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On the Dunford-Schwartz Theorem*

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1. Introduction

Let $(S, \mathcal{S}, \mu)$ be a σ-finite measure space and let $L_1 = L_1(S, \mathcal{S}, \mu)$. If T is a linear operator of L_1 to L_1 then for each positive integer n we can define the linear operator $T_n = n^{-1}\sum_{k=0}^{n-1} T^k$. DUNFORD and SCHWARTZ [4] have extended HOPF's ergodic theorem [5] as follows:

Theorem. *Let T be a linear operator of L_1 to L_1 with $\| T\|_1 \leqq 1$ and $\| T\|_\infty \leqq 1$; the latter inequality means*

$$\| Tg\|_\infty = \operatorname*{ess.\,sup.}_{s\in S} | Tg(s)| \leqq \operatorname*{ess.\,sup.}_{s\in S} |g(s)| = \|g\|_\infty, \quad g\in L_1 \cap L_\infty .$$

Then $\lim_{n\to\infty} T_n f(s)$ exists almost everywhere for $f \in L_1$.

If for an arbitrary $\varepsilon > 0$ we still require that $\| T\|_\infty \leqq 1$ but allow $\| T\|_1 = 1 + \varepsilon$ then a counterexample to the theorem may be easily found. The problem of constructing a counterexample for the case where $\| T\|_1 = 1$ but $\| T\|_\infty$ may be unbounded was recently solved in [2], where it is indicated that it is possible to obtain a counterexample for the case where $\| T\|_1 = 1$ and $\| T\|_\infty = 1 + \varepsilon$. We obtain a result (Theorem 2) from which this follows as a corollary besides giving us an approximation result which strengthens Theorem 2 [3].

Our construction takes place on the unit interval with Lebesgue measure m. The method of proof of Theorem 1 consists in applying a refinement of the technique in [2] to a regular partition as defined below. We are dealing with point transformations τ and associated mappings T_τ in L_1 (see below). We remark that the non-existence of the limit $\lim_{n\to\infty} T_{\tau n} f(x)$ for some $f \in L_1[0, 1]$ is sufficient to guarantee that the point transformation τ has no non-trivial σ-finite invariant measure absolutely continuous with respect to m. In this respect the results here can be considered to be a continuation of those of [3]. The non-existence of an invariant measure as above follows trivially from the ratio ergodic theorem [1].

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2. Preliminaries

Let $(X, \mathcal{B}, m)$ denote the measure space consisting of the unit interval, Lebesgue measurable sets, and Lebesgue measure respectively. An invertible transformation τ of X onto X is measurable if $B \in \mathcal{B}$ implies $\tau(B) \in \mathcal{B}$ and $\tau^{-1}(B) \in \mathcal{B}$ and nonsin-

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gular if $m(B) = 0$ implies $m(\tau(B)) = m(\tau^{-1}(B)) = 0$. We let $\mathscr{I}$ denote the class of transformations which are invertible, measurable, and nonsingular. It is easily seen that $\tau \in \mathscr{I}$ implies $\tau^n \in \mathscr{I}$ for each integer n. A topology is defined on $\mathscr{I}$ by the metric $\mathrm{d}(\tau, \sigma) = m(\tau \neq \sigma)$. A transformation τ is said to be antiperiodic if for each positive integer n $\tau^n(x) \neq x$ for a.e. x, i.e. $d(\tau^n, e) = 1$ where $e(x) = x$, $x \in X$. We let τ' denote the Radon-Nikodym derivative of the measure $m(\tau(\cdot))$ with respect to m. Given sets A and B of positive measure, we say τ maps A linearly onto B if $\tau(A) = B$ and $\tau'(x) = m(B)/m(A)$, $x \in A$. We define the transformation T_τ by $T_\tau f(x) = f(\tau^{-1}(x))\tau^{-1\prime}(x)$. The properties stated in the following lemma follow easily from the definitions.

Lemma 1. (1) $\| T_\tau \|_1 = 1$, $\| T_\tau \|_\infty = \| \tau^{-1\prime} \|_\infty$, *and for each positive integer* n *we have*

(2) $$(\tau^n)'(x) = \prod_{i=0}^{n-1} \tau'(\tau^i(x)),$$

(3) $$T_\tau^n f(x) = f(\tau^{-n}(x)) / \prod_{i=1}^{n} \tau'(\tau^{-i}(x)),$$

(4) $$m(\tau^n(B)) = \int_B \prod_{i=0}^{n-1} \tau'(\tau^i(x))\,dm, \quad B \in \mathscr{B}.$$

In what follows we will consider η to be a transformation which is not defined on all of X but which is one-to-one, measurable, and nonsingular on its domain of definition. In Theorem 1 we extend the definition of η so that $\eta \in \mathscr{I}$. In order to facilitate the verification of certain points in the proof of Theorem 1 we introduce the following definitions and note some elementary properties.

Definition 1. Let $B \in \mathscr{B}$ and N a positive integer. $P(B, N) = \{\eta^i(B),\ 0 \leqq i \leqq N-1\}$ is said to be a partition of $B^* = \bigcup_{i=0}^{N-1} \eta^i(B)$ if the sets $\eta^i(B)$, $0 \leqq i \leqq N-1$, are pairwise disjoint.

We say $P(B, N)$ has base B and order N and assume η is not defined on $F = \eta^{N-1}(B)$.

Definition 2. $P(B, N)$ is a linear partition if η maps $\eta^i(B)$ linearly onto $\eta^{i+1}(B)$, $0 \leqq i \leqq N-2$, and if $m(B) \geqq m(\eta^i(B))$, $1 \leqq i \leqq N-1$.

Definition 3. $P(B, N)$ is an identity partition if (1) $\prod_{i=0}^{N-2} \eta'(\eta^i(x)) = 1$ for a.e. $x \in B$ and (2) $\eta'(x) \leqq \beta$ for a.e. $x \in \eta^i(B)$, $0 \leqq i \leqq N-3$, where $1 < \beta < \infty$.

In what follows we assume all identity partitions have the same order R and satisfy Definition 3 (2) for the same number β.

Lemma 2. *Let* $P(B, R)$ *be an identity partition.*

(1) *If* $b \subset B$ *then* $P(b, R)$ *is an identity partition,* $m(b) = m(\eta^{R-1}(b))$, *and in particular* $m(B) = m(F)$.

(2) *If* $f(x) \geqq K$ *a.e. on* B *then* $T_\eta^i f(x) \geqq K\beta^{2-R}$ *a.e. on* $\eta^i(B)$, $0 \leqq i \leqq R-2$, *and* $T_\eta^{R-1} f(x) \geqq K$ *a.e. on* F.

Proof. (1) follows from Definition 3 and Lemma 1 (4). (2) follows from Definition 3 and Lemma 1 (3).

Briefly, Lemma 2 (2) implies that in transforming a function through an identity partition there can be at most a decrease in height by a factor of β^{2-R} and the function regains its height on F.

Definition 4. Given partitions $P_1(B_1, N_1)$ and $P_2(B_2, N_2)$ such that $B_1^* \cap B_2^* = \emptyset$, we extend η to map F_1 linearly onto B_2 and define the product partition

$$P_1 P_2 = P(B_1, N_1 + N_2).$$

Definition 5. Let $P_j(B_j, N_j)$, $1 \leqq j \leqq J$, be partitions such that $B_i^* \cap B_j^* = \emptyset$, $i \neq j$. We say $\prod_{j=1}^{J} P_j = P_1 \cdots P_J$ is a regular partition if each partition P_j is either a linear partition or an identity partition and if $m(B_1) \geqq m(B_j)$, $2 \leqq j \leqq J$.

The properties stated in the following lemma follow easily from the previous definitions and Lemmas 1 and 2.

Lemma 3. *Let $P(B, N)$ be a regular partition.*

(1) *If $b \subset B$ then $P(b, N)$ is a regular partition and*

$$\alpha = m(\eta^{N-1}(b))/m(b) = m(F)/m(B) \leqq 1.$$

(2) *If $f(x) \geqq K$ a.e. on B then $T_\eta^i f(x) \geqq K\beta^{2-R}$ a.e. on $\eta^i(B)$, $0 \leqq i \leqq N-1$.*

Let $P_1(B_1, N_1)$ and $P_2(B_2, N_2)$ be regular partitions such that $B_1^* \cap B_2^* = \emptyset$.

(3) *If $m(B_1) \geqq m(B_2)$ then $P_1 P_2$ is a regular partition.*

(4) *If $f(x) \geqq K$ a.e. on B_1 then $T_\eta^{N_1} f(x) \geqq K m(B_1)/m(B_2)$ a.e. on B_2.*

3. Main Results

Theorem 1. *Let $P(B, N)$ be a regular partition. Let $m(D) > 0$ where $D = X - B^*$ and let $\varepsilon > 0$. Then there exists a function $f \in L_1[0, 1]$ such that η can be extended to $F \cup D$ so that*

(1) $$\limsup_{n\to\infty} \frac{T_\eta^n f(x)}{n} = \infty \quad a.e.,$$

(2) $$\liminf_{n\to\infty} T_{\eta n} f(x) = 0 \quad a.e.,$$

(3) $$\eta^{-1\prime}(x) \leqq 1 + \varepsilon \quad on \quad B \cup D.$$

Theorem 2. *Let $\tau \in \mathscr{I}$ be antiperiodic and let $\varepsilon > 0$. Then there exists $\eta \in \mathscr{I}$ and $f \in L_1[0, 1]$ such that* (1) *and* (2) *of Theorem 1 hold and*

(3) $$d(\tau, \eta) \leqq \varepsilon,$$

(4) $$\| T_\eta \|_\infty \leqq M < \infty.$$

Moreover if τ is measure preserving we may take $M = 1 + \varepsilon$ in (4).

The addition of conclusion (4) in the preceding theorem strengthens Theorem 2 [3].

Corollary. *For each $\varepsilon > 0$ there exists a class of linear transformations T of $L_1[0, 1]$ to $L_1[0, 1]$ such that $\| T \|_1 = 1$, $\| T \|_\infty = 1 + \varepsilon$, and $\lim_{n\to\infty} T_n f(x)$ does not exist a.e. for some $f \in L_1[0, 1]$.*

Proof of Theorem 1. We first make some preliminary computations. Let q be a positive integer such that $1/q \leqq \varepsilon$. Let k be a positive integer and let $\alpha > 0$. Let a set B have measure W. Following [2] we decompose B into $k+1$ disjoint subsets $B_0, \ldots, B_k$ of measure w_k and $\alpha q w_k/(q+l)$, $1 \leqq l \leqq k$, respectively by selecting

$$w_k = W/[1 + \alpha q \sum_{l=1}^{k} 1/(q+l)]. \tag{1}$$

We note that $\lim_{k\to\infty} w_k = 0$. Let $\lambda = (1+\varepsilon)^{-1}$ and let $\xi = (1-\lambda)^{-1}$. If $L > 0$ we can then choose k so large that w_k in (1) satisfies

$$\xi \alpha^2 q w_k/(q+k) < L/8. \tag{2}$$

We set $P_1 = P$, $B_1 = B$, $F_1 = F$, $N_1 = N$, and $D_1 = D$ and proceed inductively in stages as follows. At the i-th stage we have a regular partition $P_i(B_i, N_i)$ and $m(D_i) > 0$ where $D_i = X - B_i^*$. We have η defined except on $D_i \cup F_i$ and η^{-1} is defined except on $D_i \cup B_i$. For the first step of the i-th stage we let $\alpha_i = m(F_i)/m(B_i)$, $W_i = m(B_i)$, and $L_i = m(D_i)$. We select k_i so large that w_{k_i} in (1) satisfies (2) and also $m(\bigcup_{j=0}^{N_i-1} \eta^j(B_{i,0})) \leqq 2^{-i}$. Here $B_i = \bigcup_{l=0}^{k_i} B_{i,l}$ where $m(B_{i,0}) = w_{k_i}$ and $m(B_{i,l}) = \alpha_i w_{k_i} q/(q+l)$, $1 \leqq l \leqq k_i$. Since P_i is a regular partition it follows by Lemma 3 (1) that $P_{i,l} = P(B_{i,l}, N_i)$ are regular partitions, $0 \leqq l \leqq k_i$, and Lemma 3 (3) implies $\prod_{l=0}^{k_i} P_{i,l}$ is a regular partition. Furthermore it follows from Lemma 3 (1) that $\eta^{-1'}(x) = \alpha_i(q+l+1)/(q+l) \leqq 1 + 1/(q+l) \leqq 1+\varepsilon$ on $B_{i,l+1}$, $0 \leqq l \leqq k_i - 1$. Therefore on the extension $\eta^{-1'}(x) \leqq 1+\varepsilon$ a. e. on $\bigcup_{l=1}^{k_i} B_{i,l}$.

Let $m_i = m(F_{i,k_i}) = \alpha_i m(B_{i,k_i}) = \alpha_i^2 q w_{k_i}/(q+k_i)$ by Lemma 3 (1). Hence $m_i < L_i/8$ by (2). Let s_i be the smallest positive integer such that $s_i m_i > 5L_i/8$. Let $D_{i,j}$, $1 \leqq j \leqq s_i$, denote s_i disjoint subsets of D_i each of measure m_i. Let η map $D_{i,j}$ linearly onto $D_{i,j+1}$, $1 \leqq j \leqq s_i - 1$, and let $P_{i,k_i+1} = P(D_{i,1}, s_i)$. It follows that P_{i,k_i+1} is a linear identity partition and $\prod_{l=0}^{k_i+1} P_{i,l}$ is a regular partition. Furthermore $\eta^{-1'}(x) = 1$ a.e. on $D_{i,j}, 1 \leqq j \leqq s_i$. We now have extended η to all of F_i and more than one half of D_i. Furthermore η^{-1} is extended to more than one-half of D_i and all of B_i except $B_{i,0}$.

For the second step of the i-th stage we employ the positive integers t_i defined inductively as follows:

$$t_{i+1} = t_i + s_i + n_i, \; i \geqq 1; \quad t_1 = 0. \tag{3}$$

The number n_i is defined below. We set $f(x) = f_1(x) = q\beta^{R-2} N_1 \varphi_{F_1}(x)$ (where φ_B denotes the characteristic function of B). Let $f_i(x) = T_\eta^{t_i} f_1(x) \,.\, \varphi_{F_i}(x)$. We will have $f_i(x) \equiv 0$ on D_i and have $f_i(x) \equiv H_i$ constant a.e. on F_i where $H_i \geqq q\beta^{R-2}\, i(t_i + N_i)$.We will extend η so that this holds for $i+1$. However let us first show that it follows from the extension of η in the first step of the i-th stage that $T_\eta^n f_1(x)/n \geqq i$ for some n, $t_i + 1 \leqq n \leqq t_i + k_i N_i$, a. e. on the set $X_i = \bigcup_{j=0}^{N_i-1} \eta^j(B_i - B_{i,0}) = X - (B_{i,0}^* \cup D_i)$. In fact let $f_i^*(x) = f_i(x)\varphi_{F_{i,0}}(x) \equiv H_i$

a.e. on $F_{i,0}$. We then have $T_\eta^{(l-1)N_i+1} f_i^*(x)$ with support on $B_{i,l}$, $1 \leqq l \leqq k_i$. Lemma 3 (4) implies that for $n = t_i + (l-1)N_i + 1$ we have

$$T_\eta^n f_1(x) \geqq T_\eta^{(l-1)N_i+1} f_i^*(x) = H_i m(F_{i,0})/m(B_{i,l}) \geqq \geqq \beta^{R-2} i(q+l)(t_i + N_i) > \beta^{R-2} i(t_i + l N_i)$$

a.e. on $B_{i,l}$, $1 \leqq l \leqq k_i$. Lemma 3 (2) then implies that for $n = t_i + (l-1) N_i + 1 + j$ we have $T_\eta^n f_1(x)/n > i(t_i + lN_i)/n \geqq i$ a.e. on $\eta^j(B_{i,l})$, $0 \leqq j \leqq N_i - 1$, $1 \leqq l \leqq k_i$, which completes the verification of our assertion.

We now consider how n_i is to be chosen. We select n_i subsets $D_{i,j}$, $s_i + 1 \leqq j \leqq s_i + n_i$, of measure $\lambda^{j-s_i} m_i$ respectively from the set $D_i - \bigcup_{j=1}^{s_i} D_{i,j}$ whose measure exceeds $L_i/4$ according to the definition of s_i. This is possible because of (2). Let $D_{i+1} = D_i - \bigcup_{j=1}^{s_i+n_i} D_{i,j}$ and we note that $m(D_{i+1}) > L_i/8$. Let η map $D_{i,j}$ linearly onto $D_{i,j+1}$, $s_i + 1 \leqq j \leqq s_i + n_i - 1$, and let $P_{i,k_i+2} = P(D_{i,s_i+1}, n_i)$. We now set $P_{i+1} = \prod_{l=0}^{k_i+2} P_{i,l}$ and Lemma 3 (3) implies that P_{i+1} is a regular partition. We have $\eta^{-1\prime}(x) = \lambda^{-1} = 1 + \varepsilon$ a.e. on $D_{i,j}$, $s_i + 1 \leqq j \leqq s_i + n_i$, hence $\eta^{-1\prime}(x) \leqq 1 + \varepsilon$ a.e. on the extension. Now the first requirement that n_i must satisfy is that it be so large that $T_{\eta n} f_1(x) \leqq 1/i$ on $B_i^* = \bigcup_{j=0}^{N_i-1} \eta^j(B_i)$ for $n = t_{i+1}$. This is possible because $T_\eta^j f_1(x) \equiv 0$ on B_i^* for $t_i + (k_i + 1)N_i < j \leqq t_{i+1}$. The second requirement is that n_i be so large that H_{i+1}, the value of $T_\eta^{s_i+n_i} f_i(x)$ on $D_{i,s_i+n_i} = F_{i+1}$, satisfies the condition

$$H_{i+1} \geqq q\beta^{R-2}(i+1)(t_{i+1} + N_{i+1}), \tag{4}$$

where

$$N_{i+1} = (k_i + 1)N_i + s_i + n_i. \tag{5}$$

This is possible because $H_{i+1} = H_i(1+\varepsilon)^{n_i}$ and increases exponentially with respect to n_i whereas the right side of (4) only increases linearly with respect to n_i as seen from (3) and (5).

Setting $B_{i+1} = B_{i,0}$ we have $P_{i+1} = P(B_{i+1}, N_{i+1})$. At the end of the i-th stage η^{-1} is defined except on $B_{i+1} \cup D_{i+1}$ where $m(B_{i+1} \cup D_{i+1}) < 2^{1-i}$. Furthermore η is defined except on $F_{i+1} \cup D_{i+1}$ where $m(F_{i+1} \cup D_{i+1}) \leqq 2^{-i}$. Therefore our construction implies we can write $X = X^* \cup - X^*$ where $\eta(X^*) = X^*$, η is invertible, measurable, and non-singular on X^*, and $m(X^*) = 1$. We set $\eta(x) = x$, $x \notin X^*$, hence $\eta \in \mathscr{I}$.

We have $T_\eta^n f_1(x)/n \geqq i$ for some n, $t_i + 1 \leqq n \leqq t_i + k_i N_i$, a.e. on X_i where $m(X_i) \geqq 1 - 2^{1-i}$ and $T_{\eta n} f_1(x) \leqq 1/i$ for $n = t_{i+1}$ on $B_i^* \supset X_i$. Properties (1) and (2) of Theorem 1 follow and our construction implies (3) holds.

Proof of Theorem 2. We will first construct an identity partition and then utilize Theorem 1 to obtain the desired transformation. As in the proof of Theorem 2 of [3], it follows from Lemma 6, Theorem 1, and remark 2 of [3] that there exists $\sigma \in \mathscr{I}$, $A \in \mathscr{B}$, and a positive integer N such that σ generates a partition $P(A, N)$ with $A^* = X$ and $d(\sigma, \tau) \leqq \varepsilon/2$ independently of how σ maps $\sigma^{N-1}(A)$ onto A.

We decompose A into k disjoint subsets A_j, $1 \leqq j \leqq k$, such that $m(A_j) = m(A)/k$, $1 \leqq j \leqq k$, and take k so large that $m(\bigcup_{i=0}^{N-1} \sigma^i(A_k)) \leqq \varepsilon/2$. Let $E_j = \sigma^{N-1}(A_j)$, $1 \leqq j \leqq k-1$, $E = \bigcup_{j=1}^{k-1} E_j$, and $D = \bigcup_{i=1}^{N-1} \sigma^i(A_k)$. We define $\eta(x) = \sigma(x)$ for $x \notin E \cup D \cup A_k$, hence it follows that $d(\eta, \tau) \leqq \varepsilon$ independently of how η is defined on $E \cup D \cup A_k$. Let ξ_j be an invertible measure preserving transformation of A_j onto A_{j+1}, $1 \leqq j \leqq k-1$, and let $\eta(x) = \xi_j(\eta^{-N+1}(x))$, $x \in E_j$, $1 \leqq j \leqq k-1$. Let $B = A_1$, $R = (k-1)N + 1$, and $P = P(B, R)$. We note that $F = A_k = \eta^{R-1}(B)$. The definition of η implies that if $x \in A_j$ then $\eta^N(x) = \xi_j(x) \in A_{j+1}$, $1 \leqq j \leqq k-1$. Therefore for a.e. $x \in B$ we have

$$\prod_{i=0}^{R-2} \eta'(\eta^i(x)) = \eta^{(k-1)N'}(x) = \xi_{k-1}(\xi_{k-2}(\cdots(\xi_1(x))\cdots)' = 1$$

since $\xi_j' = 1$ a.e. on A_j, $1 \leqq j \leqq k-1$. Moreover it follows from remark 1 of [3] and our construction that there exists a and b such that $0 < a \leqq \eta'(x) \leqq b < \infty$ for a.e. $x \notin F \cup D$. Letting $\beta = b$ it follows that $P(B, R)$ is an identity partition. We now apply Theorem 1 to extent η to $F \cup D$. Therefore (1), (2), and (3) of Theorem 2 are satisfied. Letting $M = \max(1/a, 1 + \varepsilon)$ we satisfy (4). If τ is measure preserving then it is easily shown that we may take $a = b = 1$ above, hence $M = 1 + \varepsilon$ in this case.

Proof of Corollary. Let τ be measure preserving in Theorem 2. It then suffices to take $T = T_\eta$.

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