Z. Wahrscheinlichkeitstheorie verw. Gebiete
40, 257–282 (1977)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

The Exact Hausdorff Measure of Irregularity Points for a Brownian Path

Norio Kôno
Institute of Mathematics, Yosida College, Kyoto University, Kyoto, Japan

1. Theorem

Let $B(t)$ denote a standard Brownian motion on the line. It is well known that the local asymptotic behaviour of sample paths at a prescribed time $t_0 \geqq 0$ and the uniform asymptotic behaviour on the interval $0 \leqq t \leqq 1$ are given by

$$\overline{\lim_{h\downarrow 0}} \frac{B(t_0+h)-B(t_0)}{(2h\log\log 1/h)^{\frac{1}{2}}}=1, \quad \text{a.s.,} \tag{1.1}$$

and

$$\overline{\lim_{\substack{|t-s|\downarrow 0\\ 0\leqq t,s\leqq 1}}} \frac{B(t)-B(s)}{(2|t-s|\log 1/|t-s|)^{\frac{1}{2}}}=1, \quad \text{a.s.,} \tag{1.2}$$

respectively.

These arguments show that there exist almost surely some time points which violate (1.1). About this phenomena, S. Orey and S.J. Taylor [6] have obtained the following results:

Set

$$E(\alpha)=\left\{0\leqq t\leqq 1;\ \overline{\lim_{h\downarrow 0}} \frac{B(t+h)-B(t)}{(2\alpha h\log 1/h)^{\frac{1}{2}}}\geqq 1\right\},$$

and

$$F(\beta)=\left\{0\leqq t\leqq 1;\ \overline{\lim_{h\downarrow 0}} \frac{B(t+h)-B(t)}{(2\beta h\log\log 1/h)^{\frac{1}{2}}}\geqq 1\right\}.$$

Then, they have

$$\dim E(\alpha)=1-\alpha \qquad (0<\alpha\leqq 1) \text{ a.s.,}$$

and

$$\begin{aligned}\phi_\gamma-m(F(\beta))&=0 \qquad (\gamma<\beta-1) \text{ a.s.,}\\ &=+\infty(\gamma>\beta-1) \text{ a.s.,}\end{aligned}$$

where $\phi_\gamma - m$ denotes the Hausdorff measure with respect to $\phi_\gamma(s) = s(\log 1/s)^\gamma$ $(\gamma > 0)$.

Motivated by their results, we investigate irregularity points of a Brownian path under slightly different formulation, but give the exact Hausdorff measure of the set according to an integral test.

Let $\{B^d(t,\omega); 0 \leqq t \leqq 1, \omega \in \Omega\}$ be a standard Brownian motion in R^d on a complete probability space $(\Omega, \mathscr{B}, P)$ and ϕ be a positive continuous function such that

$$\phi(x) \uparrow +\infty \qquad (x \downarrow 0)^1. \tag{1.3}$$

Then we know that for each $0 < t < 1$,

$$P(\exists \{(u_n, v_n)\} \in \mathscr{S}, \\ \|B^d(t+u_n, \omega) - B^d(t-v_n, \omega)\| > \sqrt{u_n + v_n}\, \phi(u_n + v_n))^2 = 0$$

if and only if

$$\int_{+0} x^{-1} \phi^{d+2}(x) e^{-\phi^2(x)/2} dx < +\infty \quad ([1]), \tag{1.4}$$

where $\mathscr{S}$ is a class of all couples $\{(u_n, v_n)\}_{n=1}^{\infty}$ of two sequences such that $u_n \geqq 0$, $v_n \geqq 0$, $\frac{1}{2} \geqq u_n + v_n > 0$ and $u_n + v_n \downarrow 0$ $(n \uparrow +\infty)$, and

$$P(\exists \delta(\omega) > 0, \quad 0 < \forall |t-s| < \delta(\omega), \\ \|B^d(t,\omega) - B^d(s,\omega)\| \leqq \sqrt{|t-s|}\, \phi(|t-s|)) = 0$$

if and only if

$$\int_{+0} x^{-2} \phi^{d+2}(x) e^{-\phi^2(x)/2} dx = +\infty. \tag{1.5}$$

Therefore under the conditions (1.3), (1.4) and (1.5)

$$E(\phi, \omega) = \{0 < t < 1;\ \exists \{(u_n, v_n)\} \in \mathscr{S} \\ \|B^d(t+u_n, \omega) - B^d(t-v_n, \omega)\| > \sqrt{u_n + v_n}\, \phi(u_n + v_n)\}$$

has Lebesgue measure 0 but is a non-empty set almost surely ([6]). So we are interested to give a more exact measure — Hausdorff measure — of this set.

We denote by $h - m(A)$ the Hausdorff measure of a measurable set A for a Hausdorff measure function $h(x)$, that is, a positive continuous function such that

$$h(x) \downarrow 0 \qquad (x \downarrow 0), \tag{1.6}$$

and

$$h(x)/x \uparrow +\infty \qquad (x \downarrow 0). \tag{1.7}$$

---

[1] We write $f(x) \downarrow 0 (x \uparrow 0)$ to indicate $f(x)$ tends to zero as x decreases to zero, and $f(x) \leqq f(y)$ for $0 < x \leqq y$ sufficiently small. Similar expressions are to be interpreted analogously

[2] We denote by $\|x\|$, $x \in R^d$ the usual Euclidean norm

Then, we will prove the following:

Theorem. *Under the conditions* (1.3), (1.4), (1.6), (1.7) *and*

$$\exists n_0, \quad \forall n \geqq n_0$$
$$\phi(2^{-n-2}) \leqq 2^{10}\, 10^{-3}\, \phi(2^{-n-2} n^5), \tag{1.8}$$

we claim that

$$h-m(E(\phi,\omega))=0 \quad (\text{resp. } +\infty) \ a.s.$$

if and only if an integral test

$$I(h,\phi)=\int_{+0} x^{-2}\,\phi^{d+2}(x)\,e^{-\phi^2(x)/2}\,h(x)\,dx$$

converges (resp. diverges).

Examples. (i) Let $\phi(x)=\sqrt{2a\log 1/x}$, $0<\alpha\leqq 1$, then

$$h-m(E(\phi,\omega))=0 \quad (\text{resp. } +\infty) \text{ a.s.}$$

if and only if

$$\int_{+0} x^{\alpha-2}(\log 1/x)^{1+d/2}\,h(x)\,dx<+\infty \qquad (\text{resp.}=+\infty),$$

in particular,

$$h_\varepsilon-m(E(\phi,\omega))=0 \qquad \text{if } \varepsilon>0,$$
$$=+\infty \qquad \text{if } \varepsilon\leqq 0 \text{ a.s.},$$

where

$$h_\varepsilon(x)=x^{1-\alpha}(\log 1/x)^{-2-\varepsilon-d/2}.$$

(ii) Let $\phi(x)=\sqrt{2\beta\log\log 1/x}$, $\beta>1$, then

$$h-m(E(\phi,\omega))=0 \quad (\text{resp. } +\infty) \text{ a.s.}$$

if and only if

$$\int_{+0} x^{-2}(\log 1/x)^{-\beta}(\log\log 1/x)^{1+d/2}\,h(x)\,dx<+\infty \qquad (\text{resp.}=+\infty),$$

in particular

$$h_\varepsilon-m(E(\phi,\omega))=0 \qquad \text{if } \varepsilon>0,$$
$$=+\infty \qquad \text{if } \varepsilon\leqq 0 \text{ a.s.},$$

where

$$h_\varepsilon(x)=x(\log 1/x)^{\beta-1}(\log\log 1/x)^{-2-\varepsilon-d/2}.$$

2. Preliminary Lemmas

Before starting the proof of our theorem, we prepare some preliminary lemmas about the normal distribution in R^{2d}, most of which are already well known.

Let (X, Y) be a random variable in R^2 having the bivariate normal density centered at the origin with $EX^2=EY^2=1$ and $EXY=\rho$, and (X_i, Y_i), $1\leqq i\leqq d$ be independent copies of (X, Y). Set

$$X^d=(X_1,\dots,X_d), \qquad Y^d=(Y_1,\dots,Y_d)$$

and

$$\Phi_d(x)=P(\|X^d\|\geqq x),$$

then we have

Lemma 1. *For all* $x\geqq\sqrt{2d}$,

$$2a_1 x^{d-2}e^{-x^2/2}>\Phi_d(x)>a_1 x^{d-2}e^{-x^2/2}, \tag{2.1}$$

where $a_1=2(\Gamma(d/2)2^{d/2})^{-1}$ *if* $d\geqq 2$ *and* $a_1=(\sqrt{2\pi})^{-1}$ *if* $d=1$.

Lemma 2. (i) *For all* $y\geqq x\geqq\sqrt{2d}$ *and* $|\rho xy|\leqq 1$,

$$P(\|X^d\|\geqq x, \|Y^d\|\geqq y)<a_2\,\Phi_d(x)\,\Phi_d(y), \tag{2.2}$$

where $a_2=(10+2^{d/2}a_1^{-1})$.

(ii) *For all* $x\geqq\sqrt{2d}$,

$$P(\|X^d\|\geqq x,\ \|Y^d\|\geqq x)<a_3\,e^{-(1-|\rho|)x^2/32}\,\Phi_d(x), \tag{2.3}$$

where $a_3=2^{d/2}\,d^{d/2}\,a_1+2^{3d/2}\,d^{d/2}$.

(iii) *For all* $y\geqq x\geqq\sqrt{2d}$ *and* $x\geqq(1+|\rho|)y/2+\sqrt{2d}$,

$$P(\|X^d\|\geqq x, \|Y^d\|\geqq y) \\ <(2a_1\,y^{d-2}+a_4)\,e^{-(2x-y-|\rho|y)^2/8}\,\Phi_d(y), \tag{2.4}$$

where $a_4=2^{3d/2}\,d^{d/2}+2a_1$.

Here, the exact values of $a_1,\dots,a_4$ are not essential in the sequel.

Since (i) and (ii) are essentially proved in [5, Lemma 1.5 and 1.6], we give a proof of (iii).

Without loss of generality, we can assume that $\rho>0$. Then we have

$$P(\|X^d\|\geqq x, \|Y^d\|\geqq y)=P\left(\|X^d\|\geqq x, \frac{1+\rho}{2\rho}y\geqq\|Y^d\|\geqq y\right) \\ +P\left(\|X^d\|\geqq x, \|Y^d\|\geqq\frac{1+\rho}{2\rho}y\right)=I_1+I_2.$$

Since X is expressed with a standard normal random variable Z independent of Y by

$$X=\rho Y+\sqrt{1-\rho^2}\,Z,$$

it follows that

$$I_1 \leqq P\left(\|Z^d\| \geqq \frac{x-\rho\|Y^d\|}{\sqrt{1-\rho^2}}, \frac{1+\rho}{2\rho}y \geqq \|Y^d\| \geqq y\right)$$
$$\leqq \Phi_d(y)\,\Phi_d\left(\frac{2x-y-\rho y}{2\sqrt{1-\rho^2}}\right) < 2a_1(y^{d-2}+1)\,e^{-(2x-y-\rho y)^2/8}\,\Phi_d(y).$$

On the other hand,

$$I_2 < \Phi_d\left(\frac{1+\rho}{2\rho}y\right)$$
$$< 2a_1\left(\frac{1+\rho}{2\rho}y\right)^{d-2} e^{-(1+\rho)^2y^2/(8\rho^2)}$$
$$< 2a_1(y/\rho)^{d-2}\,e^{-(1-\rho)y^2/(8\rho^2)-y^2/2}$$
$$< a_1\,2^{3d/2}\,d^{d/2}\,y^{d-2}\,e^{-(1-\rho)y^2/8-y^2/2}$$
$$< 2^{3d/2}\,d^{d/2}\,e^{-(1-\rho)y^2/8}\,\Phi_d(y)$$
$$< 2^{3d/2}\,d^{d/2}\,e^{-(2x-y-\rho y)^2/8}\,\Phi_d(y).$$

Summing up the inequalities for I_1 and I_2 we have (2.4).

3. Proof when $I(h,\phi)<+\infty$

To prove that the convergence of $I(h,\phi)$ implies $h-m(E(\phi,\omega))=0$ a.s., we need a lemma which is an extention to d-dimensional case of Lemma 2 in [3].

Let (S,m) denote a compact metric space which satisfies the following condition; There exist a constant c_1 and a positive integer N such that

$$N(\varepsilon,K) \leqq c_1(d(K)/\varepsilon)^N, \qquad (0<\varepsilon\leqq d(K)). \tag{3.1}$$

holds for any closed ball K of S, where $N(\varepsilon,K)$ is the minimal cardinal number of ε-nets on K (a subset A of K is said to be a ε-net on K if for any $t\in K$, there exists $s\in A$ such that $d(s,t)\leqq\varepsilon$) and $d(K)$ is the diameter of K.

Let $\{X(t); t\in S\}$ be a path continuous centered Gaussian process with $EX(t)^2=1$ for all $t\in S$ and $\{X_i(t)\}_{i=1}^d$ be independent copies of $\{X(t)\}$. We consider a d-dimensional Gaussian process

$$X^d(t)=(X_1(t),\ldots,X_d(t)).$$

Moreover we assume that there exist a positive constant η and a positive non-decreasing continuous function σ with $\sigma(0)=0$ such that

$$E(X(s)-X(t))^2 \leqq \eta^2\sigma^2(m(s,t)). \tag{3.2}$$

Then we have

Lemma 3. [4] *If there exist two positive constants c_2 and α such that*

$$\frac{\sigma(tx)}{\sigma(x)} \leqq c_2 t^\alpha \tag{3.3}$$

holds for any $0<t\leqq 1$ and $0<x\leqq d(S)$, then

$$P(\sup_{t\in S} \|X^d(t)\| \geqq x) \leqq c_1 c_3 c_5 N(\tfrac{1}{2}\sigma^{-1}(1/(\eta x)), S)\,\Phi_d(x) \tag{3.4}$$

holds for any $x \geqq 1+\sqrt{\pi}\,c_2 c_4/\sqrt{\alpha}$, where

$$c_3 = 4e^{\sqrt{\pi}\,c_2c_4/(2\sqrt{\alpha})+\pi c_2^2 c_4^2/(4\alpha)},$$

$$c_4 > \sqrt{N}(\sqrt{2}+1)2^{5/2},$$

$$c_5 = e^{4N} + \sum_{k=2}^{\infty} e^{N2^{k+1}} q_k (<+\infty),$$

$$q_k = \Phi_d\left(c_4(\sqrt{2}-1)2^{(k-3)/2} - \left(1+\frac{\sqrt{\pi}\,c_2c_4}{2\sqrt{\alpha}}\right)\bigg/\sqrt{3}\right),$$

and $\sigma^{-1}(x) = \inf\{y;\ \sigma(y)=x\}$.

We remark that c_1, c_3 and c_5 are independent of x and η.

Proof. This lemma is proved just the same manner as that of Lemma 2 in [3] by taking account of Lemma 1 and 2. First we take a closed ball K of S such that

$$0 \leqq \eta x \sigma(d(K)) \leqq 1, \tag{3.5}$$

and let $\{t_i^{(n)}\}$, $1\leqq i \leqq N(\varepsilon_n, K)$ be the minimal ε_n-net on K, where

$$\varepsilon_n = d(K)e^{-2^{n+1}}, \qquad n=0,1,2,\ldots,.$$

Setting

$$x_n = c_4\eta(\sqrt{2}-1)\,2^{(n-1)/2}\,\sigma(\varepsilon_{n-1}), \qquad n=1,2,\ldots,$$

we have

$$\sum_{k=1}^{\infty} x_k < c_4\eta \int_0^\infty \sigma(d(K)e^{-u^2})\,du$$

$$\leqq \frac{\sqrt{\pi}\,c_2c_4}{2\sqrt{\alpha}\,x} \quad \text{(in (3.3) let } t=e^{-u^2} \text{ and } x=d(K)). \tag{3.6}$$

Now let

$$A = \left\{\sup_{t\in K} \|X^d(t)\| > x + \sum_{k=1}^{\infty} x_k\right\} \qquad (x \geqq 1),$$

$$A_n^{(j,i)} = \left\{\|X^d(t_i^{(j)})\| \geqq x + \sum_{k=1}^{n} x_k\right\},$$

$$A_n = \bigcup_{\substack{1\leqq i\leqq N(\varepsilon_j, K)\\ 1\leqq j\leqq n}} A_n^{(j,i)},$$

and

$$A_n^\infty = \left\{ \max_{\substack{1 \leqq i \leqq N(\varepsilon_j, K) \\ 1 \leqq j \leqq n}} \|X^d(t_i^{(j)})\| \geqq x + \sum_{k=1}^{\infty} x_k \right\}.$$

Since we have

$$A \subset \bigcup_{n=1}^{\infty} A_n^\infty, \quad A_n^\infty \subset A_{n+1}^\infty, \quad \text{and} \quad A_n^\infty \subset A_n,$$

it follows that

$$P(A) \leqq \varliminf_{n \to \infty} P(A_n).$$

For $P(A_n)$, we have

$$\begin{aligned} P(A_n) &\leqq P(A_{n-1}) + P(A_n \cap A_{n-1}^c) \\ &\leqq P(A_{n-1}) + \sum_{i=1}^{N(\varepsilon_n, K)} P(A_n^{(n,i)} \cap A_{n-1}^{(n-1,i')c}), \end{aligned}$$

where i' is chosen such that $m(t_i^{(n)}, t_{i'}^{(n-1)}) \leqq \varepsilon_{n-1}$. To estimate the last term, we write

$$X(t_{i'}^{(n-1)}) = r X(t_i^{(n)}) + \sqrt{1-r^2}\, Z,$$

where Z is a standard normal random variable independent of $X(t_i^{(n)})$ and

$$\begin{aligned} r &= E X(t_{i'}^{(n-1)}) X(t_i^{(n)}) \\ &= 1 - E(X(t_i^{(n)}) - X(t_{i'}^{(n-1)}))^2/2 \\ &\geqq 1 - \eta^2 \sigma^2(m(t_i^{(n)}, t_{i'}^{(n-1)}))/2 \\ &\geqq 1 - \eta^2 \sigma^2(\varepsilon_{n-1})/2 \\ &\geqq 1 - \eta^2 \sigma^2(d(K))/2 \\ &\geqq 1 - 1/(2x^2) \geqq \tfrac{1}{2}. \end{aligned} \tag{3.7}$$

By the triangular inequality, we have

$$\|X^d(t_{i'}^{(n-1)})\| \geqq r \|X^d(t_i^{(n)})\| - \sqrt{1-r^2}\, \|Z^d\|.$$

Therefore we have

$$\begin{aligned} &P(A_n^{(n,i)} \cap A_{n-1}^{(n-1,i')c}) \\ &\quad \leqq P\left(\|X^d(t_i^{(n)})\| \geqq x + \sum_{k=1}^{n} x_k, \sqrt{1-r^2}\, \|Z^d\| \geqq r x_n - (1-r)\left(x + \sum_{k=1}^{n-1} x_k\right)\right) \\ &\quad \leqq \Phi_d(x)\, \Phi_d\left(\frac{r x_n - (1-r)\left(x + \sum_{k=1}^{n-1} x_k\right)}{\sqrt{1-r^2}} \right). \end{aligned}$$

Combining (3.5), (3.6) and (3.7), we have

$$\frac{r x_n}{\sqrt{1-r^2}} \geqq c_4(\sqrt{2}-1)\, 2^{(n-3)/2},$$

$$\frac{1-r}{\sqrt{1-r^2}}\left(x+\sum_{k=1}^{n-1} x_k\right) \leqq \frac{\sqrt{1-r}}{\sqrt{1+r}}\left(x+\sum_{k=1}^{\infty} x_k\right)$$

$$\leqq \eta\,\sigma(\varepsilon_{n-1})\left(x+\frac{\sqrt{\pi}\, c_2 c_4}{2\sqrt{\alpha x}}\right)\Big/\sqrt{3}$$

$$\leqq \left(1+\frac{\sqrt{\pi}\, c_2 c_4}{2\sqrt{\alpha}}\right)\Big/\sqrt{3},$$

and so

$$\Phi_d\left(\frac{r x_n-(1-r)\left(x+\sum_{k=1}^{n-1} x_k\right)}{\sqrt{1-r^2}}\right) \leqq q_n.$$

It follows that

$$P(A_n) \leqq P(A_{n-1}) + q_n N(\varepsilon_n, K)\, \Phi_d(x).$$

By induction it follows from (3.1) that

$$P(A_n) \leqq c_1\left(e^{4N} + \sum_{k=2}^{n} e^{N 2^{k+1}} q_k\right) \Phi_d(x).$$

If we choose the constant c_4 greater than $\sqrt{N}(\sqrt{2}+1)2^{5/2}$, then by virtue of Lemma 1,

$$\sum_{k=2}^{\infty} e^{N 2^{k+1}} q_k$$

converges, and so we have

$$P(A) \leqq c_1 c_5\, \Phi_d(x).$$

Finally let $\{s_i\}$, $1 \leqq i \leqq N(\frac{1}{2}\sigma^{-1}(1/(\eta x)), S)$ be the minimal $\sigma^{-1}(1/(\eta x))/2$ – net on S and set

$$K_i = \{s;\ m(s, s_i) \leqq \sigma^{-1}(1/(\eta x))/2\}.$$

Then K_i is a closed ball of S such that

$$d(K_i) \leqq \sigma^{-1}(1/(\eta x)),$$

and

$$0 \leqq \eta x \sigma(d(K_i)) \leqq 1.$$

Therefore we have

$$P\left(\sup_{t\in S}\|X^d(t)\|>x+\sum_{k=1}^{\infty}x_k\right)$$
$$\leqq \sum_{i=1}^{N(\frac{1}{2}\sigma^{-1}(1/(\eta x)),S)} P\left(\sup_{t\in K_i}\|X^d(t)\|>x+\sum_{k=1}^{\infty}x_k\right)$$
$$\leqq c_1 c_5 N(\tfrac{1}{2}\sigma^{-1}(1/(\eta x)),S)\,\Phi_d(x).$$

Setting

$$y=x+\frac{\sqrt{\pi}\,c_2c_4}{2\sqrt{\alpha x}}.$$

we have

$$y\leqq x+\sqrt{\pi}\,c_2c_4/(2\sqrt{\alpha}),$$
$$N(\tfrac{1}{2}\sigma^{-1}(1/(\eta x)),S)\leqq N(\tfrac{1}{2}\sigma^{-1}(1/(\eta y)),S),$$

and

$$\Phi_d(x)=\Phi_d(y-\sqrt{\pi}\,c_2c_4/(2\sqrt{\alpha x})$$
$$\leqq 4a_1 y^{d-2} e^{-y^2/2+\frac{\sqrt{\pi}c_2c_4y}{2\sqrt{\alpha x}}}$$
$$\leqq c_3\Phi_d(y).$$

This completes the proof of Lemma 3.

Now we begin the proof of Theorem when $I(h,\phi)<+\infty$. Let

$$[s,t]=\{u;\ s\leqq u\leqq t\},$$
$$K=\{(s,t)\in[0,1]^2;\ 0<t-s\leqq\tfrac{1}{2}\}=\bigcup_{n=1}^{\infty}\bigcup_{i=1}^{2^{n+1}-1}K_{n,i},$$
$$K_{n,i}=\{(s,t);2^{-n-1}\leqq t-s\leqq 2^{-n},i2^{-n-1}\leqq t\leqq(i+1)2^{-n-1}\},$$
$$n=1,2,\ldots,1\leqq i\leqq 2^{n+1}-1,$$
$$I_{n,i}=\{t;\ (i-2)2^{-n-1}\leqq t\leqq(i+1)2^{-n-1}\},\qquad i=2,3,\ldots,2^{n+1}-1,$$
$$I_{n,1}=\{t;\ 0\leqq t\leqq 2^{-n}\},$$

and

$$A_{n,i}=\{\omega;\ \sup_{(s,t)\in K_{n,i}}\|B^d(t,\omega)-B^d(s,\omega)\|/\sqrt{t-s}\geqq\phi(2^{-n})\}.$$

We denote by $X_{n,i}$ the indicator function of $A_{n,i}$ and for convenience we understand that $1\cdot I=I$ and $0I=\phi$ for a time interval I. Then for each ω, we set

$$I_n(\omega)=\bigcup_{i=1}^{2^{n+1}-1}X_{n,i}I_{n,i}\quad\text{and}\quad I(\omega)=\bigcap_{m=1}^{\infty}\bigcup_{n\geqq m}I_n(\omega).$$

Now we show that

$$E(\phi, \omega) \subset I(\omega). \tag{3.8}$$

In fact, for each $t \in E(\phi, \omega)$, by definition there exists $\{(u_n, v_n)\} \in \mathscr{S}$ such that

$$\|B^d(t+u_n, \omega) - B^d(t-v_n, \omega)\| \geqq \sqrt{u_n+v_n}\, \phi(u_n+v_n).$$

On the other hand, for each n there exists (m_n, i_n) such that $m_n \uparrow +\infty\,(n \uparrow +\infty)$ and $(t-v_n, t+u_n) \in K_{m_n, i_n}$. So we have

$$\begin{aligned}
&\sup_{(s', t') \in K_{m_n, i_n}} \|B^d(t', \omega) - B^d(s', \omega)\| / \sqrt{t'-s'} \\
&\qquad \geqq \|B^d(t+u_n, \omega) - B^d(t-v_n, \omega)\| / \sqrt{u_n+v_n} \\
&\qquad \geqq \phi(u_n+v_n) \geqq \phi(2^{-m_n}).
\end{aligned}$$

Since this means $\omega \in A_{m_n, i_n}$, we have

$$t \in [t-v_n, t+u_n] \subset X_{m_n, i_n} I_{m_n, i_n} \quad \text{for all } n$$

which implies (3.8).

In order to apply Lemma 3 to estimate the probability $P(A_{n,i})$, we let $K_{n,i}$ be a compact metric space with the usual two dimensional Euclidean metric, and by easy calculation, we have

$$E\left(\frac{B(t,\omega)-B(s,\omega)}{\sqrt{t-s}} - \frac{B(t',\omega)-B(s',\omega)}{\sqrt{t'-s'}}\right)^2 \leqq 2^{n+5/2} \|(s,t)-(s',t')\|.$$

Therefore setting

$$c_1 = 1, \quad c_2 = 1, \quad c_4 > 8(\sqrt{2}+1),$$
$$N = 2, \quad \alpha = 1/2 \quad \text{and} \quad \sigma(x) = \sqrt{x},$$

in Lemma 3, we have

$$\begin{aligned}
&P\Big(\sup_{(s,t) \in K_{n,i}} \|B^d(t,\omega) - B^d(s,\omega)\| / \sqrt{t-s} \geqq \phi(2^{-n})\Big) \\
&\qquad \leqq 5 \cdot 2^5 c_3 c_5 \phi^4(2^{-n}) \Phi_d(\phi(2^{-n})), \quad \text{for } \phi(2^{-n}) \geqq 1 + \sqrt{\pi}\, c_4 \sqrt{2}.
\end{aligned} \tag{3.9}$$

Combining Lemma 1 and (3.9), we have

$$\begin{aligned}
&\sum_n \sum_{i=1}^{2^{n+1}-1} P(A_{n,i})\, h(|I_{n,i}|)^3 \\
&\qquad \leqq \sum_n 5 \cdot 2^5 c_3 c_5 2^{n+1} \phi^4(2^{-n}) \Phi_d(\phi(2^{-n}))\, h(3 \cdot 2^{-n-1}) \\
&\qquad \leqq 3 \cdot 5 \cdot 2^8 a_1 c_3 c_5 \sum_n (3 \cdot 2^{-n} - 3 \cdot 2^{-n-1})(3 \cdot 2^{-n})^{-2} \phi^{d+2}(3 \cdot 2^{-n-1}) \\
&\qquad\quad \cdot e^{-\phi^2(3 \cdot 2^{-n-1})/2}\, h(3 \cdot 2^{-n-1}) \\
&\qquad \leqq 3 \cdot 5 \cdot 2^8 a_1 c_3 c_5 \int_{+0} x^{-2} \phi^{d+2}(x)\, e^{-\phi^2(x)/2}\, h(x)\, dx < +\infty.
\end{aligned}$$

---

3 We denote by $|I|$ the Lebesgue measuer of a time set I

It follows that

$$\sum_n \sum_{i=1}^{2^{n+1}-1} h(X_{n,i}|I_{n,i}|) < +\infty \quad \text{a.s.}$$

With the help of Theorem 3.2 of Roger's book [7], we have

$$h - m(I(\omega)) = 0 \quad \text{a.s.}.$$

This yields the proof of Theorem when $I(h, \phi) < +\infty$.

Remark. To prove the convergent case of $I(h, \phi)$, we have used only the conditions (1.3) and (1.6).

Remark. For a Gaussian process $\{X(t);\ 0 \leqq t \leqq 1\}$ such that

$$E(X(t) - X(s))^2 \leqq \sigma^2(|t-s|),$$

if Lemma 3 is applicable, then

$$\int_{+0} [\sigma^{-1}(\sigma(x)/\phi(x))]^{-2}\, \phi^{d-2}(x)\, e^{-\phi^2(x)/2}\, h(x)\, dx < +\infty$$

implies $h - m(E_\sigma(\phi, \omega)) = 0$ a.s., where $E_\sigma(\phi, \omega)$ is a ω-set replacing $\sigma(x)$ instead of $\sqrt{x}$ in $E(\phi, \omega)$.

4. Proof when $I(h, \phi) = +\infty$

The proof of this part is much more complicated than that of the previous section.

First we need the following lemma to impose an additional condition on $\phi(x)$.

Lemma 4. *Without loss of generality, we can assume that $\phi(x)$ satisfies the following condition:*

$$\phi(x) \leqq \sqrt{3 \log 1/x}. \tag{4.1}$$

Proof. Let

$$\bar{\phi}(x) = \min(\phi(x), \sqrt{3 \log 1/x}),$$

then by the same technique as that of Lemma 1 of [3], we can prove that $I(h, \bar{\phi}) = +\infty$. Since $\phi(x)$ and $\sqrt{3 \log 1/x}$ satisfy (1.3), (1.4) and (1.8) so does $\bar{\phi}(x)$. Moreover $\sqrt{3 \log 1/x}$ being of the upper class with respect to the uniform continuity, we have

$$E(\phi, \omega) = E(\bar{\phi}, \omega) \quad \text{a.s.}$$

This yields the proof of Lemma 4.

The next lemma allows us to impose an additional condition to $h(x)$.

Lemma 5. *Without loss of generality we can assume that $h(x)$ satisfies the following condition:*

$$x^{-1} \phi^{d+2}(x)\, e^{-\phi^2(x)/2}\, h(x) \leqq 1. \tag{4.2}$$

Proof. It is sufficient to prove that we can construct a positive continuous function $\bar{h}(x)$ such that

(i) $\bar{h}(x) \leqq h(x)$,
(ii) $\bar{h}(x) \downarrow 0 \qquad (x \downarrow 0)$,
(iii) $\bar{h}(x)/x \uparrow +\infty \quad (x \downarrow 0)$,
(iv) $I(\bar{h}, \phi) = +\infty$,
(v) $\bar{h}(x)$ satisfies (4.2).

Since we are interested in only small x, we can assume that $\phi^{d+2}(x)\, e^{-\phi^2(x)/2}$ is non-decreasing and also multiplying by a constant, if necessary, we can assume that there exists $\varepsilon_{n_0} = 2^{-n_0}$ such that

$$\varepsilon_{n_0}^{-1}\, \phi^{d+2}(\varepsilon_{n_0})\, e^{-\phi^2(\varepsilon_{n_0})/2}\, h(\varepsilon_{n_0}) \leqq 1/2.$$

By (1.3) and (1.6), we have

$$\begin{aligned} x^{-1} \phi^{d+2}(x)\, e^{-\phi^2(x)/2}\, h(x) &\leqq 2\varepsilon_{n_0}^{-1} \phi^{d+2}(\varepsilon_{n_0})\, e^{-\phi^2(\varepsilon_{n_0})/2}\, h(\varepsilon_{n_0}) \\ &\leqq 1 \qquad \text{for } \varepsilon_{n_0+1} \leqq x \leqq \varepsilon_{n_0}. \end{aligned} \tag{4.3}$$

Let n_1 be the first $n\ (>n_0)$ which violates the inequality

$$\varepsilon_n^{-1} \phi^{d+2}(\varepsilon_n)\, e^{-\phi^2(\varepsilon_n)/2}\, h(\varepsilon_n) \leqq 1/2, \qquad (\varepsilon_n = 2^{-n}). \tag{4.4}$$

If there is no such n_1, taking account of (4.3), $h(x)$ satisfies (4.2) for all $x \leqq \varepsilon_{n_0}$, so we have nothing to do. When there is such n_1, we define a new function $h_1(x)$ as follows:

$$\begin{aligned} h_1(x) &= h(x) && \text{if } \varepsilon_{n_1-1} \leqq x \leqq \varepsilon_{n_0}, \\ &= h(\varepsilon_{n_1-1})\, x/\varepsilon_{n_1-1} && \text{if } \varepsilon_{n_1} \leqq x \leqq \varepsilon_{n_1-1}, \\ &= \tfrac{1}{2} h(\varepsilon_{n_1-1})\, h(x)/h(\varepsilon_{n_1}) && \text{if } x \leqq \varepsilon_{n_1}. \end{aligned} \tag{4.5}$$

Then by (1.6) and (1.7), $h_1(x)$ satisfies (i), (ii) and (iii) for all $x \leqq \varepsilon_{n_0}$, and (iv) is also fulfilled from $I(h, \phi) = +\infty$. By definition and (4.3), $h_1(x)$ satisfies (4.2) for all $\varepsilon_{n_1+1} \leqq x \leqq \varepsilon_{n_0}$, moreover

$$\begin{aligned} &\varepsilon_{n_1-1}^{-1} \phi^{d+2}(\varepsilon_{n_1-1})\, e^{-\phi^2(\varepsilon_{n_1-1})/2}\, h_1(\varepsilon_{n_1-1}) \\ &\quad \geqq 2^{-1} \varepsilon_{n_1}^{-1} \phi^{d+2}(\varepsilon_{n_1})\, e^{-\phi^2(\varepsilon_{n_1})/2}\, h(\varepsilon_{n_1}) \\ &\quad \geqq 1/4. \end{aligned} \tag{4.6}$$

By induction assume that we have constructed $h_{m-1}(x)$ which satisfies (i), (ii), (iii) for all $x \leqq \varepsilon_{n_0}$ with (iv), (4.4) for all $n_0 \leqq n \leqq n_{m-1}$, and (4.6) for $n_{m-1} - 1$. Then let n_m be the first $n\ (>n_{m-1})$ which violates (4.4) for $h_{m-1}(x)$ instead of $h(x)$. If there is no such n, we define $\bar{h}(x)$ by $h_{m-1}(x)$ then $\bar{h}(x)$ is a desired function by taking account of (4.3). If there exists such n_m, then $h_m(x)$ is defined by

$$\begin{aligned} h_m(x) &= h_{m-1}(x) && \text{if } \varepsilon_{n_m-1} \leqq x \leqq \varepsilon_{n_0} \\ &= h_{m-1}(\varepsilon_{n_m-1})\, x/\varepsilon_{n_m-1} && \text{if } \varepsilon_{n_m} \leqq x \leqq \varepsilon_{n_m-1} \\ &= \tfrac{1}{2} h_{m-1}(\varepsilon_{n_m-1})\, h_{m-1}(x)/h_{m-1}(\varepsilon_{n_m}) && \text{if } x \leqq \varepsilon_{n_m}. \end{aligned}$$

Then by the same procedure with $h_1(x)$, $h_m(x)$ satisfies (i), (ii) and (iii) for all $x \leqq \varepsilon_{n_0}$ with (iv), (4.2) for all $\varepsilon_{n_m+1} \leqq x \leqq \varepsilon_{n_0}$ and (4.6) for $n_k - 1$, $1 \leqq k \leqq m$. Now it is enough to investigate the case that there exists an infinite sequence of such $\{n_m\}_{m=1}^{\infty}$. Let

$$\bar{h}(x) = h_m(x) \quad \text{if } \varepsilon_{n_{m+1}-1} \leqq x \leqq \varepsilon_{n_0}.$$

Then obviously $\bar{h}(x)$ satisfies (i), (ii) and (v). By (4.6) we have

$$\begin{aligned} I(\bar{h}, \phi) &= \int_{+0} x^{-2} \phi^{d+2}(x)\, e^{-\phi^2(x)/2}\, \bar{h}(x)\, dx \\ &\geqq \sum_m \int_{\varepsilon_{n_m-1}}^{\varepsilon_{n_m-2}} x^{-2} \phi^{d+2}(x)\, e^{-\phi^2(x)/2}\, h_m(x)\, dx \\ &\geqq \sum_m \tfrac{1}{2} \varepsilon_{n_m-1}^{-1} \phi^{d+2}(\varepsilon_{n_m-1})\, e^{-\phi^2(\varepsilon_{n_m-1})/2}\, h_m(\varepsilon_{n_m-1}) \\ &\geqq \sum_m \tfrac{1}{8} = +\infty. \end{aligned}$$

Since $\bar{h}(x)/x$ is monotone by definition, if we assume that $\bar{h}(x)/x$ is bounded, say by C, then we have

$$\int_{+0} x^{-2} \phi^{d+2}(x)\, e^{-\phi^2(x)/2}\, \bar{h}(x)\, dx \leqq C \int_{+0} x^{-1} \phi^{d+2}(x)\, e^{-\phi^2(x)/2}\, dx < +\infty,$$

which contradicts with (iv), which shows that (iii) is verified. This completes the proof of Lemma 5.

Now we begin the proof of our Theorem when $I(h, \phi) = +\infty$, with the help of several lemmas. We recall that by virtue of Lemma 4 and Lemma 5, we can assume that $\phi(x)$ and $h(x)$ satisfy the additional conditions (4.1) and (4.2) respectively.

For sufficiently large n, let

$$\varepsilon_n = 2^{-n}, \qquad \delta_n = a_5\, \varepsilon_{n+2}/\phi^2(\varepsilon_{n+2}), \qquad a_5 = 10^5\, d^{d/2}\, 2^{3d/2},$$

$$\Delta_n = [\varepsilon_{n+2}/\delta_n]^{4},$$

$$\begin{aligned} A_{k,i,j}^{(n)} = \{\omega;\ &\|B^d(k\,\varepsilon_n + 2\varepsilon_{n+2} + j\,\delta_n, \omega) - B^d(k\,\varepsilon_n + i\,\delta_n, \omega)\| \\ &\geqq \sqrt{2\varepsilon_{n+2} + (j-i)\,\delta_n}\; \phi(\varepsilon_{n+2})\}, \quad (1 \leqq i, j \leqq \Delta_n,\ 0 \leqq k \leqq 2^n - 1) \end{aligned}$$

$$\begin{aligned} X_{n,k} &= \text{the indicator function of } \bigcup_{1 \leqq i,j \leqq \Delta_n} A_{k,i,j}^{(n)} \text{ for } 0 \leqq k \leqq 2^n - 1, \\ &= 0 \ \text{ otherwise}, \end{aligned}$$

and

$$P(X_{n,k} = 1) = p_n.$$

Then we have

Lemma 6.

$$\begin{aligned} 2 a_1 a_5^{-2} \phi^{d+2}(\varepsilon_{n+2})\, e^{-\phi^2(\varepsilon_{n+2})/2} &\geqq \sum_{i,j} P(A_{k,i,j}^{(n)}) \\ \geqq p_n \geqq 0.8 \sum_{i,j} P(A_{k,i,j}^{(n)}) &\geqq 0.2 a_1 a_5^{-2} \phi^{d+2}(\varepsilon_{n+2})\, e^{-\phi^2(\varepsilon_{n+2})/2}. \end{aligned} \tag{4.7}$$

---

[4] We denote by $[x]$ the integral part of x

Proof. Since we have

$$\sum_{i,j} P(A^{(n)}_{0,i,j}) \geqq p_n \geqq \sum_{i,j} P(A^{(n)}_{0,i,j}) - \sum_{(i,j)\neq(i',j')} P(A^{(n)}_{0,i,j} \cap A^{(n)}_{0,i',j'}),$$

let us find the upper bound of the last term by (2.3).

Setting

$$X = \frac{B(2\varepsilon_{n+2} + j\,\delta_n) - B(i\,\delta_n)}{\sqrt{2\varepsilon_{n+2} + (j-i)\,\delta_n}},$$

$$Y = \frac{B(2\varepsilon_{n+2} + j'\,\delta_n) - B(i'\,\delta_n)}{\sqrt{2\varepsilon_{n+2} + (j'-i')\,\delta_n}},$$

we have

$$\begin{aligned} 1 - EXY &\geqq (|j-j'| + |i-i'|)\, 3^{-1}\, \varepsilon_n^{-1}\, \delta_n \\ &= (|j-j'| + |i-i'|)\, 12^{-1}\, a_5\, \phi^{-2}(\varepsilon_{n+2}). \end{aligned}$$

Applying Lemma 2, (ii), we have

$$P(A^{(n)}_{0,i,j} \cap A^{(n)}_{0,i',j'}) \leqq a_3\, e^{-a_5(|j-j'| + |i-i'|)/384}\, \Phi_d(\phi(\varepsilon_{n+2})),$$

and

$$\begin{aligned} &\sum_{(i,j)\neq(i',j')} P(A^{(n)}_{0,i,j} \cap A^{(n)}_{0,i',j'}) \\ &\qquad \leqq 4a_3 \sum_{i,j} \sum_{k=1}^{\infty} \sum_{k'=0}^{k} e^{-a_5 k/384}\, \Phi_d(\phi(\varepsilon_{n+2})) \\ &\qquad \leqq 0.2 \sum_{i,j} \Phi_d(\phi(\varepsilon_{n+2})). \end{aligned}$$

It follows that

$$P(X_{n,0} = 1) \geqq 0.8 \sum_{i,j} P(A^{(n)}_{0,i,j}).$$

The rest of (4.7) are easily obtained by (2.1).

Now for each sufficiently large n, we correspond three integers $n'' > n' > \bar{n}\ (> n)$, which are possibly choosen from (1.6), (1.7), (4.2) and $I(h, \phi) = +\infty$, satisfying the following conditions:

$$2^{25}\, h(\varepsilon_{n+2})\, \varepsilon_{n+2}^{-1} \geqq h(\varepsilon_{\bar{n}+2})\, \varepsilon_{\bar{n}+2}^{-1} \geqq 2^{24}\, h(\varepsilon_{n+2})\, \varepsilon_{n+2}^{-1}, \tag{4.8}$$

$$\frac{h(\varepsilon_{n'+2})}{\varepsilon_{n+2}\, h(\varepsilon_{n+2})} \leqq n^{-2}, \tag{4.9}$$

$$h(\varepsilon_{n'+2})\, \varepsilon_{\bar{n}+2} \geqq 4a_2\, h(\varepsilon_{\bar{n}+2})\, \varepsilon_{n'+2}, \tag{4.10}$$

and

$$2^{-5}\, h(\varepsilon_{\bar{n}+2})\, \varepsilon_{\bar{n}+2}^{-1} \leqq \sum_{m=n'}^{n''} h(\varepsilon_{m+2})\, p_m\, \varepsilon_{m+2}^{-1} \leqq 2^{-4}\, h(\varepsilon_{\bar{n}+2})\, \varepsilon_{\bar{n}+2}^{-1}. \tag{4.11}$$

Since we are interested in only large n, we assume that n satisfies the following conditions:

$$\int_{+0}^{2^{-n-1}} x^{-1}\phi^{d+2}(x)\,e^{-\phi^2(x)/2}\,dx<20^{-1}a_1^{-1}a_2^{-1}a_5^2, \tag{4.12}$$

$$\log\phi(\varepsilon_{n+2})\,(2a_1\phi^{d+2}(\varepsilon_{n+2})+a_4\phi^4(\varepsilon_{n+2}))\,e^{-8\cdot 10^{-4}\phi^2(\varepsilon_{n+2})}<0.02a_5^2, \tag{4.13}$$

and

$$\phi(\varepsilon_{n+2})>10^4 d^d. \tag{4.14}$$

Let

$$I_{n,k}=\{t;\ k\varepsilon_n+\varepsilon_{n+2}<t<k\varepsilon_n+2\varepsilon_{n+2}\},$$

$$b_{n,m}=\left[\frac{\varepsilon_{\bar{n}+2}\,h(\varepsilon_{m+2})}{h(\varepsilon_{\bar{n}+2})\,\varepsilon_{m+2}}\right]+1, \qquad n'\leqq m\leqq n'',$$

$$Y_{m,k'}^{(n)}=\prod_{b_{n,m}\geqq|s|\geqq 1}(1-X_{m,k'+s})\prod_{b_{n,m-1}\geqq|s|\geqq 1}(1-X_{m-1,[k'/2]+s})$$

$$\cdots\prod_{b_{n,n'}\geqq|s|\geqq 1}(1-X_{n',[k'2^{n'-m}]+s}),$$

$$Z_{m,k'}^{(n)}=\prod_{m-n'\geqq s\geqq 1}(1-X_{m-s,[k'2^{-s}]}),$$

$$I_m^{(n,k)}=\{X_{m,k'}\cdot Y_{m,k'}^{(n)}\cdot Z_{m,k'}^{(n)}\,I_{m,k'};\ k' \text{ varies such that } k\varepsilon_n\leqq k'\varepsilon_m<(k+1)\,\varepsilon_n\},$$

$$I^{(n,k)}=\bigcup_{n'\leqq m\leqq n''} I_m^{(n,k)},$$

$$H_m^{(n,k)}=\sum_{I_{m,k'}\in I_m^{(n,k)}} h(|I_{m,k'}|),$$

and

$$H^{(n,k)}=\sum_{m=n'}^{n''} H_m^{(n,k)},$$

here we recall that $1\cdot I=I$ and $0I=\phi$, and we exclude ϕ from a family of open intervals $I_m^{(n,k)}$ which depends on a path.

Under the above formulation we prove the following:

Lemma 7.

$$EH^{(n,k)}\leqq 2^{21}h(\varepsilon_{n+2}) \tag{4.15}$$

and

$$EH^{(n,k)}\geqq 2^{17}h(\varepsilon_{n+2}). \tag{4.16}$$

Proof. Easily we have

$$EH_m^{(n,k)}\leqq\sum_{k'=k2^{m-n}}^{(k+1)2^{m-n}-1} EX_{m,k'}\,h(|I_{m,k'}|)=\varepsilon_{n+2}\,\varepsilon_{m+2}^{-1}\,h(\varepsilon_{m+2})\,p_m,$$

and

$$EH^{(n,k)} \leqq \varepsilon_{n+2} \sum_{m=n'}^{n''} \varepsilon_{m+2}^{-1} h(\varepsilon_{m+2}) p_m \leqq 2^{21} h(\varepsilon_{n+2}), \quad \text{(by (4.8) and (4.11))}.$$

It is more complicated to prove (4.16) than that of (4.15). First we have

$$\begin{aligned} EH_m^{(n,k)} \geqq & \sum_{k'=k2^{m-n}}^{(k+1)2^{m-n}-1} h(\varepsilon_{m+2}) E\Big\{X_{m,k'}\Big(1-\sum_{j=n'}^{m-1} X_{j,[k'2^{j-m}]} \\ & -\sum_{j=n'}^{m} \sum_{b_{n,j}\geqq|s|\geqq 1} X_{j,[k'2^{j-m}]+s}\Big)\Big\} \\ = & \sum_{k'=k2^{m-n}}^{(k+1)2^{m-n}-1} h(\varepsilon_{m+2}) \Big(p_m - \sum_{j=n'}^{m-1} EX_{m,k'} X_{j,[k'2^{j-m}]} \\ & -\sum_{j=n'}^{m} \sum_{b_{n,j}\geqq|s|\geqq 1} p_m p_j\Big). \end{aligned} \tag{4.17}$$

For the third term, we have

$$\begin{aligned} \sum_{j=n'}^{m} \sum_{b_{n,j}\geqq|s|\geqq 1} p_m p_j & \leqq 2p_m \sum_{j=n'}^{m} b_{n,j} p_j \\ & \leqq 4p_m \sum_{j=n'}^{n''} \varepsilon_{\bar{n}+2} h(\varepsilon_{j+2}) h(\varepsilon_{\bar{n}+2})^{-1} \varepsilon_{j+2}^{-1} p_j \\ & \leqq 2^{-2} p_m \quad \text{(by (4.11))}. \end{aligned} \tag{4.18}$$

To estimate the second term, setting

$$X = \frac{B(\bar{k}\varepsilon_j + 2\varepsilon_{j+2} + v'\delta_j) - B(\bar{k}\varepsilon_j + u'\delta_j)}{\sqrt{2\varepsilon_{j+2} + (v'-u')\delta_j}}, \qquad (\bar{k} = [k'2^{j-m}])$$

$$Y = \frac{B(k'\varepsilon_m + 2\varepsilon_{m+2} + v\delta_m) - B(k'\varepsilon_m + u\delta_m)}{\sqrt{2\varepsilon_{m+2} + (v-u)\delta_m}},$$

for $n' \leqq j \leqq m-1$, we have

$$0 \leqq EXY \leqq \sqrt{3}/2 \qquad \text{if } j = m-1, \tag{4.19}$$

and

$$0 \leqq EXY \leqq \sqrt{3} 2^{(j-m)/2} \qquad \text{if } n' \leqq j \leqq m-2.$$

We divide into the following two cases:

Case (I). Assume that $m - 10\log\phi(\varepsilon_{m+2}) \leqq j \leqq m-1$.

By Lemma 4, we have

$$m - 10\log\phi(\varepsilon_{m+2}) \geqq m - 6\log m,$$

and

$$\varepsilon_{j+2} \leqq \varepsilon_{m+2} m^5.$$

Combining this with (1.3) and (1.8), we have

$$\phi(\varepsilon_{m+2})/\phi(\varepsilon_{j+2}) \leqq \phi(\varepsilon_{m+2})/\phi(\varepsilon_{m+2}\, m^5) \leqq 2^{10}\, 10^{-3}.$$

With the help of (4.14) and (4.19), we have

$$2^{-1}(1+r)\,\phi(\varepsilon_{m+2}) + \sqrt{2d} \leqq 0.935\,\phi(\varepsilon_{m+2}) \leqq 0.975\,\phi(\varepsilon_{m+2}) \leqq \phi(\varepsilon_{j+2}),$$

where $r = EXY$.

It follows from (2.1), (2.4) and (4.7) that

$$\begin{aligned} EX_{m,k'}\, X_{j,\bar k} &\leqq \sum_{u,v} \sum_{u',v'} P(A^{(m)}_{k',u,v} \cap A^{(j)}_{\bar k,u',v'}) \\ &\leqq \sum_{u,v} \sum_{u',v'} (2a_1 \phi^{d-2}(\varepsilon_{m+2}) + a_4)\, e^{-8\cdot 10^{-4}\phi^2(\varepsilon_{m+2})}\, \Phi_d(\phi(\varepsilon_{m+2})) \\ &\leqq 1.25 a_5^{-2}(2a_1 \phi^{d+2}(\varepsilon_{m+2}) + a_4 \phi^4(\varepsilon_{m+2}))\, e^{-8\cdot 10^{-4}\phi^2(\varepsilon_{m+2})}\, p_m. \end{aligned}$$

Therefore we have

$$\begin{aligned} &\sum_{m-10\log\phi(\varepsilon_{m+2}) \leqq j \leqq m-1} EX_{m,k'}\, X_{j,\bar k} \\ &\quad\leqq 12.5 a_5^{-2} p_m \log\phi(\varepsilon_{m+2})(2a_1 \phi^{d+2}(\varepsilon_{m+2}) + a_4 \phi^4(\varepsilon_{m+2}))\, e^{-8\cdot 10^{-4}\phi^2(\varepsilon_{m+2})} \\ &\quad\leqq 4^{-1} p_m. \qquad \text{(by (4.13))} \end{aligned} \tag{4.20}$$

Case (II). Assume that $n' \leqq j \leqq m - 10\log\phi(\varepsilon_{m+2})$.

By (4.19)

$$\begin{aligned} r\,\phi(\varepsilon_{m+2})\,\phi(\varepsilon_{j+2}) &\leqq \phi^2(\varepsilon_{m+2})\sqrt{3}\, 2^{-5\log\phi(\varepsilon_{m+2})} \\ &\leqq \sqrt{3}\,\phi^{-1}(\varepsilon_{m+2}) < 1. \end{aligned}$$

It follows from (2.1) and (2.2) that

$$\begin{aligned} EX_{m,k'}\, X_{j,\bar k} &\leqq \sum_{u,v} \sum_{u',v'} a_2\, \Phi_d(\phi(\varepsilon_{m+2}))\, \Phi_d(\phi(\varepsilon_{j+2})) \\ &\leqq 2.5 a_1 a_2 a_5^{-2} p_m \phi^{d+2}(\varepsilon_{j+2})\, e^{-\phi^2(\varepsilon_{j+2})/2}. \end{aligned}$$

Hence by (4.12), we have

$$\begin{aligned} &\sum_{n' \leqq j \leqq m-10\log\phi(\varepsilon_{m+2})} EX_{m,k'}\, X_{j,\bar k} \\ &\quad\leqq 2.5 a_1 a_2 a_5^{-2} p_m \sum_{j=n'}^{\infty} \phi^{d+2}(\varepsilon_{j+2})\, e^{-\phi^2(\varepsilon_{j+2})/2} \\ &\quad\leqq 5 a_1 a_2 a_5^{-2} p_m \int_{+0}^{2^{-n'-1}} x^{-1} \phi^{d+2}(x)\, e^{-\phi^2(x)/2}\, dx \\ &\quad\leqq 4^{-1} p_m. \end{aligned} \tag{4.21}$$

Combining (4.17), (4.18), (4.20) and (4.21), we have

$$EH_m^{(n,k)} \geqq 4^{-1} \varepsilon_{n+2} \varepsilon_{m+2}^{-1} h(\varepsilon_{m+2}) p_m,$$

and

$$EH^{(n,k)} \geqq 4^{-1} \varepsilon_{n+2} \sum_{m=n'}^{n''} \varepsilon_{m+2}^{-1} h(\varepsilon_{m+2}) p_m \geqq 2^{17} h(\varepsilon_{n+2}),$$

(by (4.8) and (4.11)).

This completes the proof of (4.16).

Lemma 8. *Let*

$$L^{(n,k)} = E(H^{(n,k)} - EH^{(n,k)})^2.$$

Then

$$L^{(n,k)} \leqq 2^{24} n^{-2} \varepsilon_{n+2} h^2(\varepsilon_{n+2}). \tag{4.22}$$

Proof. We calculate $L^{(n,k)}$ by dividing several parts:

$$\begin{aligned} L^{(n,k)} &= \sum_{m=n'}^{n''} E(H_m^{(n,k)} - EH_m^{(n,k)})^2 + 2 \sum_{n' \leqq m < m' \leqq n''} E(H_m^{(n,k)} - EH_m^{(n,k)})(H_{m'}^{(n,k)} - EH_{m'}^{(n,k)}) \\ &= \sum_{m=n'}^{n''} L_m^{(1)} + 2 \sum_{n' \leqq m < m' \leqq n''} L_{m,m'}^{(2)}. \end{aligned}$$

$$\begin{aligned} L_m^{(1)} &= E(H_m^{(n,k)} - EH_m^{(n,k)})^2 \\ &= \sum_{k'=k2^{m-n}}^{(k+1)2^{m-n}-1} h^2(\varepsilon_{m+2}) E(X_{m,k'} Y_{m,k'}^{(n)} Z_{m,k'}^{(n)} - EX_{m,k'} Y_{m,k'}^{(n)} Z_{m,k'}^{(n)})^2 \\ &\quad + 2 \sum_{k2^{m-n} \leqq k' < k'' \leqq (k+1)2^{m-n}-1} h^2(\varepsilon_{m+2}) E(X_{m,k'} Y_{m,k'}^{(n)} Z_{m,k'}^{(n)} \\ &\quad - EX_{m,k'} Y_{m,k'}^{(n)} Z_{m,k'}^{(n)})(X_{m,k''} Y_{m,k''}^{(n)} Z_{m,k''}^{(n)} - EX_{m,k''} Y_{m,k''}^{(n)} Z_{m,k''}^{(n)}) \\ &= L_{m,1}^{(1)} + L_{m,2}^{(1)}. \end{aligned}$$

Obviously we have

$$L_{m,1}^{(1)} \leqq \sum_{k'=k2^{m-n}}^{(k+1)2^{m-n}-1} h^2(\varepsilon_{m+2}) EX_{m,k'} = \varepsilon_{m+2}^{-1} \varepsilon_{n+2} h^2(\varepsilon_{m+2}) p_m. \tag{4.23}$$

Since $X_{m,k'} Y_{m,k'}^{(n)} Z_{m,k'}^{(n)}$ is independent of $X_{m,k''} Y_{m,k''}^{(n)} Z_{m,k''}^{(n)}$ if $|k'-k''| > 4 b_{n,n'} \varepsilon_{n'} \varepsilon_m^{-1}$, it follows that

$$\begin{aligned} L_{m,2}^{(1)} &\leqq 2 \sum_{\substack{k2^{m-n} \leqq k' < k'' \\ \leqq k' + 4 b_{n,n'} \varepsilon_{n'} \varepsilon_m^{-1} \\ < (k+1) 2^{m-n}}} h^2(\varepsilon_{m+2}) EX_{m,k'} X_{m,k''} \\ &\leqq 8 b_{n,n'} \varepsilon_{n'} \varepsilon_n \varepsilon_m^{-2} h^2(\varepsilon_{m+2}) p_m^2. \end{aligned} \tag{4.24}$$

For $L^{(2)}_{m,m'}$, setting

$$L^{(2)}_{m,m'} = \sum_{k'=k2^{m-n}}^{(k+1)2^{m-n}-1} \sum_{k''=k2^{m'-n}}^{(k+1)2^{m'-n}-1} h(\varepsilon_{m+2})\, h(\varepsilon_{m'+2})\, E\bar{X}^{(n)}_{m,k'} \bar{X}^{(n)}_{m',k''},$$

where

$$\bar{X}^{(n)}_{m,k'} = X_{m,k'} Y^{(n)}_{m,k'} Z^{(n)}_{m,k'} - EX_{m,k'} Y^{(n)}_{m,k'} Z^{(n)}_{m,k'}$$

and

$$\bar{X}^{(n)}_{m',k''} = X_{m',k''} Y^{(n)}_{m',k''} Z^{(n)}_{m',k''} - EX_{m',k''} Y^{(n)}_{m',k''} Z^{(n)}_{m',k''},$$

we estimate $E\bar{X}^{(n)}_{m,k'} \bar{X}^{(n)}_{m',k''}$ by dividing into three cases:

Case (I). If $k''\varepsilon_{m'} - (k'+1)\varepsilon_m > 4b_{n,n'}\varepsilon_{n'}$ or $k'\varepsilon_m - (k''+1)\varepsilon_{m'} > 4b_{n,n'}\varepsilon_{n'}$, then we have $EX^{(n)}_{m,k'} \bar{X}^{(n)}_{m',k''} = 0$ because $\bar{X}^{(n)}_{m,k'}$ is independent of $\bar{X}^{(n)}_{m',k''}$.

Case (II). If $0 \leqq k''\varepsilon_{m'} - (k'+1)\varepsilon_m \leqq 4b_{n,n'}\varepsilon_{n'}$ or $0 \leqq k'\varepsilon_m - (k''+1)\varepsilon_{m'} \leqq 4b_{n,n'}\varepsilon_{n'}$, then we have

$$E\bar{X}^{(n)}_{m,k} \bar{X}^{(n)}_{m',k''} \leqq EX_{m,k'} X_{m',k''} = p_m p_{m'}.$$

Case (III). In case of $k'\varepsilon_m \leqq k''\varepsilon_{m'} < (k''+1)\varepsilon_{m'} \leqq (k'+1)\varepsilon_m$, it is more difficult than the previous ones. If $1 \leqq m' - m \leqq 10 \log \phi(\varepsilon_{m'+2})$, by the same argument as the case (I) of the proof of Lemma 7, we have

$$\begin{aligned} E\bar{X}^{(n)}_{m,k'} \bar{X}^{(n)}_{m',k''} &\leqq EX_{m,k'} X_{m',k''} \\ &\leqq 1.25 p_{m'} a_5^{-2} (2a_1 \phi^{d+2}(\varepsilon_{m'+2}) + a_4 \phi^4(\varepsilon_{m'+2})) e^{-8\cdot 10^{-4} \phi^2(\varepsilon_{m'+2})}. \end{aligned}$$

On the contrary, if $m' - m > 10 \log \phi(\varepsilon_{m'+2})$, then by the same argument as the case (II) of the proof of Lemma 7, we have

$$\begin{aligned} E\bar{X}^{(n)}_{m,k'} \bar{X}^{(n)}_{m',k''} &\leqq EX_{m,k'} X_{m',k''} \\ &\leqq 2.5 a_1 a_2 a_5^{-2} p_{m'} \phi^{d+2}(\varepsilon_{m+2}) e^{-\phi^2(\varepsilon_{m+2})/2} \\ &\leqq 12.5 a_2 p_m p_{m'}. \end{aligned}$$

Summing up all cases, if $1 \leqq m' - m \leqq 10 \log \phi(\varepsilon_{m'+2})$, then

$$\begin{aligned} L^{(2)}_{m,m'} \leqq & \sum_{k'=k2^{m-n}}^{(k+1)2^{m-n}-1} (8b_{n,n'}\varepsilon_{n'}\varepsilon_{m'}^{-1} h(\varepsilon_{m+2})\, h(\varepsilon_{m'+2}) p_m p_{m'} \\ &+ 1.25 \varepsilon_m \varepsilon_{m'}^{-1} h(\varepsilon_{m+2})\, h(\varepsilon_{m'+2}) p_{m'} a_5^{-2} (2a_1 \phi^{d+2}(\varepsilon_{m'+2}) \\ &+ a_4 \phi^4(\varepsilon_{m'+2})) e^{-8\cdot 10^{-4}} \phi^2(\varepsilon_{m'+2})) \\ \leqq & 8b_{n,n'}\varepsilon_{n'}\varepsilon_n \varepsilon_m^{-1} \varepsilon_{m'}^{-1} h(\varepsilon_{m+2})\, h(\varepsilon_{m'+2}) p_m p_{m'} \\ &+ 1.25 \varepsilon_n \varepsilon_{m'}^{-1} h(\varepsilon_{m+2})\, h(\varepsilon_{m'+2}) p_{m'} a_5^{-2} (2a_1 \phi^{d+2}(\varepsilon_{m'+2}) \\ &+ a_4 \phi^4(\varepsilon_{m'+2})) e^{-8\cdot 10^{-4} \phi^2(\varepsilon_{m'+2})}, \end{aligned} \tag{4.25}$$

and if $m'-m>10\log\phi(\varepsilon_{m'+2})$, then

$$
\begin{aligned}
L^{(2)}_{m,m'} \leqq & \sum_{k'=k2^{m-n}}^{(k+1)2^{m-n}-1} (8b_{n,n'}\varepsilon_{n'}\varepsilon_{m'}^{-1}h(\varepsilon_{m+2})h(\varepsilon_{m'+2})p_m p_{m'} \\
& + 12.5a_2\varepsilon_m\varepsilon_{m'}^{-1}h(\varepsilon_{m+2})h(\varepsilon_{m'+2})p_m p_{m'}) \\
= & 8b_{n,n'}\varepsilon_{n'}\varepsilon_n\varepsilon_m^{-1}\varepsilon_{m'}^{-1}h(\varepsilon_{m+2})h(\varepsilon_{m'+2})p_m p_{m'} \\
& + 12.5a_2\varepsilon_n\varepsilon_{m'}^{-1}h(\varepsilon_{m+2})h(\varepsilon_{m'+2})p_m p_{m'}. \qquad (4.26)
\end{aligned}
$$

Combining (4.10), (4.13), (4.23), (4.24), (4.25) and (4.26), we have

$$
\begin{aligned}
L^{(n,k)} \leqq & \sum_{m=n'}^{n''} (\varepsilon_{m+2}^{-1}\varepsilon_{n+2}h^2(\varepsilon_{m+2})p_m + 8b_{n,n'}\varepsilon_{n'}\varepsilon_n\varepsilon_m^{-2}h^2(\varepsilon_{m+2})p_m^2) \\
& + 2\sum_{\substack{n'\leqq m<m'\leqq n'' \\ m'-m\leqq 10\log\phi(\varepsilon_{m'+2})}} \{8b_{n,n'}\varepsilon_{n'}\varepsilon_n\varepsilon_m^{-1}\varepsilon_{m'}^{-1}h(\varepsilon_{m+2})h(\varepsilon_{m'+2})p_m p_{m'} \\
& + 1.25\varepsilon_n\varepsilon_{m'}^{-1}h(\varepsilon_{m+2})h(\varepsilon_{m'+2})p_{m'}a_5^{-2}(2a_1\phi^{d+2}(\varepsilon_{m'+2}) \\
& + a_4\phi^4(\varepsilon_{m'+2}))e^{-8\cdot 10^{-4}\phi^2(\varepsilon_{m'+2})}\} \\
& + 2\sum_{\substack{n'\leqq m<m'\leqq n'' \\ m'-m>10\log\phi(\varepsilon_{m'+2})}} (8b_{n,n'}\varepsilon_{n'}\varepsilon_n\varepsilon_m^{-1}\varepsilon_{m'}^{-1}h(\varepsilon_{m+2})h(\varepsilon_{m'+2})p_m p_{m'} \\
& + 12.5a_2\varepsilon_n\varepsilon_{m'}^{-1}h(\varepsilon_{m+2})h(\varepsilon_{m'+2})p_m p_{m'}) \\
\leqq & \varepsilon_{n+2}h(\varepsilon_{n'+2})\left(\sum_{m=n'}^{n''}\varepsilon_{m+2}^{-1}h(\varepsilon_{m+2})p_m\right) \\
& + (8b_{n,n'}\varepsilon_{n'}\varepsilon_{n+2} + 12.5a_2\varepsilon_{n+2}\varepsilon_{n'+2})\left(\sum_{m=n'}^{n''}\varepsilon_{m+2}^{-1}h(\varepsilon_{m+2})p_m\right)^2 \\
& + 2^{-1}\varepsilon_{n+2}h(\varepsilon_{n'+2})\left(\sum_{m=n'}^{n''}\varepsilon_{m+2}^{-1}h(\varepsilon_{m+2})p_m\right) \\
\leqq & 2\varepsilon_{n+2}h(\varepsilon_{n'+2})\left(\sum_{m=n'}^{n''}\varepsilon_{m+2}^{-1}h(\varepsilon_{m+2})p_m\right) \\
& + 9b_{n,n'}\varepsilon_{n'}\varepsilon_{n+2}\left(\sum_{m=n'}^{n''}\varepsilon_{m+2}^{-1}h(\varepsilon_{m+2})p_m\right)^2 \\
\leqq & 2^{24}n^{-2}\varepsilon_{n+2}h^2(\varepsilon_{n+2}),
\end{aligned}
$$

(by (4.8)–(4.11) and the definition of $b_{n,n'}$). This yields Lemma 8.

By virtue of (4.16), (4.22) and Čebyšev's inequality, we have

$$
\begin{aligned}
& P(0\leqq \exists k\leqq 2^n-1, |H^{(n,k)}-EH^{(n,k)}|\geqq \tfrac{1}{2}EH^{(n,k)}) \\
& \quad \leqq 2^{-10}n^{-2}
\end{aligned}
$$

which is a convergent sequence of n. Therefore by Borel-Cantelli lemma, there exists almost surely finite integer $n(\omega)$ such that for all $n\geqq n(\omega)$ and all $0\leqq k\leqq 2^n-1$,

$$\begin{aligned} H^{(n,k)} &\geqq \tfrac{1}{2} E H^{(n,k)} \\ &\geqq 2^{-3} \varepsilon_{n+2} \sum_{m=n'}^{n''} \varepsilon_{m+2}^{-1} h(\varepsilon_{m+2}) p_m \\ &\geqq 2^{-8} \varepsilon_{n+2} \varepsilon_{\bar{n}+2}^{-1} h(\varepsilon_{\bar{n}+2}) \qquad \text{(by (4.11))} \\ &\geqq 2^{16} h(\varepsilon_{n+2}) \qquad \text{(by (4.8))} \\ &\geqq 2^{14} h(\varepsilon_n) \qquad \text{(by (1.7)).} \end{aligned} \tag{4.27}$$

The final step will be cleared if for any positive number $M(>1)$, we can construct a subset N of $E(\phi, \omega)$ almost surely such that the Hausdorff outer measure $h-m^*(N)$ is greater than M. With the help of Jarnik's method [2] for diophantine approximation theory, we will accomplish this procedure.

First by (1.7) we can choose sufficiently large $n(\geqq n(\omega))$ such that

$$h(\varepsilon_{n+2}) \varepsilon_{n+2}^{-1} \geqq 2^{13} M. \tag{4.28}$$

Now let

$$\mathfrak{J}_1 = \{I^{(n,k)};\ 0 \leqq k \leqq 2^n - 1\}.$$

We recall that the $Y_{m,k'}^{(m)}$ and $Z_{m,k'}^{(n)}$ in the definition of $I_m^{(n,k)}$ force all the intervals of $I^{(n,k)}$ (they are at least two by (4.27)) to be disjoint-even separated by a certain distance. Since they are contained in $[k\varepsilon_n + \varepsilon_{n'+2},\ (k+1)\varepsilon_n - 2\varepsilon_{n'+2}]$, all the intervals of $\mathfrak{J}_1$ are also disjoint-even separated by a certain distance.

$$\sum_{I \in \mathfrak{J}_1} h(|I|) = \sum_{k=0}^{2^n-1} H^{(n,k)} \geqq 2^{14} \varepsilon_{n+2}^{-1} h(\varepsilon_{n+2}) \geqq 2^{27} M. \tag{4.29}$$

For each $I \in \mathfrak{J}_1$, there exist $n(I)$ and $k(I)$ such that

$$I = \{t;\ k(I)\varepsilon_{n(I)} < t < (k(I)+1)\varepsilon_{n(I)}\}.$$

Therefore setting

$$\mathfrak{J}(I) = I^{(n(I), k(I))},$$

and

$$\mathfrak{J}_2 = \bigcup_{I \in \mathfrak{J}_1} \mathfrak{J}(I),$$

we have

$$\sum_{I' \in \mathfrak{J}(I)} h(|I'|) = H^{(n(I), k(I))} \geqq 2^{14} h(|I|). \tag{4.30}$$

By induction we can construct a family of open intervals

$$\mathfrak{J}_{n+1} = \bigcup_{I \in \mathfrak{J}_n} \mathfrak{J}(I),$$

which consists of mutually disjoint open intervals-even separated by a certain distance as is explained above.

By definition, for any $I \in \mathfrak{J}_{n+1}$ there exists the unique $I' \in \mathfrak{J}_n$ such that $I \subset I'$. Let

$$V_n = \bigcup_{I \in \mathfrak{J}_n} I, \qquad N = \bigcap_{n=1}^{\infty} V_n,$$

then obviously we have

$$\overline{V_{n+1}} \subset V_n, \qquad N \subset E(\phi, \omega),$$

and so N is compact.

Let $\mathfrak{A}_\delta$ be any covering of N by open intervals, each element of which has the length less than δ. Since N is compact, there exists a finite sub-covering $\mathfrak{A}_\delta^f$ of $\mathfrak{A}_\delta$. If δ is less than the minimum of the distances between the elements of $\mathfrak{J}_1$, each element W of $\mathfrak{A}_\delta^f$ has non-empty intersection with only one element from $\mathfrak{J}_1$. We denote by W' the intersection of W with such element from $\mathfrak{J}_1$ and let $\mathfrak{A}_\delta'^f$ be a set of all elements of W'. Then $\mathfrak{A}_\delta'^f$ is a finite covering of N by open intervals such that each element of $\mathfrak{A}_\delta'^f$ is included by a unique open interval of $\mathfrak{J}_1$.

Let

$$\Lambda(\mathfrak{A}_\delta) = \sum_{I \in \mathfrak{A}_\delta} h(|I|),$$

then clearly

$$\Lambda(\mathfrak{A}_\delta) \geqq \Lambda(\mathfrak{A}_\delta^f) \geqq \Lambda(\mathfrak{A}_\delta'^f).$$

Now we define some terminologies.

Definition 1. We say that a time set W meets with another time set I when $W \cap I \neq \phi$.

Definition 2. We say that an open interval W is of degree n when W is included by an open interval I of $\mathfrak{J}_n$, and meets with at least two different elements of $\mathfrak{J}(I)$.

Definition 3. We say that an open interval is normal when its degree is defined.

We remark that the degree for an open interval is uniquely defined if it is possibly defined, in fact if an open interval W is of degree n, then there exists the unique I of $\mathfrak{J}_n$ which includes W, and it is impossible that W meets with two different intervals of $\mathfrak{J}_l$ for any $l \leqq n$.

Since W in the cover is an open set, there exists n such that W will meet with at least two intervals of some $\mathfrak{J}_n$. Let n be the minimal such value. Then restrict W to the intervals of $\mathfrak{J}_{n-1}$ it meets. Now the restricted interval W' is of degree $n-1$ and nothing was lost. The family $\mathfrak{A}_\delta^{*f}$ of all such W' is also a covering of N by open intervals such that

$$\Lambda(\mathfrak{A}_\delta'^f) \geqq \Lambda(\mathfrak{A}_\delta^{*f}).$$

The above arguments conclude that it is sufficient to consider only finite coverings which consist of normal intervals.

Definition 4. We say that a point P attaches to a normal interval W of degree n which is included by $J \in \mathfrak{J}_n$ when there exists $I \in \mathfrak{J}(J)$ such that I meets with W and includes P.

We remark that every point P of $W \cap N$ attaches to the normal interval W.

Definition 5. A finite family of normal intervals $\mathfrak{A}_\delta^e$ having the lengths less than δ, is said to be an estimating system of N when each point of N attaches to some elements of $\mathfrak{A}_\delta^e$.

We remark that the previous $\mathfrak{A}_\delta^{*f}$ is an estimating system of N.

Definition 6. An estimating system is said to be irreducible when $\mathfrak{A}_\delta^e$ does not contain proper sub-family which is an estimating system.

We remark that any estimating system contains at least one irreducible estimating system.

Definition 7. An estimating system is said to be of degree l when it contains at least one open interval of degree l but does not contain any open intervals of degree n, $n \geqq l+1$.

Since each estimating system consists of only finite normal intervals, we can always define our degree.

For an estimating system $\mathfrak{A}_\delta^e$ of degree l, let

$$\bar{A}(\mathfrak{A}_\delta^e) = \sum_{\substack{w \in \mathfrak{A}_\delta^e \\ \text{of degree } \leqq (l-1)}} h(|w|) + 2^{-14} \sum_{\substack{w \in \mathfrak{A}_\delta^e \\ \text{of degree } l}} h(|w|) \quad \text{if } l > 1,$$

and

$$\bar{A}(\mathfrak{A}_\delta^e) = 2^{-14} \sum_{w \in \mathfrak{A}_\delta^e} h(|w|) \quad \text{if } l = 1,$$

then

$$\begin{aligned} h - m^*(N) &\geqq \liminf_{\delta \downarrow 0} \inf_{\mathfrak{A}_\delta^e} A(\mathfrak{A}_\delta^e) \\ &\geqq \liminf_{\delta \downarrow 0} \inf_{\mathfrak{A}_\delta^e} \bar{A}(\mathfrak{A}_\delta^e), \end{aligned}$$

where "inf" runs among all irreducible estimating system of N having the lengths less than δ.

Now we prove two key lemmas in the Jarnik's method.

Lemma 9. *Let W be a normal interval of degree l, which is included by an open interval J of $\mathfrak{I}_l$ and we assume that W meets with intervals I_j of $\mathfrak{I}(J)$, $j = 1, 2, \ldots, t \geqq 2$ (by definition). Then*

$$|W| \geqq 2^{-27} |J| \, h(|J|)^{-1} \sum_{j=1}^{t} h(|I_j|). \tag{4.31}$$

Proof. Reordering $\{I_j\}_{j=1}^t$ if necessary, we can assume that $\{I_j\}_{j=1}^t$ are placed in order in the interval $J = \{t;\, k\varepsilon_n < t < (k+1)\varepsilon_n\}$. Assume that $I_1 \in I_m^{(n,k)}$ and $I_2 \in I_{m'}^{(n,k)}$, $m \leqq m'$ then the distance $d(I_1, I_2)$ between I_1 and I_2 is greater than $b_{n,m}\varepsilon_m$ and we have

$$\begin{aligned} d(I_1, I_2) &\geqq 2^{-2}(b_{n,m}\varepsilon_m + b_{n,m'}\varepsilon_{m'}) \quad (\text{by } b_{n,m}\varepsilon_m \geqq 2^{-1} b_{n,m'}\varepsilon_{m'}) \\ &\geqq (h(\varepsilon_{m+2}) + h(\varepsilon_{m'+2}))\varepsilon_{\bar{n}+2} h(\varepsilon_{\bar{n}+2})^{-1} \\ &\geqq (h(|I_1|) + h(|I_2|)) 2^{-25} \varepsilon_{n+2} h(\varepsilon_{n+2})^{-1} \quad (\text{by } (4.8)) \\ &\geqq 2^{-27} |J| \, h(|J|)^{-1} (h(|I_1| + h(|I_2|)). \end{aligned}$$

It follows that

$$|W| \geqq \sum_{j=1}^{t-1} d(I_j, I_{j+1}) \geqq 2^{-27}|J|\, h(|J|)^{-1} \sum_{j=1}^{t} h(|I_j|).$$

Lemma 10. *Let $\mathfrak{A}_\delta^e$ be an irreducible estimating system of N of degree $l(>1)$, then there exists an estimating system $\mathfrak{A}_{\delta'}^{\prime e}$ of degree $(l-1)$ such that*

$$\bar{A}(\mathfrak{A}_\delta^e) \geqq \bar{A}(\mathfrak{A}_{\delta'}^{\prime e}).$$

Since each normal interval of degree l (there exists at least one by definition) is contained in a unique open interval of $\mathfrak{I}_{l-1}$ (not $\mathfrak{I}_l$!), let $J_1, \ldots, J_t$, $t \geqq 1$ be different such intervals of $\mathfrak{I}_{l-1}$. Set

$$\mathfrak{A}_{\delta'}^{\prime e} = \mathfrak{A}_\delta^e \cup \{J_i\}_{i=1}^t - \bigcup_{i=1}^{t} \{W \subset J_i;\ W \in \mathfrak{A}_\delta^e \text{ of degree } l \text{ or } l-1\}.$$

We show that $\mathfrak{A}_{\delta'}^{\prime e}$ is an estimating system of N of degree $(l-1)$. In fact each J_i is a normal interval of degree $(l-1)$, so we need only to prove that $\mathfrak{A}_{\delta'}^{\prime e}$ is an estimating system of N. If $W \in \mathfrak{A}_\delta^e$ is of degree l, for each point P of N which attaches to W, there exists $I \in \mathfrak{I}_{l+1}$ such that I meets with W and $P \in I$, but W is included by some J_i and I is included by J_i, so P attaches to J_i. On the contrary, if $W \in \mathfrak{A}_\delta^e$ is of degree $(l-1)$ included by J_i, then for each point P on N which attaches to W, there exists $I \in \mathfrak{I}_l$ such that I meets with W and $P \in I$, but again I is included by J_i, so P attaches to J_i. This concluds that $\mathfrak{A}_{\delta'}^{\prime e}$ is an estimating system of N.

Next we show that any normal interval W of $\mathfrak{A}_\delta^e$ to which a point P of $N \cap J_i$ attaches is of degree $(l-1)$ or l. In fact suppose that W is of degree $m < l-1$, then there exists an interval $J \in \mathfrak{I}_{m+1}$ such that $P \in J$ and J meets with W, but this implies that J_i is included by J because of $m+1 \leqq l-1$. Therefore every point P of $N \cap J_i$ attaches to W. On the other hand, by definition there exists at least one $W' \subset J_i$ and in $\mathfrak{A}_\delta^e$, W' of degree l. If P attaches to W', then $P \in J_i$ since W' is of degree l and J_i is in $\mathfrak{I}_{l-1}$. Therefore every point P of N which attaches to W', attaches also to W. This follows that the family that excludes W' from $\mathfrak{A}_\delta^e$ is also an estimating system, which contradicts the irreducibility of $\mathfrak{A}_\delta^e$.

Let $W_1 \ldots W_m$ be normal intervals of degree $(l-1)$ included by J_i (it may be $m=0$) and assume that $\mathfrak{I}(J_i)$ consists of $\{U_1 \ldots U_{\bar{a}}, U_{\bar{a}+1}, \ldots, U_a\}$, where U_j meets with at least one of $\{W_k\}_{k=1}^m$ for $1 \leqq j \leqq \bar{a}$, but does not meet with any W_k, $1 \leqq k \leqq m$ for $\bar{a} < j \leqq a$. By (4.30), we have

$$\sum_{j=1}^{a} h(|U_j|) \geqq 2^{14} h(|J_i|).$$

We proceed by dividing into two cases.

(I) If $\sum_{j=1}^{\bar{a}} h(|U_j|) \geqq 2^{13} h(|J_i|)$, by Lemma 9 we have

$$|W_k| \geqq 2^{-27} |J_i|\, h(|J_i|)^{-1} \sum_{(k)}{}' h(|U_j|),$$

where $\sum_{(k)}'$ sums up all $U_j's$ which meet with W_k, and

$$\sum_{k=1}^{m} h(|W_k|) \geqq h\left(\sum_{k=1}^{m} |W_k|\right) \quad \text{(by (1.7))}$$
$$\geqq h\left(2^{-27}|J_i|\, h(|J_i|)^{-1} \sum_{k=1}^{m} \sum_{(k)}' h(|U_j|)\right)$$
$$\geqq h\left(2^{-27}|J_i|\, h(|J_i|)^{-1} \sum_{j=1}^{\bar{a}} h(|U_j|)\right)$$
$$\geqq h(2^{-14}|J_i|)$$
$$\geqq 2^{-14} h(|J_i|).$$

(II) On the contrary we assume that

$$\sum_{j=\bar{a}+1}^{a} h(|U_j|) \geqq 2^{13} h(|J_i|).$$

By definition, U_s; $\bar{a} < s \leqq a$ does not meet with any W_k, $1 \leqq k \leqq m$, so any point P of $N \cap U_s$ can not attach to a normal interval of degree $(l-1)$, this follows that there exists at least one normal interval V of degree l included by U_s and P attaches to V, because every point of $N \cap J_i$ must attach to a normal interval of degree l or $(l-1)$. Now let $V_1 \dots V_b$, $b \geqq 1$ be the elements of $\mathfrak{A}_\delta^e$ of degree l included by U_s, then by Lemma 9,

$$|V_k| \geqq 2^{-27}|U_s|\, h(|U_s|)^{-1} \sum_{(k)}' h(|I|),$$

where $\sum_{(k)}'$ sums up all $I \in \mathfrak{J}(U_s)$ which meets with V_k, and

$$\sum_{k=1}^{b} h(|V_k|) \geqq h\left(\sum_{k=1}^{b} |V_k|\right)$$
$$\geqq h\left(2^{-27}|U_s|\, h(|U_s|)^{-1} \sum_{k=1}^{b} \sum_{(k)}' h(|I|)\right)$$
$$\geqq h(2^{-27}|U_s|\, h(|U_s|)^{-1} \sum_{I \in \mathfrak{J}(U_s)} h(|I|))$$
$$\geqq h(2^{-13}|U_s|) \geqq 2^{-13} h(|U_s|) \quad \text{(by (4.30))}.$$

Finally we have

$$\sum' h(|V|) \geqq \sum_{\bar{a} < s \leqq a} 2^{-13} h(|U_s|) \geqq h(|J_i|),$$

where $\sum'$ sums up all $V \in \mathfrak{A}_\delta^e$ of degree l included by J_i. Taking account of the definition of $\bar{\Lambda}$ we conclude Lemma 10.

At last we have reached the final step of the proof. With the help of Lemma 10, it is sufficient to estimate $\bar{\Lambda}(\mathfrak{A}_\delta^e)$ of an estimating system of degree 1. By (4.31)

$$\begin{aligned}\bar{\Lambda}(\mathfrak{A}_\delta^e)&=2^{-14}\sum_{W\in\mathfrak{A}_\delta^e} h(|W|)\\
&\geqq 2^{-14}\sum_{\substack{W\in\mathfrak{A}_\delta^e\\ W\subset J\in\mathfrak{J}_1}} h(2^{-27}|J|\,h(|J|)^{-1}{\textstyle\sum}' h(|I_j|))\\
&\geqq 2^{-14}\sum_{J\in\mathfrak{J}_1} h(2^{-27}|J|\,h(|J|)^{-1}\sum_{\substack{W\subset J\\ W\in\mathfrak{A}_\delta^e}}{\textstyle\sum}' h(|I_j|))\\
&\geqq 2^{-14}\sum_{J\in\mathfrak{J}_1} h(2^{-27}|J|\,h(|J|)^{-1}\sum_{I\in\mathfrak{J}(J)} h(|I|))\\
&\geqq 2^{-14}\sum_{J\in\mathfrak{J}_1} h(2^{-13}|J|)\\
&\geqq 2^{-27}\sum_{J\in\mathfrak{J}_1} h(|J|)\geqq M,\end{aligned}$$

where $\sum'$ sums up all $I_j\in\mathfrak{J}(J)$ which meets with W, and each point of $N\cap I_j$ must attach to at least one $W\subset J$.

This completes the proof!

The author would like to thank the referee for his helpful comments and suggestions.

References

1. Jain, N.C., Taylor, S.J.: Local asymptotic laws for Brownian motion. Ann. of Probab. **1-4**, 527–549 (1973)
2. Jarnik, V.: Über die simultanen diophantischen Approximationen. Math. Z. **33**, 505–543 (1931)
3. Kôno, N.: Asymptotic behavior of sample functions of Gaussian random fields. J. Math. Kyoto Univ. **15-3**, 671–707 (1975)
4. Kôno, N.: Mouvement brownien a valeurs dans R^d: classes de modules de continuite uniformes sur une partie localement compacte de $[0, 1]\times[0, 1]$. C.R. Acad. Sc. Paris, Sér. A–B, t. 280, 1233–1236 (1975)
5. Orey, S., Pruitt, W.: Sample functions of the N-parameter Wiener process. Ann. of Probab. **1-1**, 138–163 (1973)
6. Orey, S., Taylor, S.J.: How often on a Brownian path does the law of iterated logarithm fail? Proc. London math. Soc. **28**, 174–192 (1974)
7. Rogers, C.A.: Hausdorff measures. Cambridge at the University Press (1970)

Received August 3, 1976