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On Admissible Translates of Infinitely Divisible Distributions*

William N. Hudson and Howard G. Tucker

§ 1. Introduction and Summary

A probability measure μ over the Borel subsets $\mathfrak{B}^1$ of the real line $\mathbb{R}^1$ is said to have the real number t as an admissible translate if μ_t is absolutely continuous with respect to μ, where μ_t is defined by $\mu_t(B)=\mu(B-t)$ for all $B\in\mathfrak{B}^1$ and $B-t=\{b-t\colon b\in B\}$. Interest in admissible translates for stochastic processes has existed for over ten years, possibly as a natural extension of the change of location problem in statistics. Some attention in connection with this problem has been paid to stochastically continuous processes with independent increments and even to infinitely divisible distributions over a Hilbert space. It appears that one step towards a solution of the problem of finding the set of all admissible translates of such processes or distributions is to solve it for an infinitely divisible distribution function over the real line.

Possibly the first result along these lines is due to I.I.Gikhman and A.V. Skorokhod [3], in which they give sufficient conditions for a fixed number to be an admissible translate of an infinitely divisible distribution function F. The general problem of admissible translates of a measure over a Hilbert space was considered in 1970 by Skorokhod [8], where he gives a brief history of the problem and references. It is (or will be) clear that there is no problem concerning admissible translates of an infinitely divisible distribution function if a Gaussian component is present; thus, from here on in our discussion we assume this component missing. Another result related to this problem in an obvious way is due to M. Sharpe [9] who showed that if F is an absolutely continuous infinitely divisible distribution function with a continuous density and whose characteristic function $\hat{F}(u)$ is in $L_p(-\infty,+\infty)$ for all $p>0$, then the set over which the density is positive is an unbounded interval. F.W.Steutel [10] improved this result by dropping entirely the integrability hypothesis on $\hat{F}(u)$. From a different direction the present authors complemented the Sharpe-Steutel result in [6] by showing that if F is absolutely continuous with a density which is not necessarily continuous but whose Lévy spectral measure has an unbounded absolutely continuous component, then the density of F is positive a.e. over its support which is necessarily an unbounded interval; the set of admissible translates in this case is obvious. This result was in turn extended to the multivariate case by W.N.Hudson and J.D.Mason in [5].

In this paper the result of the present authors referred to above is improved to read as follows: if F is absolutely continuous and infinitely divisible, then F

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has a density which is positive over its support which is necessarily an unbounded interval.

A word or two is now in order as for notation. The same symbol will in general be used for a non-decreasing function over a subset of $\mathbb{R}^1$ and for the Lebesgue-Stieltjes measure it determines. For example, if F is a distribution function, then for any real number x, $F(x)=F((-\infty, x])$, and if $A\in\mathfrak{B}^1$, then $F(A)=\int_A F(dx)$. If M is a Lévy spectral function or measure, then for real $x>0$, $M(x)=-M([x,\infty))$ and for $x<0$, $M(x)=M((-\infty, x])$, and $M(A)=\int_A M(dx)$ for every Borel set A. If X is a random variable, then F_X will denote the distribution function determined by X, i.e., $F_X(x)=P[X\leqq x]$ and $F_X(A)=P[X\in A]=\int_A F_X(dx)$ for every $A\in\mathfrak{B}^1$. Lebesgue measure over the real line will be denoted by λ, and $\mu\ll\nu$ means that the measure μ is absolutely continuous with respect to the measure ν. If α and β are two probability measures, $\alpha*\beta$ will denote their convolution.

§ 2. Background on Admissible Translates

In this section some background is provided for the problem under consideration, and concrete examples are included. Along with several lemmas there is included Theorem 1 which gives sufficient conditions for an absolutely continuous distribution function to have a positive density over it support.

Lemma 1. *If U and V are independent random variables, and if the real number v is an admissible translate for F_V, then it is an admissible translate for F_{U+V}.*

Proof. Suppose X and Y are independent random variables such that $F_X\ll F_U$ and $F_Y\ll F_V$. Then $F_X\times F_Y\ll F_U\times F_V$ holds over $(\mathbb{R}^2,\mathfrak{B}^2)$. Consider the measurable mapping T: $\mathbb{R}^2\to\mathbb{R}^1$ defined by $T(x,y)=x+y$. If $A\in\mathfrak{B}^1$, and if $F_U\times F_V T^{-1}(A)=0$, then $F_X\times F_Y T^{-1}(A)=0$. But $F_X\times F_Y T^{-1}=F_{X+Y}$ and $F_U\times F_V T^{-1}=F_{U+V}$, and thus $F_{X+Y}\ll F_{U+V}$. Now take $X=U$ and $Y=V+v$, and the lemma follows.

We now consider the simplest cases of admissible translates of an infinitely divisible distribution function F. If F is normal with mean μ and variance $\sigma^2>0$, then every real number is an admissible translate of F. By lemma 1, we immediately conclude an assertion made in section 1 that if F is infinitely divisible and has a Gaussian component, every real number is an admissible translate of F. If F has a Poisson distribution, then the set of *all* admissible translates is easily seen to be the set of all positive integers. A first non-trivial result would be the following.

Proposition 1. *If F is an infinitely divisible distribution function without a Gaussian component, and if its Lévy spectral measure M has a discontinuity at $b\neq 0$, then b is an admissible translate of F.*

Proof. One may represent F as the convolution of two distributions G and H whose characteristic functions are

$$\hat{G}(u)=\exp\left\{iu\left(\gamma-\frac{b}{1+b^2}M(\{b\})\right)+\int_{x\notin\{0,b\}}\left(e^{iux}-1-\frac{iux}{1+x^2}\right)M(dx)\right\}$$

and

$$\hat{H}(u)=\exp[M(\{b\})(e^{iub}-1)].$$

Thus $F = G * H$, where H is the distribution of bY, where Y has a Poisson distribution. Lemma 1 and the observation made above about the Poisson distribution show that b is an admissible translate of F.

Thus it is easy to see that for F infinitely divisible, the set of all admissible translates of F contains the additive semi-group generated by the points at which its Lévy spectral measure M is discontinuous. In an attempt to find the set of all admissible translates of F, one might inquire if every point of increase of M is an admissible translate of F. One consequence of the following proposition is that one can easily find F and M where no point of increase of M is an admissible translate of F.

Proposition 2. *If F is an infinitely divisible distribution without a Gaussian component and with Lévy spectral measure M, and if F is not continuous, then the set of admissible translates of F is the additive semi-group generated by the points of discontinuity of M.*

Proof. Since F is not continuous, then by a known result (see [1], [2] and [4]), $0 < K = M(\mathbb{R}^1 \setminus \{0\}) < \infty$. Without loss of generality we may write

$$\hat{F}(u) = \exp \int_{-\infty}^{\infty} (e^{iux} - 1)\, M(dx).$$

Then, as is well-known, one may write

$$F = \sum_{n \geqq 0} e^{-K} \frac{M^{*n}}{n!},$$

where M^{*n} denotes the convolution of M with itself n times. Let $t \neq 0$ be a real number not in the semi-group generated by the discontinuities of M. Then $M^{*n}(\{t\}) = 0$ for all $n \geqq 0$, and thus $F(\{t\}) = 0$. But the t-translate of F, F_t, satisfies $F_t(\{t\}) = F(\{0\}) = e^{-K} > 0$, and thus t is not admissible. Invoking Proposition 1, we obtain our conclusion.

Thus, the only real problem is that of determining the set of admissible translates for a continuous F which we cannot do completely here. The following lemma provides the crucial step in the proof of theorem 1. Recall that λ denotes Lebesgue measure over $(\mathbb{R}^1, \mathfrak{B}^1)$.

Lemma 2. *Let $y > 0$ be an admissible translate for a probability measure φ over $(\mathbb{R}^1, \mathfrak{B}^1)$, and assume $\varphi \ll \lambda$. If I is any bounded interval, then*

$$\lambda\left(\left\{x\colon \frac{d\varphi}{d\lambda}(x) > 0\right\} \cap I\right) \leqq \lambda\left(\left\{x\colon \frac{d\varphi}{d\lambda}(x) > 0\right\} \cap (I + y)\right).$$

Proof. It is sufficient to prove

$$y + \left(\left\{x\colon \frac{d\varphi}{d\lambda}(x) > 0\right\} \cap I\right) \subset \left\{x\colon \frac{d\varphi}{d\lambda}(x) > 0\right\} \cap (I + y)$$

except for a set of Lebesgue measure zero. But

$$y + \left(\left\{x\colon \frac{d\varphi}{d\lambda}(x) > 0\right\} \cap I\right) = \left\{x + y\colon \frac{d\varphi}{d\lambda}(x) > 0\right\} \cap (I + y).$$

Thus it is sufficient to prove

$$\left\{x+y\colon \frac{d\varphi}{d\lambda}(x)>0\right\}\subset\left\{x\colon \frac{d\varphi}{d\lambda}(x)>0\right\} \quad \text{a.e.} \tag{1}$$

Defining φ_y by $\varphi_y(A)=\varphi(A-y)$ for all $A\in\mathfrak{B}^1$, our hypothesis becomes $\varphi_y\ll\varphi$, which implies that

$$\left\{x\colon \frac{d\varphi_y}{d\lambda}(x)>0\right\}\subset\left\{x\colon \frac{d\varphi}{d\lambda}(x)>0\right\} \quad \text{a.e.} \tag{2}$$

But by the definition of φ_y,

$$\left\{x\colon \frac{d\varphi_y}{d\lambda}(x)>0\right\}=y+\left\{x\colon \frac{d\varphi}{d\lambda}(x)>0\right\}=\left\{x+y\colon \frac{d\varphi}{d\lambda}(x)>0\right\}. \tag{3}$$

Now (2) and (3) imply (1).

Theorem 1 which follows is the fundamental lemma needed to prove Theorem 2. However, it is an interesting result in its own right which should be compared with Skorokhod's remarkable result [8] that goes as follows:

Skorokhod's Lemma. *Let φ be a probability measure over $(\mathbb{R}^1, \mathfrak{B}^1)$ such that $\varphi_y\ll\varphi$ for all $y>0$. Then $\varphi\ll\lambda$, and φ has a positive density a.e. over its support which is necessarily an interval of the form $[a,\infty)$ or $(-\infty,\infty)$.*

Now in our Theorem 1 we put part of Skorokhod's conclusion into his hypothesis, namely, $\varphi\ll\lambda$, while at the same time weakening his hypothesis to read: $\varphi_{y_n}\ll\varphi$ for some sequence $y_n\downarrow 0$. The proof of Theorem 1 will involve the notion of metric density, about which we should state this much (see [7], pp 222–224): if $A\in\mathfrak{B}^1$, then there is a subset A_0 of A such that $\lambda(A_0)=0$ and such that every $x\in A\setminus A_0$ has metric density 1 with respect to A, i.e.,

$$\lim_{\varepsilon\downarrow 0}(2\varepsilon)^{-1}\,\lambda\big((x-\varepsilon, x+\varepsilon)\cap A\big)=1.$$

Theorem 1. *Let φ be a probability measure over $(\mathbb{R}^1, \mathfrak{B}^1)$, and assume $\varphi\ll\lambda$. Let $\{x_n\}$ be a sequence of positive numbers such that $x_n\downarrow 0$ and each x_n is an admissible translate of φ. Then $d\varphi/d\lambda>0$ a.e. $[\lambda]$ over its support which is an interval of the form $[a,\infty)$ or $(-\infty,\infty)$.*

Proof. Let $D=\left\{x\colon \frac{d\varphi}{d\lambda}(x)>0\right\}$. Let $a=\inf\{x\colon \lambda(D\cap(-\infty,x])>0\}$. Define $D^c=[a,\infty)\setminus D$ if $a>-\infty$ and $=\mathbb{R}^1\setminus D$ if $a=-\infty$. We wish to prove $\lambda(D^c)=0$. We assume to the contrary that $\lambda(D^c)\neq 0$. Then there exist points in D^c of metric density 1. Let $z\in D^c$ be one such point, $z>a$. Then there exists an $\varepsilon>0$ small enough so that

$$(2\varepsilon)^{-1}\,\lambda\big(D^c\cap(z-\varepsilon, z+\varepsilon)\big)>0.9 \tag{4}$$

Let $x\in D$ be a point of metric density 1 with respect to D and such that $a<x<z$. Next choose $\delta>0$ small enough to satisfy both

$$0<\delta<\varepsilon/10 \quad\text{and}\quad \lambda\big(D\cap(x-\delta, x+\delta)\big)>0.5(2\delta)=\delta. \tag{5}$$

Let y be any of the x_n's which satisfy $0<y<\delta/10$. Next define

$$k_1=\min\{k\colon x+ky-\delta>z-\varepsilon\}, \qquad k_2=\min\{k\colon x+ky-\delta>x+k_1 y+\delta\},$$

and, for every integer $j\geqq 2$, let $k_j=\min\{k\colon x+ky-\delta>x+k_{j-1}y+\delta\}$. Let $n=\max\{j\colon x+k_j y+\delta<z+\varepsilon\}$ and denote $I_j=(x+k_j y-\delta, x+k_j y+\delta)$. Now $I_1, \ldots, I_n$ are disjoint, $n\geqq 1$, and

$$\bigcup_{j=1}^{n} I_j\subset(z-\varepsilon, z+\varepsilon). \tag{6}$$

Since y is an admissible translate of φ, we obtain by Lemma 2 for every j

$$\lambda(D\cap I_j)\geqq\lambda(D\cap(x-\delta, x+\delta))>\delta. \tag{7}$$

Hence by (5), (6) and (7),

$$\lambda(D\cap(z-\varepsilon, z+\varepsilon))\geqq\sum_{j=1}^{n}\lambda(D\cap I_j)>n\delta. \tag{8}$$

An easy computation shows that $n\geqq 9$, from which it follows from (8) that

$$\lambda(D\cap(z-\varepsilon, z+\varepsilon))\geqq 0.9\,\varepsilon, \tag{9}$$

while (4) implies that

$$\lambda(D^c\cap(z-\varepsilon, z+\varepsilon))>1.8\,\varepsilon. \tag{10}$$

Thus, (9) contradicts (10), and hence (4), from which we obtain the theorem.

§ 3. Admissible Translates in the Absolutely Continuous Case

We now solve the problem of admissible translates in the case of an absolutely continuous infinitely divisible distribution function F. It is already known [13] that in this case the support of F is an unbounded interval of the form $(-\infty, a]$, $[a, \infty)$, or $(-\infty, \infty)$, and one can always determine the finite number a when it exists [11]. In Theorem 2 we shall prove that the density of F is positive a.e. over its support. Thus, if the support of F is $[a, \infty)$, the set of all admissible translates is the set of all positive real numbers, when the support of F is $(-\infty, a]$, then the set of all admissible translates is the set of all negative real numbers, and in the remaining case it is $\mathbb{R}^1$.

Theorem 2. *Let F be an absolutely continuous infinitely divisible distribution function with characteristic function*

$$\hat{F}(u)=\exp\left\{i\gamma u+\int_{x\neq 0}\left(e^{iux}-1-\frac{iux}{1+x^2}\right)M(dx)\right\}.$$

Then the density of F is positive a.e. $[\lambda]$ over the support of F which is necessarily an unbounded interval.

Proof. If every deleted neighborhood of zero contains discontinuity points of M, then the conclusion follows from Proposition 1 and Theorem 1. Hence we shall assume there exists a deleted neighborhood of zero over which M is continuous. Of course, M is also inbounded. Without loss of generality we shall assume $M(+0)=-\infty$. Let S denote the set of all admissible translates of F. According to Theorem 1 it is sufficient to show that there exists a sequence $\{x_n\}\subset S$ such that $x_n\downarrow 0$ as $n\to\infty$. This is accomplished by the following two claims.

Claim 1⁰. If there is no sequence $\{x_n\}\subset S$ such that $x_n\downarrow 0$ as $n\to\infty$, then for some Borel set A and number $\delta>0$,

(i) $F(A)=0$ and (ii) $F(A-y)>0$ for all $y\in(0,\delta)$.

Proof of Claim 1⁰. Define $F_x(B)=F(B-x)$ for $B\in\mathfrak{B}^1$. By hypothesis there exists a $\delta>0$ such that for all $x\in(0,\delta)$, F_x is not absolutely continuous with respect to F. Hence for every $x\in(0,\delta)$ there is a Borel set A_x such that

$$F(A_x)=0 \quad \text{but } F(A_x-x)>0. \tag{11}$$

Now, for every $x\in(0,\delta)$, define a function $g_x(\cdot)$ by $g_x(y)=F(A_x-y)$. Since F is absolutely continuous, it follows that $g_x(y)$ is continuous in y. Now let

$$N_x=\{y: g_x(y)>0\}. \tag{12}$$

Since $g_x(y)$ is continuous in y, then N_x is open, and (11) implies that $x\in N_x$ for all $x\in(0,\delta)$. Hence $(0,\delta)\subset\bigcup\{N_x: 0<x<\delta\}$. Since $(0,\delta)$ is sigma-compact, there is a sequence $\{t_n\}\subset(0,\delta)$ such that $(0,\delta)\subset\bigcup_n N_{t_n}$. Set

$$A=\bigcup_n A_{t_n}. \tag{13}$$

By (11), $F(A)\leqq\sum_n F(A_{t_n})=0$, so $F(A)=0$. Let $y\in(0,\delta)$ be arbitrary; then for some n, $y\in N_{t_n}$. By (12) and (13), $F(A-y)\geqq F(A_{t_n}-y)>0$, which proves claim 1⁰.

Claim 2⁰. There exist two infinitely divisible distribution functions G and H such that (i) $F=G*H$, (ii) F and H are equivalent, and (iii) the support of G is $[0,\infty)$.

Proof of Claim 2⁰. Let $x_1>0$ be such that M is continuous over $(0,x_1)$, and select $y_1\in(0,x_1)$ so that $M(x_1)-M(y_1)=1/2$. For $n=2,3,\ldots$, let $x_n=y_{n-1}/2$ and $0<y_n<x_n$ be such that $M(x_n)-M(y_n)=1/2^n$. It is easy to determine two Lévy spectral measures, M_G and M_H, so that $M_G+M_H=M$ and

$$\frac{dM}{dM_H}(x)=\begin{cases}2 & \text{if } x\in\bigcup\limits_{n=1}^{\infty}[y_n,x_n]\\ 1 & \text{if } x\notin\bigcup\limits_{n=1}^{\infty}[y_n,x_n].\end{cases}$$

Now take as G and H those distribution functions whose corresponding characteristic functions are

$$\hat{G}(u)=\exp\int_0^\infty (e^{iux}-1)\,M_G(dx)$$

and

$$\hat{H}(u)=\exp\left\{i\left(\gamma-\int\frac{x}{1+x^2}M_G(dx)\right)u+\int_{x\neq 0}\left(e^{iux}-1-\frac{iux}{1+x^2}\right)M_H(dx)\right\}.$$

Since $M_G((0,\delta))>0$ for every $\delta>0$, it follows from the theorems in [11] and [13] that the support of G is $[0,\infty)$, thus implying (iii). In order to verify (ii), we use Theorem 4 of [6] which asserts in our case here that F and H are equivalent measures if the following four conditions are satisfied:

$$M(\mathbb{R}^1 \setminus \{0\}) = M_H(\mathbb{R}^1 \setminus \{0\}) = \infty, \tag{a}$$

$$M \ll M_H \quad \text{and} \quad M_H \ll M, \tag{b}$$

$$\int_{x \neq 0} (1-\sqrt{dM_H/dM})^2\, dM < \infty \tag{c}$$

and

$$\int_{x \neq 0} (1-\sqrt{dM/dM_H})^2\, dM_H < \infty,$$

and

$$\int_{x \neq 0} \frac{x}{1+x^2}\, M_G(dx) = \int_{x \neq 0} \frac{x}{1+x^2}\,(M-M_H)(dx).$$

Conditions (a), (b) and (d) are immediately seen to be true, and the value of each integral in (c) is $(1-\sqrt{2})^2/2$. This proves conclusion (ii), and finishes the proof of *claim* 2^0. We now prove the theorem. Let A and $\delta>0$ be as in Claim 1^0 and let G and H be as in Claim 2^0. By (i) of Claim 2^0,

$$F(A) = \int H(A-y)\, G(dy). \tag{14}$$

By claims 1^0 and 2^0, since H and F are equivalent, and since $F(A-y)>0$ for all $y \in (0,\delta)$, then $H(A-y)>0$ for *all* $y \in (0,\delta)$. By (14) and (iii) of Claim 2^0, we obtain $F(A)>0$. But this contradicts conclusion (i) of Claim 1^0. Hence the hypothesis of Claim 1^0 is not true, i.e., there does exist a sequence $\{x_n\} \subset S$ such that $x_n \downarrow 0$. Now we apply Theorem 1 to conclude that $dF/d\lambda>0$ a.e. $[\lambda]$ over the support of F.

In order to show that Theorem 2 extends both the first three theorems in [6] and the Sharpe-Steutel result referred to earlier (but does not contain the latter), we must give some indication that (i) there exists an F without a Gaussian component but which is absolutely continuous and is determined by an M-function which is continuous singular, and (ii) there exist absolutely continuous infinitely divisible distribution functions without a continuous density. An example that satisfies (i) is given in [12]. Examples which satisfy (ii) can be found among the distributions of class L; see e.g., Theorem 4 in [14].

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William N. Hudson
Dept. of Mathematics
University of Utah
Salt Lake City, Utah 84112
USA

Howard G. Tucker
Math. Dept.
University of California
Irvine, Cal. 92664
USA

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