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On the Uniqueness of Pre-image Measures

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A necessary and sufficient condition for the uniqueness of the pre-image measure in the situation studied by Yershov [8] and other authors [4, 5] is given. This very natural condition, different from that in [8], has been discovered by Eisele [2] under more restrictive assumptions. Our main theorem extends this result to the situation considered by Yershov.

Introduction

Let $(E,\mathscr{E})$ and $(F,\mathscr{F})$ be measurable spaces and $f\colon E\to F$ be a surjective measurable function. Given a probability measure ν on $\mathscr{F}$ we look for a pre-image of ν with respect to f, i.e. a probability measure μ on $\mathscr{E}$ which solves the equation $\nu=f(\mu)$. Such an equation has been called stochastic equation by Yershov.

Yershov [8] proved that a pre-image always exists if E and F are assumed to be analytic topological spaces with $\mathscr{E}$ and $\mathscr{F}$ their respective Borel fields (see also Landers and Rogge [4], Lubin [5]). For this case Yershov also gave an uniqueness condition.

Under the more restrictive requirements that E and F be compact metric spaces and f be continuous, Eisele [2] gave another uniqueness condition – namely: There exists exactly one solution of the equation $\nu=f(\mu)$, if and only if the set

$$Y=\{y\in F\colon\ \#f^{-1}(y)\geqq 2\},$$

which is universally measurable, is a ν-zero set.

The aim of this article is, to extend this result to the situation considered by Yershov.

The main theorem of this paper was proved independently by both authors. A partial result was published by the second author in [6]; the corollaries are due to the first author.

1. Notation

A topological space E is called analytic if it is a Hausdorff space which is the continuous image of a Polish space. A subset A of E is called analytic if it is an analytic space in its subspace topology; it is called coanalytic if $E \setminus A$ is analytic. If $(E, \mathscr{E}, \mu)$ is a measure space we write $\mathscr{E}_\mu$ for the completion of $\mathscr{E}$ with respect to μ and use the symbol μ for the completed measure, too. $\mathscr{E}_*$ denotes the universal completion of $\mathscr{E}$. Let E and F be topological spaces, $\mathscr{B}(E)$ and $\mathscr{B}(F)$ their respective Borel fields. A function $f\colon E \to F$ is called universally measurable if it is measurable with respect to $\mathscr{B}(E)_*$ and $\mathscr{B}(F)$; for such a function and any measure μ on $\mathscr{B}(E)$ the image measure with respect to f is well defined and denoted by $f(\mu)$. The graph of a function f is denoted by $\mathrm{gr}(f)$.

2. The Theorem

We shall prove the following theorem:

Theorem. *Let E and F be analytic topological spaces and $f\colon E \to F$ be a surjective Borel measurable function. Then the set $Y = \{y \in F\colon \# f^{-1}(y) \geqq 2\}$ is universally measurable and for any probability measure ν on $\mathscr{B}(F)$ the following statements are equivalent:*

(i) *Y is a ν-zero set.*

(ii) *There exists exactly one probability measure μ on $\mathscr{B}(E)$ such that $\nu = f(\mu)$.*

In [8] the space E is assumed to be a metric analytic topological space. Since the Borel structure of any analytic topological space is isomorphic with that of a metric analytic topological space (see e.g. [3], III.2.3) and the definition of an analytic topological space given in [8] is consistent with our definition in the metric case, the existence of at least one measure μ with $f(\mu) = \nu$ is known from Yershov's theorem ([8], Thm. 3.1; see also the results in [4, 5]).

Our proof of the theorem is based on the following known inverse function lemma given in Hoffmann-Jørgensen's book [3], and we claim that the existence of a pre-image measure is an immediate consequence of part (ii) of this lemma:

Lemma (*cf.* [3], III.11.7). *Let E and F be analytic topological spaces. If $f\colon E \to F$ is a surjective Borel measurable function, then*

(i) *there exists a function $s\colon F \to E$ with coanalytic graph such that $y = f(s(y))$ for all $y \in F$,*

(ii) *there exists a universally measurable function $t\colon F \to E$ such that $y = f(t(y))$ for all $y \in F$.*

3. Proof of the Theorem

Our proof is divided into four steps.

1. Let $s\colon F \to E$ be a function as described in (i) of the lemma. Since $\mathrm{gr}(s)$ is coanalytic, the set

$\mathrm{gr}'(s) = \{(x, y) \in E \times F\colon x = s(y)\}$

is coanalytic, too, and the inclusion $\mathrm{gr}'(s) \subset \mathrm{gr}(f)$ holds. Since $\mathrm{gr}(f)$, as the graph of a Borel measurable function, is analytic, $\mathrm{gr}(f)\setminus\mathrm{gr}'(s)$ is analytic and its projections on E and F

$$\pi_E(\mathrm{gr}(f)\setminus\mathrm{gr}'(s))=E\setminus s(F) \tag{$*$}$$

and

$$\pi_F(\mathrm{gr}(f)\setminus\mathrm{gr}'(s))=\{y\in F\colon \# f^{-1}(y)\geqq 2\}=Y$$

are analytic, too. Thus Y is universally measurable (cf. [3], III.6.1).

2. The restriction of f to the analytic space $E\setminus s(F)$ is a surjective Borel measurable function onto the analytic space Y and therefore by (ii) of the lemma, there exists a universally measurable function $t\colon Y\to E\setminus s(F)$ such that $y=f(t(y))$ for all $y\in Y$. Let us define a function $u\colon F\to E$ by

$$u(y)=\begin{cases} s(y) & \text{for all } y\in F\setminus Y\\ t(y) & \text{for all } y\in Y.\end{cases} \tag{$**$}$$

For $B\in\mathscr{B}(E)$ the intersection $(B\times F)\cap\mathrm{gr}(f)$ is analytic and therefore

$$u^{-1}(B)=t^{-1}(B)\cup[\pi_F((B\times F)\cap\mathrm{gr}(f))\cap(F\setminus Y)]\in\mathscr{B}(F)_*,$$

since analytic sets are universally measurable. Thus $\mu=u(v)$ is a pre-image of v with respect to f, i.e. $v=f(\mu)$.

3. To prove the equivalence let first $v(Y)=0$. Then there exists a set $Y'\in\mathscr{B}(F)$ such that $Y'\supset Y$ and $v(Y')=0$. Let α be the measure on $\mathscr{A}=f^{-1}(\mathscr{B}(F))$ defined by

$$\alpha(f^{-1}(C))=v(C) \qquad \text{for all } C\in\mathscr{B}(F).$$

Then $\alpha(f^{-1}(Y'))=0$ and the restriction of f to the analytic space $E\setminus f^{-1}(Y')$ is one-to-one and onto $F\setminus Y'$. Therefore it is a Borel isomorphism between these spaces (see [3], III.7.4). Thus each subset of $f^{-1}(Y')$ is a set in $\mathscr{A}_\alpha$ and each Borel subset of $E\setminus f^{-1}(Y')$ is a set in $\mathscr{A}$. Therefore we have $\mathscr{B}(E)\subset\mathscr{A}_\alpha$. Thus from Satz 1A of [1], it follows that any measure μ on $\mathscr{B}(E)$ with $v=f(\mu)$ is equal to the restriction of the (completed) measure α to $\mathscr{B}(E)$. Thus the pre-image measure is unique.

4. To complete the proof let $v(Y)=\eta>0$ and $\mu=u(v)$ where u is defined as in ($**$). Then $\mu(E\setminus s(F))=\eta$, since by ($*$) $E\setminus s(F)$ is universally measurable. Therefore a set $B_0\in\mathscr{B}(E)$ exists such that $B_0\subset E\setminus s(F)$ and $\mu(B_0)=\eta$. The restriction of f to $E\setminus B_0$ is a Borel measurable function from the analytic space $E\setminus B_0$ to the analytic space F. Since $s(F)\subset E\setminus B_0$, this function is surjective. Now, by the lemma, there exists a universally measurable function $v\colon F\to E$ such that $y=f(v(y))$ for all $y\in F$, and therefore $\mu'=v(v)$ is a second pre-image measure of v with respect to f. But μ' is different from μ, since $\mu'(B_0)=0$, whereas $\mu(B_0)=\eta>0$. This completes the proof of our theorem.

4. Corollaries

The following corollaries result from an application of our theorem to an important class of measurable spaces, the Blackwell spaces in the sense of [7], III. For proofs consult III.16 in [7] and Satz 1B in [1].

Corollary 1. *Let $(E,\mathscr{E})$ be a Blackwell space. $\mathscr{A}$ be a countably generated sub-σ-field of $\mathscr{E}$, and let α be a probability measure defined on $\mathscr{A}$. If X is the union of all atoms of $\mathscr{A}$ consisting of more than one point, then $X\in\mathscr{A}_*$ and the following statements are equivalent:*

(i) *X is an α-zero set.*

(ii) *There exists exactly one probability measure μ on $\mathscr{E}$ extending α.*

Corollary 2. *Let $(E,\mathscr{E})$ be a Blackwell space, $(F,\mathscr{F})$ be a measurable space with countably generated $\mathscr{F}$, and let $f\colon E\to F$ be a measurable function. Then $f(E)\in\mathscr{F}_*$. Let moreover ν be a probability measure defined on $\mathscr{F}$ with $\nu(f(E))=1$ and let Y be the union of all atoms of $\mathscr{F}$ having an f-pre-image that consists of more than one point. Then $Y\in\mathscr{F}_*$ and the following statements are equivalent:*

(i) *X is a ν-zero set.*

(ii) *There exists exactly one probability measure μ on $\mathscr{E}$ such that $\nu=f(\mu)$.*

5. Remarks

1. As mentioned in the introduction, our theorem is a generalization of the theorem in [2]. There E is assumed to be a compact metrizable space, F to be a compact Hausdorff space and f to be continuous and surjective. Then, by definition, both spaces are analytic and f is Borel measurable. Moreover, the space F is compact and metrizable in this situation, and the functions u and v in our proof can be chosen to be Borel measurable.

2. We call special attention to the result that in the case of non-uniqueness there exist different pre-image measures with respect to f being defined as image measures of ν with respect to "inverse" functions of f. Such pre-image measures have special properties, e.g. they are extremal in the set of all pre-image measures.

3. We claim that our theorem is proved independently of Yershov's uniqueness theorem (see [8], Thm. 3.2). However, since $\mathscr{B}(F)$ is countably generated, step 4 in our proof can be done by applying this theorem as follows: Let μ be unique, then $\mathscr{A}_\alpha\supset\mathscr{B}(E)$ for $\mathscr{A}$ and α defined as in step 3 of the proof. Thus $\mathscr{A}_\alpha\supset\mathscr{B}(E)_*$ and $D=s(F)\cap f^{-1}(Y)\in\mathscr{A}_\alpha$, since D as an intersection of a coanalytic and an analytic set is universally measurable. Now $\varnothing$ is the only set in $\mathscr{A}$ which is a subset of D and any set in $\mathscr{A}$ having D as a subset has $f^{-1}(Y)$ as a subset, too. Thus we have shown $\nu(Y)=\alpha(f^{-1}(Y))=0$.

4. It should be mentioned that the function s with coanalytic graph in the proof of our theorem cannot be used without further consideration to define a measure on $\mathscr{B}(E)$. This can be seen by an example (see [3], III.16, Ex. 5) depending on Gödel's axiom of constructibility.

5. The following example makes clear that even if the function f in our theorem is assumed to be continuous, the requirement of Hausdorff separation for F cannot be ommitted. Let $E=I$ be the unit interval with its natural topology. Let F be the unit interval with the finite complement topology, i.e. the nonempty open sets are of the form $U=I\setminus A$ where A is finite. Take f to be the

identity on I. Then $\mathscr{B}(F)=\{B\subset I\colon B$ or $I\setminus B$ is countable$\}$ and there are many Lebesgue-Stieltjes measures on $\mathscr{B}(E)$ extending the measure ν given on $\mathscr{B}(F)$ by $\nu=0$ for countable sets and $\nu=1$ for other sets.

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