Z. Wahrscheinlichkeitstheorie verw. Gebiete 29, 7–19 (1974)

Reproducing Kernel Hilbert Spaces and the Law of the Iterated Logarithm for Gaussian Processes*

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In this paper, we find the limit set of a sequence $((2\log n)^{-\frac{1}{2}} X_n(t), n \geqq 3)$ of Gaussian processes in $C[0,1]$, where the processes $X_n(t)$ are defined on the same probability space and have the same distribution. Our result generalizes the theorems of Oodaira and Strassen, and we also apply it to obtain limit theorems for stationary Gaussian processes, moving averages of the type $\int_0^t f(t-s)\,dW(s)$, where $W(s)$ is the standard Wiener process, and other Gaussian processes. Using certain properties of the unit ball of the reproducing kernel Hilbert space of $X_n(t)$, we derive the usual law of the iterated logarithm for Gaussian processes. The case of multidimensional time is also considered.

1. Introduction

In [8], Oodaira proved the following version of the law of the iterated logarithm for a certain class of Gaussian processes. Let $X(t)$, $t \geqq 0$, be a separable real-valued Gaussian process with $X(0)=0$, $EX(t)=0$ and continuous covariance kernel $R(s,t)=EX(s)X(t)$ satisfying:

(1) For any $T>0$, there exists a continuous nondecreasing function $g_T(h)$ such that for all t, $t+h \in [0,T]$,

$$|R(t+h,t+h)-2R(t+h,t)+R(t,t)| \leqq g_T(h) \to 0 \quad \text{as } h \to 0,$$

$$(g_T(1))^{-\frac{1}{2}} \int_1^\infty (g_T(e^{-u^2}))^{\frac{1}{2}}\,du \leqq C < \infty,$$

and $R(T,T)/g_T(1) \uparrow \infty$ as $T \to \infty$.

(2) There exists a positive function $v(r)$, $r>0$, such that

$$v(r) \uparrow \infty \quad \text{and} \quad R(rs,rt)=v(r)\,R(s,t) \quad \text{for all } r>0,\ s,t \geqq 0.$$

Let $Z_n(t)=X(nt)/(2R(n,n)\log\log n)^{\frac{1}{2}}$, $t \in [0,1]$. Oodaira's result states that under the above assumptions, the set of limit points of the sequence of functions $(Z_n(t), n \geqq 3)$ in $C[0,1]$ is with probability one contained in the set $K^* = \{h \in H(R_1)\colon \|h\|_H \leqq 1/\sigma(1)\}$, where $\sigma^2(t)=R(t,t)$, $C[0,1]$ is the space of continuous functions on $[0,1]$ with the usual sup norm $\|\cdot\|_C$, $H(R_1)$ is the reproducing kernel Hilbert space corresponding to the kernel $R(s,t)$, $0 \leqq s$, $t \leqq 1$, and $\|\cdot\|_H$ de-

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* Research supported by the Office of Naval Research under Contract Number N00014-67-A-0108-0018 at Columbia University.

notes the norm of $H(R_1)$. Oodaira also proves that with probability one, the set of limit points of $(Z_n(t), n \geqq 3)$ in $C[0,1]$ coincides with K^* if furthermore

(3) $R(s,t)$ has a representation of the form $R(s,t) = \int_0^{t \wedge s} Q(t,\lambda)\, Q(s,\lambda)\, d\lambda$, $s, t \geqq 0$, where $\int_0^t Q^2(t,\lambda)\, d\lambda < \infty$ for all $t \geqq 0$ and there is a function $u(r)$ such that $Q(rt, r\lambda) = u(r)\, Q(t,\lambda)$ for all $r > 0$, $t, \lambda \geqq 0$ and $v(r) = r u^2(r) \uparrow \infty$ as $r \uparrow \infty$, and further

(4) $$\sup_{0 \leqq t \leqq 1} \int_0^\delta Q^2(t,\lambda)\, d\lambda \to 0 \quad \text{as } \delta \to 0.$$

Oodaira's result is a generalization of an earlier result of Strassen [10] for the standard Wiener process $W(t)$, $t \geqq 0$, in which case $R(s,t) = t \wedge s$ and

$$K^* = \left\{ h \in C[0,1]: h(0) = 0,\ h \text{ is absolutely continuous and } \int_0^1 \left(\frac{dh}{dt}\right)^2 dt \leqq 1 \right\}.$$

As is well known, Strassen's result gives the usual law of the iterated logarithm for Brownian motion as an immediate corollary, and so does Oodaira's result for the Gaussian process $X(t)$ (see Section 4 below); thus we have

(5) $$\limsup_{t \to \infty} X(t)/\{2R(t,t) \log\log t\}^{\frac{1}{2}} = 1 \quad \text{a.e.}$$

Now suppose that $U(t)$, $t \geqq 0$, is a separable Gaussian process with mean 0 and continuous covariance $R(s,t)$ such that

$$\lim_{t \to \infty} R(t,t) = \sigma^2 > 0, \qquad \lim_{T \to \infty} \sup_{|t-s| > T} R(s,t) \leqq 0$$

and for any s, t with $|t-s| \leqq 1$, $E\big(U(t) - U(s)\big)^2 \leqq \psi^2(|t-s|)$, where ψ is a non-decreasing and continuous function satisfying $\int_1^\infty \psi(e^{-x^2})\, dx < \infty$. Nisio [7] has proved that

(6) $$\limsup_{t \to \infty} U(t)/(2 \log t)^{\frac{1}{2}} = \sigma \quad \text{a.e.}$$

In this case, if imitating Oodaira and Strassen, one defines $V_n(t) = U(nt)/(2 \log n)^{\frac{1}{2}}$, $t \in [0,1]$, then it turns out that the set of limit points of $V_n(t)$ in $C[0,1]$ is empty with probability one. To see this, we note that by [7], $\lim_{n \to \infty} \max_{0 \leqq t \leqq 1} |V_n(t)| = \sigma$ a.e., and so with probability one, the constant zero function cannot be a limit point of $V_n(t)$ in $C[0,1]$. Now let I be a non-empty interval with rational end-points and let $\delta > 0$ be a rational number. Then with probability one, for every $f \in C[0,1]$ such that $f \geqq \delta$ on I, f cannot be a limit point of $V_n(t)$ in $C[0,1]$ since

(7) $$\lim_{n \to \infty} \min_{t \in I} V_n(t) = -\lim_{n \to \infty} \max_{t \in I} (-V_n(t)) = -\sigma \quad \text{a.e.}$$

(cf. [5] and [7]). Similarly since $\lim_{n \to \infty} \max_{t \in I} V_n(t) = \sigma$ a.e., it follows that with probability one, for every $g \in C[0,1]$ such that $g \leqq -\delta$ on I, g cannot be a limit point of $V_n(t)$ in $C[0,1]$.

We note that in the case of the Wiener process $W(t)$, the sequence $(n^{-\frac{1}{2}} W(nt), n \geqq 1)$ is a sequence of identically distributed copies of the process $W(t)$, $t \in [0,1]$. In Oodaira's theorem, condition (2) implies that for each $n \geqq 1$, the process

$v^{-\frac{1}{2}}(n)\,X(nt)$ has the same distribution as the process $X(t)$. $t\in[0,1]$, which corresponds to the reproducing kernel Hilbert space $H(R_1)$. For the process $U(t)$ considered in the preceding paragraph, assuming $U(t)$ to be stationary, then the sequence $(U(n+t),\, n\geqq 1)$ is a sequence of identically distributed copies of the process $U(t)$, $t\in[0,1]$. This suggests that instead of considering $V_n(t)$ as defined in the previous paragraph, one should define $\tilde{V}_n(t)=U(n+t)/(2\log n)^{\frac{1}{2}}$, $t\in[0,1]$. It turns out that under some mixing condition, the set of limit points of the process $\tilde{V}_n(t)$ in $C[0,1]$ coincides a.e. with the unit ball in the reproducing kernel Hilbert space corresponding to the process $U(t)$, $t\in[0,1]$.

In Section 2, we shall prove an intrinsic form of the Oodaira-Strassen theorem. Section 3 shows that this result can be applied to obtain Strassen's theorem for $W(t)$ and Oodaira's theorem for the process $X(t)$ and that it also gives the limit set of the process $\tilde{V}_n(t)$ as an immediate corollary. Section 4 considers some properties of reproducing kernel Hilbert spaces and derives the usual law of the iterated logarithm for Gaussian processes from the corresponding version involving the unit balls in the reproducing kernel Hilbert spaces. In Section 5, we extend our results to Gaussian processes with multidimensional time.

2. The Limit Set of a Sequence of Gaussian Processes in $C[0,1]$

We shall call a family $\mathscr{F}$ of random variables on the same probability space *Gaussian* if for any finite number of random variables $Y_1,\dots,Y_m\in\mathscr{F}$, $(Y_1,\dots,Y_m)$ has a multivariate normal distribution. By the conditional expectaion of a random variable Z given $\mathscr{F}$, denoted by $E(Z|\mathscr{F})$, we shall mean the conditional expectation of Z given the σ-field generated by $\mathscr{F}$. If H is a Hilbert space, we shall call the set of all elements with norm $\leqq 1$ the unit ball of H.

Theorem 1. *Suppose $X(t)$, $t\in[0,1]$, is a separable real-valued Gaussian process with mean 0 and continuous covariance $R(s,t)$ satisfying*

(A) $$E(X(t)-X(s))^2\leqq\psi^2(|t-s|),\qquad t,s\in[0,1]$$

where ψ is a continuous nondecreasing function on $[0,1]$ such that $\int_1^\infty \psi(e^{-u^2})\,du<\infty$. Let $(X_n(t),\, n\geqq 1)$ be a sequence of Gaussian processes defined on the same probability space and having the same distribution as the process $X(t)$, and let

$$Y_n(t)=(2\log n)^{-\frac{1}{2}}\,X_n(t).$$

Then with probability one, the sequence $(Y_n(t),\, n\geqq 3)$ is relatively compact in $C[0,1]$ and its set of limit points in $C[0,1]$ is contained in the unit ball K of the reproducing kernel Hilbert space $H(R)$ corresponding to the process $X(t)$. Letting $\mathscr{F}_n=\{X_j(t)\colon t\in[0,1],\, 1\leqq j\leqq n\}$ and $\mathscr{F}=\bigcup_{n=1}^{\infty}\mathscr{F}_n$, suppose furthermore that

(B) *$\mathscr{F}$ is a Gaussian family of random variables such that*

(8) $$\lim_{\substack{n\to\infty\\ m-n\to\infty}} E\{E(X_m(t)|\mathscr{F}_n)\}^2=0\qquad \text{for each } t\in[0,1].$$

Then with probability one, the set of limit points of $(Y_n(t),\, n\geqq 3)$ in $C[0,1]$ coincides with the set K.

Lemma 1. *Suppose $X(t)$, $t\in[0,1]$, is a Gaussian process with mean 0 and continuous covariance $R(s,t)$ satisfying condition* (A), *and let $X_n(t)$, $Y_n(t)$, $H(R)$ be as defined in Theorem 1. Let $(e_j(t), j\geqq 1)$ be a complete orthonormal system* (CONS) *in $H(R)$, and let $\psi_n\colon H(R)\to L_2(X_n)$ be the isometric isomorphism (defined by $\psi_n(R(t,\cdot))=X_n(t)$) between $H(R)$ and the closed linear manifold $L_2(X_n)$ spanned by $\{X_n(t), t\in[0,1]\}$. Let $\xi_n^{(j)}=\psi_n(e_j)$ be the Gaussian random variable corresponding to e_j. Then given $\varepsilon>0$, there exists $k_0=k_0(\varepsilon)$ such that for $k\geqq k_0$,*

$$P\left[\sup_{0\leqq t\leqq 1}\left|Y_n(t)-(2\log n)^{-\frac{1}{2}}\sum_{j=1}^{k}\xi_n^{(j)}e_j(t)\right|<\varepsilon \text{ for all large } n\right]=1. \tag{9}$$

Proof. The idea of the proof is similar to that of Lemma 1 in [8]. Let

$$U_n^{(k)}(t)=X_n(t)-\sum_{j=1}^{k}\xi_n^{(j)}e_j(t).$$

Then $EU_n^{(k)}(t)=0$ and making use of the fact that

$$E\xi_n^{(j)}X_n(s)=\langle e_j, R(s,\cdot)\rangle=e_j(s),$$

it can be shown that

$$EU_n^{(k)}(t)\,U_n^{(k)}(s)=R(s,t)-\sum_{j=1}^{k}e_j(s)\,e_j(t). \tag{10}$$

We shall denote the expression in (10) by $\Gamma^{(k)}(t,s)$. Since $\sum_{j=1}^{k}e_j^2(t)$ converges to $R(t,t)$ uniformly in $t\in[0,1]$ as $k\to\infty$, we can choose k_0 and p sufficiently large such that for $k\geqq k_0$,

$$\varepsilon_k=\varepsilon\left\{\left(\sup_{0\leqq t\leqq 1}\Gamma^{(k)}(t,t)\right)^{\frac{1}{2}}+4\int_1^\infty\psi(p^{-u^2})\,du\right\}^{-1}>1. \tag{11}$$

Let $A_n^{(k)}=\left[\sup_{0\leqq t\leqq 1}|U_n^{(k)}(t)|\geqq\varepsilon(2\log n)^{\frac{1}{2}}\right]$. Since

$$\begin{aligned}E\left(U_n^{(k)}(t)-U_n^{(k)}(s)\right)^2&=E\left(X(t)-X(s)\right)^2-\sum_{j=1}^{k}\left(e_j(t)-e_j(s)\right)^2\\&\leqq\psi^2(|t-s|),\end{aligned}$$

we can apply Fernique's lemma (cf. [2]) to obtain that for $\varepsilon_k(2\log n)^{\frac{1}{2}}>(1+4\log p)^{\frac{1}{2}}$,

$$PA_n^{(k)}\leqq 4p^2\int_{\varepsilon_k(2\log n)^{\frac{1}{2}}}^{\infty}e^{-u^2/2}\,du.$$

Hence for $k\geqq k_0$, it follows from (11) that $\sum_n PA_n^{(k)}<\infty$, and so by the Borel-Cantelli lemma we obtain the desired conclusion. Q.E.D.

Proof of Theorem 1. It is well known that under assumption (A), the processes $X_n(t)$ have continuous sample paths a.e. Also, the set ρK is a compact subset of $C[0,1]$ for all $\rho>0$ (cf. [8]). To prove that under assumption (A), K contains with probability one all the limit points of $Y_n(t)$, given $\varepsilon>0$, we can choose k by Lemma 1 such that (9) holds. Letting

(12) $$Z_n^{(k)}(t)=(2\log n)^{-\frac{1}{2}}\sum_{j=1}^{k}\xi_n^{(j)}\,e_j(t),$$

it can be shown as in Section 5 of [8] that $P[Z_n^{(k)}$ belongs to $K_{\varepsilon\Gamma}$ for all large $n]=1$, where K_η denotes the η-neighborhood of K in $C[0,1]$ and $\Gamma=\sup_{0\le t\le 1} R^{\frac{1}{2}}(t,t)$. Therefore $P[Y_n$ belongs to $K_{\varepsilon+\varepsilon\Gamma}$ for all large $n]=1$, and so $\{Y_n\colon n\geqq 3\}$ is relatively compact in $C[0,1]$.

We now prove that under assumptions (A) and (B), K is contained with probability one in the set of limit points of $Y_n(t)$ in $C[0,1]$. Because of the compactness of K in $C[0,1]$, it suffices to show that given any $g\in K$ and $0<\varepsilon<\frac{1}{2}$,

$$P[\|Y_n-g\|_C\leqq\varepsilon(6+\|g\|_C)\text{ i.o.}]=1$$

(cf. [8], p. 296). Letting $h=(1-\varepsilon)g$, we have $\|g-h\|_C=\varepsilon\,\|g\|_C$, and so we need only show that $P[\|Y_n-h\|_C\leqq 6\varepsilon\text{ i.o.}]=1$. Since h has the expansion $h(t)=\sum_{j=1}^{\infty}h_j\,e_j(t)$ with the above series converging uniformly in $t\in[0,1]$, we can choose $k\geqq k_0$ such that

$$\left\|h(t)-\sum_{j=1}^{k}h_j\,e_j(t)\right\|_C<\varepsilon,$$

where k_0 is given by Lemma 1. Defining $Z_n^{(k)}(t)$ as in (12), we need only show that

$$P\left[\left\|Z_n^{(k)}(t)-\sum_{j=1}^{k}h_j\,e_j(t)\right\|_C\leqq 4\varepsilon\text{ i.o.}\right]=1.$$

For $j=1,\ldots,k$, we define

$$B_n^{(j)}=[\|\{\xi_n^{(j)}-(2\log n)^{\frac{1}{2}}\,h_j\}\,e_j(t)\|_C\leqq(4\varepsilon/k)(2\log n)^{\frac{1}{2}}],$$
$$C_n^{(j)}=[|\xi_n^{(j)}-(2\log n)^{\frac{1}{2}}\,h_j|\leqq(4\varepsilon/k\Gamma)(2\log n)^{\frac{1}{2}}]$$

where $\Gamma=\sup_{0\le t\le 1}R^{\frac{1}{2}}(t,t)\geqq\|e_j\|_C$ (see (26) in Section 4). It is easy to see that $B_n^{(j)}\supset C_n^{(j)}$ and

$$\left[\left\|Z_n^{(k)}(t)-\sum_{j=1}^{k}h_j\,e_j(t)\right\|_C\leqq 4\varepsilon\right]\supset\bigcap_{j=1}^{k}B_n^{(j)}.$$

Therefore it suffices to show that $P\left[\bigcap_{j=1}^{k}C_n^{(j)}\text{ i.o.}\right]=1$. (We remark that it is not enough just to show that $P[C_n^{(j)}\text{ i.o.}]=1$ for each fixed $j=1,\ldots,k$, and so the proof in [8] in fact needs additional argument.)

Let $\psi\colon H(R)\to L_2(X)$ be the isometric isomorphism defined by $\psi(R(t,\cdot))=X(t)$, and let $\xi^{(i)}=\psi(e_i)$. Noting that $\xi^{(1)},\ldots,\xi^{(k)}$ are orthogonal with mean 0 and variance 1, we can choose $a_1^{(j)},\ldots,a_{m_j}^{(j)}$ and $t_1^{(j)},\ldots,t_{m_j}^{(j)}$ such that defining $\zeta^{(j)}=\sum_{i=1}^{m_j}a_i^{(j)}X(t_i^{(j)})$, we have

(13) $\zeta^{(1)},\ldots,\zeta^{(k)}$ are orthogonal and for $j=1,\ldots,k$,

$$E(\xi^{(j)}-\zeta^{(j)})^2<(\varepsilon/k\Gamma)^2,\qquad E|\zeta^{(j)}|^2>1-\varepsilon.$$

To simplify the notation, we shall write $\zeta^{(j)} = \sum_{i=1}^{m} a_i^{(j)} X(t_i)$, where $a_i^{(j)}$ may equal 0. Define $\zeta_n^{(j)} = \sum_{i=1}^{m} a_i^{(j)} X_n(t_i)$. Then for all n,

$$E(\xi_n^{(j)} - \zeta_n^{(j)})^2 = E(\xi^{(j)} - \zeta^{(j)})^2 < (\varepsilon/k\Gamma)^2,$$

and a simple application of the Borel-Cantelli lemma shows that $P[D_n^{(j)}$ for all large $n] = 1$, $j = 1, \ldots, k$, where

$$D_n^{(j)} = [|\xi_n^{(j)} - \zeta_n^{(j)}| \leqq (\varepsilon/k\Gamma)(2\log n)^{\frac{1}{2}}].$$

By (8), we can choose n_0 and $\nu \geqq 1$ such that for $n \geqq n_0$,

$$(14) \quad \sum_{i=1}^{m} \sum_{p=1}^{m} |a_i^{(j)} a_p^{(j)}| \, E^{\frac{1}{2}}\{E(X_{\nu n}(t_i) | \mathscr{F}_{\nu(n-1)})\}^2 E^{\frac{1}{2}}\{E(X_{\nu n}(t_p) | \mathscr{F}_{\nu(n-1)})\}^2 < \tfrac{1}{2}(\varepsilon/k\Gamma)^2, \quad j = 1, \ldots, k.$$

Let $U_{\nu n}^{(j)} = E(\zeta_{\nu n}^{(j)} | \zeta_{\nu n_0}^{(1)}, \ldots, \zeta_{\nu n_0}^{(k)}; \ldots; \zeta_{\nu(n-1)}^{(1)}, \ldots, \zeta_{\nu(n-1)}^{(k)})$, $V_{\nu n}^{(j)} = \zeta_{\nu n}^{(j)} - U_{\nu n}^{(j)}$. Then $E\zeta_{\nu n}^{(j)} = EU_{\nu n}^{(j)} = EV_{\nu n}^{(j)} = 0$, and

$$E|U_{\nu n}^{(j)}|^2 = E\left\{\sum_{i=1}^{m} a_i^{(j)} E(X_{\nu n}(t_i) | \zeta_{\nu n_0}^{(1)}, \ldots, \zeta_{\nu n_0}^{(k)}; \ldots; \zeta_{\nu(n-1)}^{(1)}, \ldots, \zeta_{\nu(n-1)}^{(k)})\right\}^2 < \tfrac{1}{2}(\varepsilon/k\Gamma)^2.$$

The last inequality above follows from the Schwarz inequality, (14) and the fact that

$$E\{E(X_{\nu n}(t_i) | \zeta_{\nu n_0}^{(1)}, \ldots, \zeta_{\nu n_0}^{(k)}; \ldots; \zeta_{\nu(n-1)}^{(1)}, \ldots, \zeta_{\nu(n-1)}^{(k)})\} \leqq E\{E(X_{\nu n}(t_i) | \mathscr{F}_{\nu(n-1)})\}^2.$$

Since $U_{\nu n}^{(j)}$ is normal, we obtain $\sum_n P(E_{\nu n}^{(j)}) < \infty$, $j = 1, \ldots, k$, where

$$E_{\nu n}^{(j)} = [|U_{\nu n}^{(j)}| \geqq (\varepsilon/k\Gamma)(2\log \nu n)^{\frac{1}{2}}].$$

Letting $\bar{E}_{\nu n}^{(j)}$ denote the complement of $E_{\nu n}^{(j)}$, we therefore have $P[\bar{E}_{\nu n}^{(j)}$ for all large $n] = 1$, $j = 1, \ldots, k$.

We note that $C_{\nu n}^{(j)} \supset (D_{\nu n}^{(j)} \cap \bar{E}_{\nu n}^{(j)} \cap F_{\nu n}^{(j)})$, where we define the events

$$F_{\nu n}^{(j)} = [|V_{\nu n}^{(j)} - (2\log \nu n)^{\frac{1}{2}} h_j| \leqq (2\varepsilon/k\Gamma)(2\log \nu n)^{\frac{1}{2}}],$$
$$G_{\nu n}^{(j)} = [|\zeta_{\nu n}^{(j)} - (2\log \nu n)^{\frac{1}{2}} h_j| \leqq (\varepsilon/k\Gamma)(2\log \nu n)^{\frac{1}{2}}].$$

Hence it suffices to prove $P\left[\bigcap_{j=1}^{k} F_{\nu n}^{(j)} \text{ for infinitely many } n\right] = 1$. Since $(V_{\nu n}^{(1)}, \ldots, V_{\nu n}^{(k)})$, $n \geqq n_0$, is a sequence of independent vectors, we need only show that

$$\sum_n P\left(\bigcap_{j=1}^{k} F_{\nu n}^{(j)}\right) = \infty.$$

We note that

$$P\left(\bigcap_{j=1}^{k} F_{\nu n}^{(j)}\right) \geqq P\left(\bigcap_{j=1}^{k} (G_{\nu n}^{(j)} \cap \bar{E}_{\nu n}^{(j)})\right) \geqq P\left(\bigcap_{j=1}^{k} G_{\nu n}^{(j)}\right) - \sum_{j=1}^{k} P(E_{\nu n}^{(j)}).$$

Since $\sum_{j=1}^{k} \sum_{n}^{\infty} P(E_{\nu n}^{(j)}) < \infty$, it remains to prove that $\sum_n P\left(\bigcap_{j=1}^{k} G_{\nu n}^{(j)}\right) = \infty$.

Let Φ denote the distribution function of the $N(0,1)$ distribution, and set $\sigma_j^2 = E|\zeta^{(j)}|^2$. Then $\sigma_j^2 > 1-\varepsilon$ and $E|\zeta_n^{(j)}|^2 = \sigma_j^2$ for all n. Since $\zeta_{vn}^{(1)}, \ldots, \zeta_{vn}^{(j)}$ are orthogonal,

$$\begin{aligned} P\left(\bigcap_{j=1}^{k} G_{vn}^{(j)}\right) &= \prod_{j=1}^{k} P(G_{vn}^{(j)}) \\ &\geqq \prod_{j=1}^{k} \{\Phi(\sigma_j^{-1}(2\log vn)^{\frac{1}{2}}(|h_j| + (2\varepsilon/k\Gamma))) - \Phi(\sigma_j^{-1}(2\log vn)^{\frac{1}{2}}|h_j|)\} \qquad (15) \\ &\geqq C(\log n)^{-\frac{k}{2}} \exp\left(-\left(\sum_{j=1}^{k} \sigma_j^{-2} h_j^2\right)\log n\right). \end{aligned}$$

Now h belongs to $(1-\varepsilon)K$, and so $\sum_{j=1}^{\infty} h_j^2 \leqq 1-\varepsilon$. Hence $\sum_{j=1}^{k} \sigma_j^{-2} h_j^2 \leqq 1$, and from (15), it then follows that $\sum_n P\left(\bigcap_{j=1}^{k} G_{vn}^{(j)}\right) = \infty$. Q.E.D.

Remark. By an obvious modification in the proof, Theorem 1 can be extended to vector-valued Gaussian processes. Let $X(t) = (X^{(1)}(t), \ldots, X^{(k)}(t))$, where $X^{(i)}(t)$, $t \in [0,1]$, $i = 1, \ldots, k$, are independent separable real-valued Gaussian processes with means 0 and continuous covariances $R_i(s,t)$ such that assumption (A) of Theorem 1 is satisfied by each $X^{(i)}(t)$. Let $H = H(R_1) \times \cdots \times H(R_n)$ be the product Hilbert space endowed with the inner product

$$\langle (f_1, \ldots, f_k), (g_1, \ldots, g_k) \rangle_H = \sum_{i=1}^{k} \langle f_i, g_i \rangle_{H(R_i)},$$

and let K be the unit ball of H. If $(X_n(t), n \geqq 1)$ is a sequence of Gaussian processes defined on the same probability space and having the same distribution as $X(t)$, then the set of limit points of the sequence $((2\log n)^{-\frac{1}{2}} X_n(t), n \geqq 3)$ in $C[0,1]$ can be described in terms of K as in Theorem 1.

3. Some Applications of Theorem 1

We shall now derive Oodaira's theorem from Theorem 1. Let $X(t)$, $t \geqq 0$, be the Gaussian process described in Section 1 and define

$$Z_n(t) = X(nt)/(2R(n,n)\log\log n)^{\frac{1}{2}}, \qquad t \in [0,1].$$

We shall denote the set of limit points of the sequence of functions $(Z_n(t), n \geqq 3)$ in $C[0,1]$ by $\mathscr{L}(Z_n)$, and let K be the unit ball of the reproducing kernel Hilbert space $H(R_1)$ corresponding to the kernel $R(s,t)$, $s, t \in [0,1]$. Using conditions (1) and (2) together with Fernique's lemma, Oodaira ([8], p. 291–293) has proved that with probability one, the sequence $(Z_n(t), n \geqq 3)$ is equicontinuous, and given $\varepsilon > 0$, we can choose $c > 1$ such that defining $n_r = [c^r]$, we have

$$P[\sup_{n_r \leqq n \leqq n_{r+1}} \|Z_n - Z_{n_r}\|_C < \varepsilon \text{ for all large } r] = 1. \qquad (16)$$

Defining $X_r(t) = v^{-\frac{1}{2}}(n_r) X(n_r t)$, we obtain from Theorem 1 that $\mathscr{L}((2\log r)^{-\frac{1}{2}} X_r) \subset K$ a.e., and so $\mathscr{L}(Z_{n_r}) \subset (1/\sigma(1))K = K^*$ a.e., since $\log\log n_r \sim \log r$ and $R(n,n) = v(n)\sigma^2(1)$. From (16), we then obtain that with probability one, $\mathscr{L}(Z_n) \subset K_\varepsilon^*$, where K_ε^* denotes the ε-neighborhood of K^* in $C[0,1]$. Since ε is arbitrary and K^* is closed in $C[0,1]$, $\mathscr{L}(Z_n)\subset K^*$ a.e. Now suppose further that conditions (3) and (4) also hold. We note that condition (3) implies $X(t)=\int_0^t Q(t,\lambda)\,dW(\lambda)$, and for any $t\in[0,1]$ and $m>r$, we have

$$\begin{aligned} E\{E(X_m(t)|X(s),0\leqq s\leqq n_r)\}^2 &\leqq E\{E(X_m(t)|W(s),0\leqq s\leqq n_r)\}^2 \\ &=(v(n_m))^{-1}\,E\left\{\int_0^{n_r\wedge n_m t} Q(n_m t,\lambda)\,dW(\lambda)\right\}^2 \\ &\leqq (v(n_m))^{-1}\int_0^{n_r} Q^2(n_m t,\lambda)\,d\lambda \\ &=\int_0^{n_r/n_m} Q^2(t,\lambda)\,d\lambda\to 0 \qquad \text{as } m-r\to\infty. \end{aligned}$$

Hence condition (B) of Theorem 1 is satisfied and so $\mathscr{L}((2\log r)^{-\frac{1}{2}}X_r)=K$ a.e. Therefore $\mathscr{L}(Z_n)=K^*$ a.e. It is clear from the above derivation that Oodaira's theorem still remains true if we drop condition (4).

Theorem 1 can also be applied to find the limit set of the processes $\tilde{V}_n(t)$ introduced in Section 1. The result is stated in the following corollary.

Corollary 1. *Let $X(t)$, $t\geqq 0$, be a real-valued separable stationary Gaussian process with mean 0 and continuous covariance $R(s,t)$ satisfying assumption (A). Let $\tilde{V}_n(t)=(2\log n)^{-\frac{1}{2}}X(n+t)$, $t\in[0,1]$. Then with probability one, the sequence $(\tilde{V}_n(t), n\geqq 3)$ is relatively compact in $C[0,1]$ and its set of limit points in $C[0,1]$ is contained in the unit ball K of the reproducing kernel Hilbert space $H(R)$ corresponding to the kernel $R(s,t)$, $0\leqq s,t\leqq 1$. If furthermore*

$$\lim_{t-s\to\infty} E\{E(X(t)|X(\tau),0\leqq\tau\leqq s)\}^2=0, \tag{17}$$

then with probability one, the set of limit points of $(\tilde{V}_n(t), n\geqq 3)$ in $C[0,1]$ coincides with the set K.

Proof. Set $X_n(t)=X(n+t)$, $t\in[0,1]$, in Theorem 1. Q.E.D.

In [5], we have studied the limiting behavior of moving averages of the type $\int_0^t f(t-s)\,dW(s)$ with $0<\int_0^\infty f^2(t)\,dt<\infty$, which arise in time series analysis and other statistical applications. Such moving averages are nonstationary Gaussian. Suppose

(18) $\exists$ *a continuous nondecreasing function ψ on $[0,1]$ such that*

$$\int_1^\infty \psi(e^{-u^2})\,du<\infty \qquad \textit{and for all } t\geqq 0,\ 0\leqq x\leqq 1,$$

$$\int_t^{t+x} f^2(u)\,du+\left\{\int_0^\infty (f(u)-f(u+x))^2\,du\right\}^{\frac{1}{2}}\leqq\psi^2(x).$$

Then we have (cf. [5])

$$\limsup_{t\to\infty} (2\log t)^{-\frac{1}{2}} \int_0^t f(t-s)\,dW(s) = \left\{\int_0^\infty f^2(t)\,dt\right\}^{\frac{1}{2}} \quad \text{a.e.} \tag{19}$$

In fact, Theorem 1 can be applied to describe the limiting behavior (19) in terms of limit sets in $C[0,1]$.

Corollary 2. *Let f be a continuous function on $[0,\infty)$ such that $0<\int_0^\infty f^2(t)\,dt<\infty$ and (18) is satisfied. For $s,t\in[0,1]$, define $R(s,t)=\int_0^\infty f(u)f(u+|t-s|)\,du$ and let $H(R)$ be the reproducing kernel Hilbert space corresponding to the kernel R. Let $U(t)=\int_0^t f(t-s)\,dW(s)$, $t\geqq 0$, and define $Y_n(t)=(2\log n)^{-\frac{1}{2}}\,U(n+t)$, $t\in[0,1]$. Then with probability one, the sequence $(Y_n(t), n\geqq 3)$ is relatively compact in $C[0,1]$ and its set of limit points in $C[0,1]$ coincides with the unit ball K of $H(R)$.*

Proof. Since f is continuous, the kernel R is continuous. Let $(W(t), t\leqq 0)$ be Brownian motion independent of $(W(t), t\geqq 0)$, and define $X(t)=\int_{-\infty}^t f(t-s)\,dW(s)$. Then $X(t)$ is a stationary Gaussian process with covariance $R(s,t)$. We note that for $t>s$,

$$\begin{aligned}
E\{E(X(t)|X(\tau), 0\leqq\tau\leqq s)\}^2 &\leqq E\{E(X(t)|W(\tau),\tau\leqq s)\}^2\\
&= E\left\{\int_{-\infty}^{s} f(t-\lambda)\,dW(\lambda)\right\}^2\\
&= \int_{t-s}^{\infty} f^2(u)\,du \to 0 \quad \text{as } t-s\to\infty.
\end{aligned}$$

Also by (18), assumption (A) of Theorem 1 is satisfied (cf. [5]). Hence defining $V_n(t)=(2\log n)^{-\frac{1}{2}}\,X(n+t)$, $t\in[0,1]$, it follows from Corollary 1 that the sequence $(V_n(t), n\geqq 3)$ is relatively compact in $C[0,1]$ and its set of limit points in $C[0,1]$ coincides with K a.e. It remains then to prove

$$\lim_{n\to\infty} \|V_n - Y_n\|_C = 0 \quad \text{a.e.} \tag{20}$$

To prove (20), we let $Z_n(t)=X(n+t)-U(n+t)=\int_{-\infty}^{0} f(n+t-s)\,dW(s)$, $t\in[0,1]$. Then $EZ_n^2(t)=\int_{n+t}^{\infty} f^2(u)\,du$ and for $0\leqq s<t\leqq 1$, we have from (18),

$$\begin{aligned}
E(Z_n(t)-Z_n(s))^2 &= -\int_{n+s}^{n+t} f^2(u)\,du + 2\int_{n+s}^{\infty} f(u)\big(f(u)-f(t-s+u)\big)\,du\\
&\leqq 2\left(\int_0^\infty f^2(u)\,du\right)^{\frac{1}{2}} \left\{\int_0^\infty (f(u)-f(t-s+u))^2\,du\right\}^{\frac{1}{2}}\\
&\leqq 2\left(\int_0^\infty f^2(u)\,du\right)^{\frac{1}{2}} \psi^2(t-s).
\end{aligned}$$

Hence for any $\varepsilon>0$, an application of Fernique's lemma shows that

$$\sum P[\|Z_n\|_C>\varepsilon(2\log n)^{\frac{1}{2}}]<\infty,$$

and so by the Borel-Cantelli lemma, (20) follows. Q.E.D.

The condition (8) on the asymptotic independence of the sequence of processes $X_m(t)$ is equivalent to each of the following mixing conditions:

(21) For each $t\in[0,1]$, $\lim_{\substack{n\to\infty\\ m-n\to\infty}} \sup\{E(UX_m(t)): U\in\mathscr{G}_n, EU^2=1\}=0$, where $\mathscr{G}_n$ denotes the closed linear manifold generated by $\mathscr{F}_n$.

(22) For each $t\in[0,1]$,

$$\lim_{\substack{n\to\infty\\ m-n\to\infty}} \sup\{|P(A_1\cap A_2)-P(A_1)P(A_2)|: A_1\in\mathscr{B}(X_m(t)), A_2\in\mathscr{B}(\mathscr{F}_n)\}=0,$$

where $\mathscr{B}(\mathscr{F}_n)$ denotes the σ-field generated by $\mathscr{F}_n$.

Noting that $EX_m^2(t)=EX_1^2(t)$ for all m, the equivalence of (8) and (21) is obvious, while the equivalence of (21) and (22) follows from the inequalities $\alpha_{m,n}\leqq r_{m,n}\leqq 2\pi\alpha_{m,n}$, where $\alpha_{m,n}$ denotes the supremum in (22) and $r_{m,n}E^{\frac{1}{2}}X_1^2(t)$ is the supremum in (21) (cf. [9], p. 135). In Oodaira's theorem on the Gaussian process $X(t)$ introduced in Section 1, we choose the subsequence $n_r=[c^r]$ so that we have the asymptotic independence of the sequence of processes $(v^{-\frac{1}{2}}(n_r)X(n_r t),\ r\geqq 1)$. Since $\log r\sim\log\log n_r$, this choice of the subsequence suggests why we have the iterated logarithm behavior. In the following corollary, to achieve asymptotic independence, we have to choose a subsequence $n_k=[\eta k^2\log\log k]$ where $\eta>0$. Noting that $2\log k\sim\log n_k$, we obtain from Theorem 1 the following corollary which corresponds to Strassen's version of a theorem of Kiefer. Kiefer's theorem ([3], Theorem 4) states that if $W(t)$ is the standard Wiener process and $\beta>0$, then

$$\limsup_{t\to\infty}\{\max_{|\tau-t|<(2\beta t\log\log t)^{\frac{1}{2}}}|W(t)-W(\tau)|/[2\beta t(\log t)^2\log\log t]^{\frac{1}{2}}\}=1 \quad \text{a.e.}$$

Corollary 3. *Let $W(t)$, $t\geqq 0$, be the standard Wiener process and let $\beta>0$, $g(t)=(2\beta t\log\log t)^{\frac{1}{2}}$. For $t\in[-1,1]$ and $n\geqq 5$, define*

$$X_n(t)=(W(n+tg(n))-W(n))/(g(n))^{\frac{1}{2}}, \tag{23}$$

$$Y_n(t)=X_n(t)/(\log n)^{\frac{1}{2}}. \tag{24}$$

Then with probability one, the sequence $(Y_n(t), n\geqq 5)$ is relatively compact in $C[-1,1]$ and its set of limit points in $C[-1,1]$ coincides with the set

$$K_1=\left\{h\in C[-1,1]: h(0)=0,\ h \text{ is absolutely continuous and } \int_{-1}^{1}\left(\frac{dh}{dt}\right)^2 dt\leqq 1\right\}. \tag{25}$$

The details of the proof of Corollary 3 are given in [6], where we use Theorem 1 to show that more generally, if $g(t)$ is a positive nondecreasing function on $[0,\infty)$ and $X_n(t)$ is defined by (23), K_1 is defined by (25) and $\mathscr{L}(f_n)$ denotes the set of limit points of a sequence of functions $(f_n, n\geqq n_0)$ in $C[-1,1]$, then

(i) $\mathscr{L}((2\log n)^{-\frac{1}{2}}X_n)=K_1$ a.e. if $g(t)=o(t^\beta)$ for any $\beta>0$;

(ii) $\mathscr{L}(\{2(1-\alpha)\log n\}^{-\frac{1}{2}} X_n)=K_1$ a.e. if $g(t)=t^\alpha \psi(t)$ with $0<\alpha<1$, $\psi(t)+(\psi(t))^{-1}=o(t^\beta)$ for any $\beta>0$ and $\lim_{\rho\to 1}\lim_{t\to\infty} g(\rho t)/g(t)=1$;

(iii) $\mathscr{L}((2\log\log n)^{-\frac{1}{2}} X_n)=K_1$ a.e. if $g(t)=\alpha t(1+o(1))$ for some $0<\alpha<1$.

4. Properties of Reproducing Kernel Hilbert Spaces and the Usual Law of the Iterated Logarithm

Let $X(t)$, $t\in[0,1]$, be a Gaussian process with mean 0 and continuous covariance $R(s,t)$. Let $H(R)$ be the reproducing kernel Hilbert space corresponding to the kernel $R(s,t)$, and let K be the unit ball of $H(R)$. Since R is continuous, $H(R)$ is contained in $C[0,1]$ and K is compact in $C[0,1]$. For any $f\in K$, we have the following properties:

(26) $$\|f\|_C \leqq \sup_{0\leqq t\leqq 1} R^{\frac{1}{2}}(t,t),$$

(27) $$(f(t)-f(s))^2 \leqq E(X(t)-X(s))^2, \quad 0\leqq s,\ t\leqq 1,$$

(28) $\forall t\in[0,1]$, $|f(t)|\leqq R^{\frac{1}{2}}(t,t)$ with equality iff $f=\alpha R(t,\cdot)$ with $\alpha=R(t,t)=0$ or $\alpha^2 R(t,t)=1$.

These properties can be proved by using the reproducing property. For example, to prove (28), we note that

$$\begin{aligned}|f(t)| &= |\langle f, R(t,\cdot)\rangle| \leqq \|f\|_H\, \|R(t,\cdot)\|_H \\ &\leqq |\langle R(t,\cdot), R(t,\cdot)\rangle|^{\frac{1}{2}} = R^{\frac{1}{2}}(t,t).\end{aligned}$$

Let $\sigma(t)=R^{\frac{1}{2}}(t,t)$, $t\in[0,1]$. In view of (28), σ is an upper envelope of the functions in K. In the case of the Wiener process, it is well known that $\sigma\notin K$. In general, it is easy to see from (28) that $\sigma\in K$ iff there exists $t_0\in[0,1]$ such that for all $t\in[0,1]$, $R(t_0,t)=\sigma(t_0)\,\sigma(t)$, i.e., $\sigma(t_0)\,X(t)=\sigma(t)\,X(t_0)$ a.e. By making use of (26) and (28), we can obtain the usual law of the iterated logarithm for Gaussian processes from the corresponding version involving limit sets.

For example, let us consider Oodaira's theorem for the Gaussian process $X(t)$ introduced in Section 1. Let $Z_n(t)=X(nt)/(2R(n,n)\log\log n)^{\frac{1}{2}}$, $t\in[0,1]$ and let $K^*=\{h\in H(R_1)\colon \|h\|_H\leqq 1/\sigma(1)\}=(1/\sigma(1))K$. Under assumptions (1) and (2), Oodaira's theorem states that K^* contains the set of limit points of the relatively compact sequence $(Z_n(t), n\geqq 3)$ in $C[0,1]$ with probability 1. By property (26), this implies that

(29) $$\limsup_{n\to\infty} \|Z_n\|_C \leqq (1/\sigma(1)) \sup_{0\leqq t\leqq 1} \sigma(t) = 1 \quad \text{a.e.}$$

The last equality in (29) follows from the fact $R(t,t)=v(t)\,R(1,1)$ and v is increasing. Since $R(s,t)$ is continuous on $[0,1]\times[0,1]$ and $R(t,t)=v(n)\,R(t/n,t/n)$ for $n-1\leqq t\leqq n$, it follows easily from (29) that

(30) $$\limsup_{t\to\infty} X(t)/(2R(t,t)\log\log t)^{\frac{1}{2}} \leqq 1 \quad \text{a.e.}$$

Now assume further that condition (3) holds. Then by Oodaira's theorem, with probability 1, K^* coincides with the set of limit points of $(Z_n(t), n\geqq 3)$ in $C[0,1]$.

By property (28), $\max_{f\in K} |f(1)|=\sigma(1)$, and so $\max_{f\in K^*} |f(1)|=1$. Hence

$$\limsup_{n\to\infty} X(n)/(2R(n,n)\log\log n)^{\frac{1}{2}}=1 \quad \text{a.e.} \tag{31}$$

In view of (30) and (31), we obtain the usual law of the iterated logarithm (5) for the process $X(t)$.

5. Gaussian Processes with Multi-Dimensional Time

Theorem 1 can be easily extended to Gaussian processes with multi-dimensional time parameter. Let I_k denote the k-dimensional unit cube $[0,1]\times\cdots\times[0,1]$. First we have the following generalization of Fernique's inequality:

Lemma 2. *Let $X(t)$, $t\in I_k$, be a real-valued separable Gaussian process with mean* 0. *Assume that*

$$\begin{aligned} E\{X(t_1,\ldots,t_k)-X(t_1,\ldots,t_{i-1},s_i,t_{i+1},\ldots,t_k)\}^2 &\leqq \psi^2(|t_i-s_i|),\\ t_i,s_i\in[0,1],\ i&=1,\ldots,k, \end{aligned} \tag{32}$$

where ψ is a continuous nondecreasing function on $[0,1]$ *such that $\int_1^\infty \psi(e^{-u^2})\,du<\infty$. Suppose $\Gamma=\sup_{t\in I_k} E^{\frac{1}{2}}(X^2(t))$ is a finite positive number. Then with probability* 1, *the process $X(t)$, $t\in I_k$, has continuous sample paths. For $x\geqq(4k\log n)^{\frac{1}{2}}$ and $n>e^2$,*

$$P\left[\sup_{t\in I_k}|X(t)|\geqq x\left\{\Gamma+\frac{\sqrt{2}k}{\sqrt{2}-1}\int_1^\infty \psi(n^{-u^2})\,du\right\}\right]\leqq 4n^{2k}\int_x^\infty e^{-u^2/2}\,du. \tag{33}$$

Lemma 2 can be proved by a modification of the argument given by Marcus [11] in the case $k=1$. We shall let $C(I_k)$ denote the space of continuous functions on I_k with the usual sup norm metric. A straightforward modification of the proof of Theorem 1 gives the following theorem.

Theorem 2. *Suppose $X(t)$, $t\in I_k$, is a separable real-valued Gaussian process with mean* 0 *and continuous covariance $R(s,t)$ such that condition* (32) *is satisfied. Let $(X_n(t), n\geqq 1)$ be a sequence of Gaussian processes defined on the same probability space and having the same distribution as the process $X(t)$, and let $Y_n(t)=(2\log n)^{-\frac{1}{2}} X_n(t)$. Then with probability one, the sequence $(Y_n(t), n\geqq 3)$ is relatively compact in $C(I_k)$ and its set of limit points in $C(I_k)$ is contained in the unit ball K of the reproducing kernel Hilbert space $H(R)$ of the process $X(t)$. If furthermore, assumption* (B) *of Theorem 1 is satisfied (where we replace* $[0,1]$ *by I_k), then with probability one, the set of limit points of $(Y_n(t), n\geqq 3)$ in $C(I_k)$ coincides with the set K.*

Recently, in connection with certain limit theorems for the sample distribution function, there has been considerable interest in the Gaussian process $\xi(t_1,t_2)$ with two-dimensional time, mean 0 and covariance

$$E\xi(t_1,t_2)\,\xi(s_1,s_2)=\min(t_1,s_1)[\min(t_2,s_2)-t_2 s_2] \quad \text{(cf. [4], [12]).}$$

The reproducing kernel Hilbert space of the process $\{\xi(t_1,t_2), t_1,t_2\in[0,1]\}$ is the direct product $H_1\otimes H_2$, where H_1 is the reproducing kernel Hilbert space corresponding to the standard Wiener process, while H_2 corresponds to the Brownian bridge with covariance $\Gamma(s,t)=\min(s,t)-st$ (cf. Section 8 of [1]). A CONS of $H_1 \otimes H_2$ is $\{e_{n,m}: n=0,1,\ldots, m=1,2,\ldots\}$, where

$$e_{n,m}(t_1,t_2)=2\sin\left((n+\tfrac{1}{2})\pi t_1\right)\sin(m\pi t_2).$$

It is easy to see that the unit ball of $H_1 \otimes H_2$ is

$$(34)\quad K=\left\{h\colon h(t_1,t_2)=\int_0^{t_1}\int_0^{t_2} f(u,v)\,du\,dv,\ t_1,t_2\in[0,1),\ \text{where } \int_0^1\int_0^1 f^2(u,v)\,du\,dv\leqq 1\right\}.$$

In the same way as we derived Oodaira's theorem from Theorem 1, we can obtain from Theorem 2 that with probability one, the sequence $\left(\{2n\log\log n\}^{-\frac{1}{2}}\xi(nt_1,t_2),\ n\geqq 3\right)$ is relatively compact in $C(I_2)$ and its set of limit points in $C(I_2)$ coincides with K (cf. [4], p. 32).

Note Added in Proof. A student of mine at Columbia, Gian-Carlo Mangano, has recently succeeded in weakening the asymptotic independence condition (8) of Theorem 1. By using a different approach, he has proved that Theorem 1 still holds if we replace condition (8) by:

$$(8')\qquad \lim_{r\to\infty}\max_{|m-n|>r}|EX_m(t)X_n(s)|=0 \qquad \text{for any } s,t\in[0,1].$$

This and other related results are contained in his Ph. D. thesis, "On Strassen-type laws of the iterated logarithm for Gaussian random variables with values in abstract spaces", which is now in preparation.

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(Received May 12, 1973)