Then (2) is obtained through an easy summation.

4. Integration

Let $\mathscr{D}\subset\mathscr{S}$ be a strictly g-convex bounded domain with piecewise smooth boundary $\partial\mathscr{D}$, $\mathfrak{M}_{\mathscr{D}}=\{g\in\mathfrak{M};\,g\cap\mathscr{D}\neq\emptyset\}$.

Every $(g_1,\ldots,g_n)\subset(\mathfrak{M}_{\mathscr{D}})^n$ defines a set of needles $\{\Delta_i\}_1^n$, $\Delta_i=g_i\cap D$ with a.e. nondegenerate $\{P_i\}$. The measure M_n on $(\mathfrak{M}_{\mathscr{D}})^n$ is defined by

$$dM_n=d\mu_1,\ldots,d\mu_n,\qquad \text{where each } \mu_i \text{ coincides with } \mu \text{ on } \mathfrak{M}_{\mathscr{D}}.$$

Let us integrate (2) with respect to M_n.

Firstly

$$\int \mu(E)\, dM_n = \int d\mu \int I_n(g)\, dM_n = \int (2\chi)^n\, d\mu,$$

$\chi = \chi(g)$ is the length of $g \cap \mathscr{D}$.

Then with ρ_1 for the length of Δ_1 we have

$$2 \int \sum_T \rho\, I_{n-1}(g)\, dM_n = n \int 2\rho_1\, I_{n-1}(g_1)\, dM_n = n \int (2\chi)^n\, d\mu.$$

Furthermore

$$\int \Big[\sum_{W^-} \rho\, I_{n-2}(g) - \sum_{W^+} \rho\, I_{n-2}(g)\Big]\, dM_n = 2n(n-1) \int \rho_{12}\, I_{n-2}(g_{12}) [I_{W^-} - I_{W^+}]\, dM.$$

In this writeup ρ_{12} is the distance between an "arbitrary" endpoint of Δ_1 and an "arbitrary" endpoint of Δ_2, g_{12} is the g-line through these points, $I_{W^\pm}(g_1, g_2)$ is the indicator of the set $W^\pm$.

After summing up these terms and dividing the result by $n-1$, we obtain for $n \geqq 2$

$$\int (2\chi)^n\, d\mu = n \int 2\rho_{12}\, I_{n-2}(g_{12}) [I_{W^+} - I_{W^-}]\, dM_n. \tag{15}$$

Using (1) and

$$\int I_{n-2}(g_n)\, d\mu_3, \ldots, d\mu_n = (2\rho_{12})^{n-2}$$

transform the right hand side of (15) to

$$\tfrac{1}{4} n \int (2\rho_{12})^{n-1} [I_{W^+} - I_{W^-}] \sin\psi_1 \sin\psi_2\, d\psi_1\, d\psi_2\, dl_1\, dl_2.$$

One finds readily that

$$\int_0^\pi \int_0^\pi \sin\psi_1 \sin\psi_2 [I_{W^+} - I_{W^-}]\, d\psi_1\, d\psi_2 = 4 \cos\alpha_1 \cos\alpha_2,$$

the angles α_1 and α_2 are shown on Fig. 2.

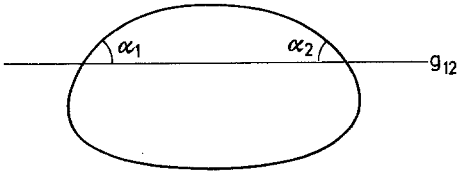

Fig. 2

So we arrive at the final result

$$\int \chi^n\, d\mu = \frac{n}{2} \iint_{(\partial\mathscr{D})^2} \rho_{12}^{n-1} \cos\alpha_1 \cos\alpha_2\, dl_1\, dl_2, \qquad n > 1. \tag{16}$$

In fact (16) holds when $\partial\mathscr{D}$ possess parts of g-lines. Let $f(x) = a_2 x^2 + a_3 x^3 + \cdots + a_n x^n$. For such polygons (16) implies

$$\int f(\chi)\, d\mu = \tfrac{1}{2} \iint_{(\partial\mathscr{D})^2} f'(\rho) \cos\psi_1 \cos\psi_2\, dl_1\, dl_2, \tag{17}$$

f' is the derivative of f.

It follows from Weierstrass' approximation theorem that (17) remains valid for every f, $f(0)=0$ with continuous f'. For nonnegative f' (17) may be written as an inequality

$$\int f(\chi)\, d\mu \leqq \tfrac{1}{2} \iint_{(\partial \mathscr{D})^2} f'(\rho)\, dl_1\, dl_2, \tag{18}$$

equality holds for the halfsphere.

The case $f(x)=x$ is of special interest. In this case we have by [1] for simple domains on the surfaces in $\mathbb{R}^3$

$\int \chi\, d\mu = \pi A$, where A is the area of $\mathscr{D}$, and (18) takes the form of the isoperimetric inequality

$\pi A \leqq \frac{1}{2} H^2$ when H is the length of $\partial \mathscr{D}$.

Another class of inequalities may be derived for convex domains on a fixed $\mathscr{S}$. For example we may compare (17) in the case of $f(x)=x$ namely,

$$\pi A = \tfrac{1}{2} \iint_{(\partial \mathscr{D})^2} \cos \alpha_1 \cos \alpha_2\, dl_1\, dl_2 \tag{19}$$

with the dual equation which follows from (7):

$$\int h\, d\mu = \tfrac{1}{2} \iint_{(\partial \mathscr{D})^2} \sin \alpha_1 \sin \alpha_2\, dl_1\, dl_2. \tag{20}$$

Adding up (19) and (20) we obtain

$$\pi A + \int h\, d\mu = \tfrac{1}{2} H^2 - \iint_{(\partial \mathscr{D})^2} \sin^2 \frac{\alpha_1 - \alpha_2}{2}\, dl_1\, dl_2, \tag{21}$$

or in a weaker form

$$\pi \mathscr{S} + \int h\, d\mu \leqq \tfrac{1}{2} H^2. \tag{22}$$

An easy calculation shows that (22) reduces to the classical, isoperimetric inequality in the case where $\mathscr{S}$ has constant curvature. It is clear from (22) that equality hold in (22) for any g-convex $\mathscr{D} \subset \mathscr{S}$ for which $\alpha_1 = \alpha_2$ for almost all $g \in \mathfrak{M}_{\mathscr{D}}$. All g-circles have this property when $\mathscr{S}$ has constant curvature. Another obvious example is the g-circle with the center at the axis of rotation, in the case when $\mathscr{S}$ is a surface of rotation.

In the general case, it might be interesting to clarify the situation.

References

1. Santaló, L. A.: Introduction to Integral Geometry. Paris: Hermann 1953
2. Sylvester, J. J.: On a funicular solution of Buffon's "Problem of the needle" in its most general form. Acta Math. **14**, 185–205 (1891)
3. Ambartzumian, R. V.: The solution to the Buffon-Sylvester problem in R^3. Z. Wahrscheinlichkeitstheorie verw. Gebiete **27**, 53–74 (1973)

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