Z. Wahrscheinlichkeitstheorie verw. Gebiete
44, 145–154 (1978)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

Birthing Markov Processes at Random Rates

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Summary. This paper studies processes constructed by birthing the trajectories of a given Markov process along time according to random probabilities. Getoor has considered the case where the random probabilities are determined by comultiplicative functionals and proved for right processes that the post-birth process has the Markov property. Here randomizations of comultiplicative functionals are described which give rise to conditionally Markov processes. The main argument is developed for general Markov processes and the transition probabilities of the new process, including those from the pre-birth state, are explicited.

1. Introduction

Meyer, Smythe and Walsh [5], Getoor [1] and Millar [6, 7] studied processes obtained by birthing Markov processes according to comultiplicative functionals and randomized coterminal times. Comultiplicative functionals, introduced in [1], are processes satisfying properties suggested by those of $1_{(L,\infty)}$ where L is a coterminal time. Such properties are dual of those of a multiplicative functional in the sense that, if we consider a space of finite trajectories, comultiplicative functionals may be seen as multiplicative functionals on the reverse process. Getoor uses this duality to birth Markov processes with comultiplicative functionals in a manner dual to that by which multiplicative functionals are used to kill processes. This naturally generalizes the results of [5] but does not include the processes considered in [6] and [7] involving times that are not cooptional. In this paper we describe a class of processes constructed through a kind of optional decision about cooptional processes, that comprises comultiplicative functionals together with processes like those determined by randomized coterminal times such as the time of the minimum or the maximum.

We work with a space of trajectories. The idea of birthing a process randomly through time is clarified by the use of a birth operator and the

0044-3719/78/0044/0145/$02.00

embedding of the original space into a larger space where its trajectories are born at all possible times. After a section on notation, the properties of the birth operator are discussed in Section 2. The notions of comultiplicative functional and randomized comultiplicative functional are studied in Sections 4 and 5. In Section 6 we determine the distribution of the process born according with a randomized comultiplicative functional. Finally Section 7 generalizes Sections 4 of [1] and [6].

2. General Notations

E and E_δ will denote separable metric spaces with Borel σ-fields $\mathscr{E}$ and $\mathscr{E}_\delta$ respectively. E_δ is obtained from E by adjoining an isolated point δ.

Ω_0 will be the set of all right continuous paths ω from $[0, \infty)$ to E and Ω will be the set of all right continuous ω: $[0, \infty) \to E_\delta$ such that $\omega_s = \delta$ for all $s \leqq t$ if $\omega_t = \delta$.

$X_t: \Omega \to E_\delta$ will be the t^{th} coordinate map. $\mathscr{F}_t^0 = \sigma(X_s, s \leqq t)$. $\mathscr{F} = \sigma(X_s, 0 \leqq s < \infty)$. $\mathscr{F}^0 = \mathscr{F}|_{\Omega_0}$. $\mathscr{E}^*$, $\mathscr{E}_\delta^*$, $\mathscr{F}^*$ and $\mathscr{F}_t$ will be the universal completions of E, E_δ, $\mathscr{F}$ and $\mathscr{F}_t^0$ respectively. $\mathscr{F}_{t+}^0 = \bigcap_{u>t} \mathscr{F}_u^0$ and $\mathscr{F}_{t+} = \bigcap_{u>t} \mathscr{F}_u$.

3. The Birth Operator

In addition to the well known shift operator θ, we will make use of the killing operator, denoted by k, defined in [5], and its dual, the birth operator $b = \{b_t: \Omega \to \Omega\}_{t \in [0, \infty)}$ defined through

$$[b_s(\omega)]_t = \begin{cases} \omega_t & \text{if } t \geqq s \\ \delta & \text{if } t < s. \end{cases}$$

The measurability of the birth operator will be important in the construction of the probability space on which the processes birthed at given rates will be defined. In order to prove it we will consider each subset of Ω decomposed according with the birth time of each element.

Let $[\delta] = b_\infty(\omega)$ be the trajectory in Ω with all coordinates equal to δ. Analogously, let $[\delta]_s$ denote the $\omega \in E^{[0,s)}$ with $\omega_t = \delta$ for all $t \in [0, s]$. For each $\Lambda \subset \Omega$, $[\delta]_s \times \Lambda$ will denote the subset of Ω whose projection on $E^{[0,s)}$ is $\{[\delta]_s\}$ and whose projection on $E^{[s,\infty)}$ is Λ, i.e., the set of $\omega \in \Omega$ such that $\omega_r = \delta$ for all $r < s$ and $(\omega_{s+t})_{t \geqq 0} \in \Lambda$.

It is easy to see that

$$\Omega = \Omega_0 \cup \Big[\bigcup_{s \in (0,\infty)} ([\delta]_s \times \Omega_0)\Big] \cup \{[\delta]\}.$$

This may be extended to general measurable subsets of Ω in the following manner.

Proposition 3.1. *For all* $\Lambda \in \mathscr{F}$, $\Lambda = \Lambda_0 \cup \Big[\bigcup_{s \in (0,\infty)} ([\delta]_s \times \Lambda_s)\Big] \cup [\Lambda \cap \{[\delta]\}]$ *where for all* $s \in \mathbb{R}_+$, $\Lambda_s \in \mathscr{F}^0$.

Proof. Let $\Lambda_0 = \Lambda \cap \Omega_0$ and $\Lambda_s = \theta_s[\Lambda \cap ([\delta]_s \times \Omega_0)]$, for all $s \in (0, \infty)$. $\Lambda \subset \Omega$ implies

$$\Lambda = \Lambda \cap \Omega = [\Lambda \cap \Omega_0] \cup [\Lambda \cap (\bigcup_{s \in (0,\infty)} [\delta]_s \times \Omega_0)] \cup [\Lambda \cap \{[\delta]\}]$$
$$= \Lambda_0 \cup [\bigcup_{s \in (0,\infty)} ([\delta]_s \times \Lambda_s)] \cup [\Lambda \cap \{[\delta]\}].$$

Clearly $\Lambda_s \subset \Omega_0$ for all $s \in \mathbb{R}_+$.

It is also clear that $\Lambda_0 \in \mathscr{F}$.

$[\delta]_s \times \Omega_0 = [X_t = \delta$ for all $t \in [0, s] \cap Q$; $X_s \in E] \in \mathscr{F}$ implies $\Lambda \cap ([\delta]_n \times \Omega_0) \in \mathscr{F}$ so that $\Lambda_s = \theta_s[\Lambda \cap ([\delta]_s \times \Omega_0)] \in \mathscr{F}$. This concludes the proof.

We will want to look at Ω as the image of Ω_0 by b. The following theorem deals with the measurability aspects of this approach.

Theorem 3.1. b: $[0, \infty] \times \Omega_0 \to \Omega$ *is* $\mathscr{B} \times \mathscr{F}^0 | \mathscr{F}$*-measurable.*

$$(t, \omega) \mapsto b_t(\omega)$$

Proof. Since $\mathscr{F}$ is generated by $\{\Lambda \subset \Omega$: Λ is a cylinder of $E_\delta^{\mathbb{R}_+}\}$ it is enough to show that $b^{-1}(\Lambda \cap \Omega) \in \mathscr{B} \times \mathscr{F}^0$ for all Λ, cylinder of $E_\delta^{\mathbb{R}_+}$.

Let $s_1, \ldots, s_n$ be the indices of the nontrivial coordinates of one such Λ.

Let $m = \inf\{i$: $\delta \notin s_i^{\text{th}}$ coordinate of $\Lambda\}$ with the convention $\inf \emptyset = \infty$.

For the sake of compactness, let us also adopt the symbols $s_0 = 0$, $s_\infty = \infty$, $[\delta]_\infty \times \Lambda_\infty = \{[\delta]\}$.

Then $\Lambda \cap \Omega = \Lambda_0 \cup [\bigcup_{s \in (0, s_m]} ([\delta]_s \times \Lambda_s)]$ with $\Lambda_0 = \Lambda \cap \Omega_0 \in \mathscr{F}$ and $[\delta]_s \times \Lambda_s = \Lambda \cap \Omega \cap [X_r = \delta$ for all $r \in [0, s)$: $X_s \in E_\Delta] \in \mathscr{F}$ for all $s \in (0, \infty]$ so that $\Lambda_s \in \mathscr{F}^0$ for all $s \in [0, \infty)$ and $\Lambda_s = b_{s_i}^{-1}([\delta]_{s_i - s} \times \Lambda_{s_i})$ if $s \in (s_{i-1}, s_i)$ for all $i \in \{1, \ldots, n, \infty\}$.

Thus $b^{-1}(\Lambda \cap \Omega) = (\{0\} \times \Lambda_0) \cup \left[\bigcup_{i=1}^{m} (s_{i-1}, s_i)\right] \times b_{s_i}^{-1}([\delta]_{s_i} \times \Lambda_{s_i})\Big]$ with $\Lambda_{s_i} \in \mathscr{F}$ for all $i \in \{1, \ldots, n\}$, so that $b^{-1}(\Lambda \cap \Omega) \in \mathscr{B} \times \mathscr{F}^0$.

4. Comultiplicative Functionals

A process $\{n_t\}_{t \geq 0}$ $\mathscr{E}^*$-measurable is a comultiplicative functional (comf) if and only if

(i) $0 \leq n_t \leq 1$ for all $t \geq 0$

(ii) $n_s \circ \theta_t = n_{s+t}$ for all $t \geq 0$, $s > 0$

(iii) $n_s = n_s \circ k_t \times n_t$ for all $t \geq s > 0$

(iv) $n_t \circ k_s = 1$ for all $t > s$.

If $\{n_t\}$ is right continuous, properties (ii) and (iii) extend to $s = 0$.

Left continuous comf are more likely to arise. For instance, for all coterminal time L, $n_t = 1_{[L<t]}$ defines a left continuous comf. With $\leq$ instead of $<$ we would still have (i), (ii) and (iv) but instead of (iii) we would have

(iii)′ $n_s = \lim_{u \downarrow t} n_s \circ k_u \times n_t$ for all $t \geq s \geq 0$.

Proposition 4.1. *If* $\{n_t\}_{t\geqq 0}$ *is a comf and* $n_{t^+}=\lim_{u\downarrow t} n_u$ *for all* $t\geqq 0$, *then* $\{n_{t^+}\}_{t\geqq 0}$ *satisfies* (iii)'.

Proof. For all $u>s\geqq 0$,

$$n_{s^+}=\lim_{v\downarrow s} n_v=\lim_{v\downarrow s} n_v\circ k_u\times n_u \qquad \text{by (iii)}$$

$$=n_{s^+}\circ k_u\times n_u.$$

Then for all $t\geqq s\geqq 0$,

$$n_{s^+}=\lim_{u\downarrow t} n_{s^+}\circ k_u\times \lim_{u\downarrow t} n_u$$

$$=\lim_{u\downarrow t} n_{s^+}\circ k_u\times n_{t^+} \qquad \text{Q.E.D.}$$

From Proposition 4.1 it follows immediately that every right continuous comf satisfies (iii)'.

More general processes are described replacing condition (ii) by

(ii)' $n_t\circ b_t=n_t$ for all $t\geqq 0$, or equivalently

(ii)'' $n_t\circ b_s=n_t$ for all $t\geqq s\geqq 0$.

5. Randomized Comultiplicative Functionals

Consider a set H of $\mathscr{F}^*|\mathscr{B}$-measurable functions from Ω into $[0, 1]$ and a process (Z_t) with values in H. Assume the existence of a σ-field $\mathscr{H}$ in H such that

a) Z_t is $\mathscr{F}_{t^+}|\mathscr{H}$-measurable for all $t\geqq 0$ and

b) $(h,\omega)\mapsto h(\omega)$ defines a $\mathscr{H}\times\mathscr{F}^*|\mathscr{B}$-measurable function on $H\times\Omega$.

Let $n_t(\omega)=[Z_t(\omega)\circ\theta_t](\omega)$ for all $\omega\in\Omega$ and all $t\geqq 0$.

For instance, $n_t=1_{[T\leqq t]}$ if $Z_t(\omega)$: $\omega'\mapsto 1_{[T(\phi(\omega,\omega'))\leqq t]}$ where

$$[\phi(\omega,\omega')]_s=\begin{cases}\omega_s & \text{if } s<t\\ \omega'_{s-t} & \text{if } s\geqq t\end{cases} \qquad \text{for all } \omega,\ \omega'\in\Omega$$

for any random time T.

Let $n(s,t)=\lim_{u\downarrow t} n_s\circ k_u$.

Assumption 5.1. $n_s=n(s,t)\times n_t$ *for all* $s\leqq t$.

Assumption 5.2. $Z_t\neq Z_t\circ b_s\Rightarrow n_t=n_s$ *for all* $s\leqq t$.

Assumption 5.3. n *is* $\mathscr{B}\times\mathscr{F}^*|\mathscr{B}$-*measurable.*

Any $\mathscr{F}^*$-measurable process n satisfying conditions (i), (ii)' and (iii)' of the last section can be described in the above terms with $Z_t(\omega)$ the same for all ω. In fact, in this case, $Z_t(\omega)=n_t\circ\theta_t^-$ for all ω, with θ_t^- defined by

$$[\theta_t^-(\omega)]_s=\begin{cases}\omega_{s-t} & \text{if } s\geqq t\\ \delta & \text{if } s<t\end{cases} \qquad \text{for all } \omega.$$

If, in addition, (ii) is satisfied then $Z_t(\omega)=n_0$ for all t and ω.

6. The Distribution of the Process Birthed at a Random Rate

Let P denote a Borel transition function on $(E, \mathscr{E}^*)$. Let μ denote a distribution on $(E, \mathscr{E})$ and P^μ (or E^μ) a distribution on $(\Omega, \mathscr{F}^*)$ that makes $(X_t)_{t \geqq 0}$ Markov with respect to $\{\mathscr{F}_t\}_{t \geqq 0}$ with transition function P and initial distribution μ. P_n^μ will denote the P^μ-distribution of the process obtained by birthing (X_t) at different times at a rate given by n. For example, if $n_t = 1_{[L \leqq t]}$ for a random time L, then P_n^μ will be the P^μ-distribution of the right continuous process

$$\tilde{X}_t = \begin{cases} X_t & \text{if } t \geqq L \\ \delta & \text{if } t < L. \end{cases}$$

P_n^μ is obtained from P^μ by relocating the mass at each $\omega \in \Omega_0$ along $\{b_t(\omega)\}_{t \in [0,\infty]}$ according to the measure α_ω defined on $[0, \infty]$ by $\alpha_\omega([0,t]) = \lim_{u \downarrow t} n_u(\omega)$.

More formally, P_n^μ is defined by $P_n^\mu(\Lambda) = E^\mu Z_\Lambda$ where Z_Λ: $\omega \mapsto \alpha_\omega(\{t: b_t(\omega) \in \Lambda\})$ for all $\Lambda \in \mathscr{F}$. Equivalently, since $Z_\Lambda(\omega) = \int_{[0,\infty]} 1_\Lambda \circ b_t(\omega)\, dn_t(\omega)$, E_n^μ is defined by

$$E_n^\mu(Y) = E^\mu \int_{[0,\infty]} Y \circ b_t \, dn_t \tag{$*$}$$

for all Y: $\Omega \to \mathbb{R}$, $\mathscr{F}$-measurable, bounded.

For all n: $[0,\infty) \times \Omega_0 \to [0,1]$, $\mathscr{B} \times \mathscr{M}$-measurable for some σ-field $\mathscr{M} \supset \mathscr{F}^0$ and with nondecreasing trajectories, $(\omega, [0,t]) \mapsto n_t(\omega)$ defines a transition probability from $(\Omega_0, \mathscr{M})$ to $([0,\infty], \mathscr{B})$, so that, if P^μ is a distribution on $\mathscr{M}$, $P^\mu \otimes dn_t$ is a distribution on $\mathscr{B} \times \mathscr{M}$ and $E^\mu \int_{[0,\infty]} W_t \, dn_t$ is defined for all W $\mathscr{B} \times \mathscr{M}$-measurable, bounded. Since, by Theorem 3.1, b is $\mathscr{B} \times \mathscr{M} | \mathscr{F}$-measurable, $Y \circ b$ is $\mathscr{B} \times \mathscr{M} | \mathscr{B}$-measurable for all Y $\mathscr{F} | \mathscr{B}$-measurable. Thus ($*$) defines a probability on $\mathscr{F}$ for all nondecreasing n: $[0,\infty) \times \Omega_0 \to [0,1]$ $\mathscr{B} \times \mathscr{M}$-measurable, for every $\mathscr{M} \supset \mathscr{F}^0$ to which P^μ may be extended (in particular, for $\mathscr{M} = \mathscr{F}^*$).

From now on, Z *and* n will be as in Section 5. Y and f will be bounded random variables $\mathscr{F}^*$- and $\mathscr{E}_\delta^*$-measurable respectively. $\{\mathscr{M}_t\}$ will be a family of σ-fields with $\mathscr{F}_t^0 \subset \mathscr{M}_t \subset \mathscr{F}_{t^+}$ for all t, such that $(Z_t)_{t \geqq 0}$ is adapted to $\{\mathscr{M}_t\}_{t \geqq 0}$ and $(X_t)_{t \geqq 0}$ is Markov with respect to $\{\mathscr{M}_t\}_{t \geqq 0}$.

Let $K^{t,x,h}(Y) = \dfrac{E^{t,x}(Y \times h)}{E^{t,x}(h)}$ for all $h \in H$, $x \in E$, $t \in [0,\infty)$, with the convention $\frac{0}{0} = 0$. $E^{t,x}$ is defined as usual through $E^{t,x} f(X_s) = P_{t,s} f(x)$.

$E^{t,X_t}(Y)$ will denote the mapping $\omega \mapsto E^{t,X_t(\omega)}(Y)$. Also $K^{t,X_t,Z_t}(Y)$ will denote the mapping $\omega \mapsto K^{t,X_t(\omega),Z_t(\omega)}(Y)$. The superscript μ will be omitted in the expressions involving E^μ, P^μ, E_n^μ and P_n^μ.

Let $K^{t,\delta,Z_t}(Y) = \dfrac{E_n([X_t = \delta]; Y \circ \theta_t)}{P_n[X_t = \delta]}$ for all $t \in [0,\infty)$.

Let $K_{s,t}(x,f) = K^{s,x,Z_s}[f(X_t)]$ for all s, $t \in [0,\infty)$ and all $x \in E_\delta$.

Lemma 6.1. $E[Y \circ \theta_t \times n_t | \mathscr{M}_t](\omega) = E^{t,X_t(\omega)}[Y \times Z_t(\omega)]$ *for P-a.a.* ω.

Proof. $E[Y \circ \theta_t \times n_t | \mathscr{M}_t](\omega) = E\{[Y \times Z_t(\omega)] \circ \theta_t | \mathscr{M}_t\}(\omega)$
$= E^{t,X_t(\omega)}[Y \times Z_t(\omega)]$ for P-a.a. ω.

In fact for all $\phi\colon H\times\Omega\to\mathbb{R}$, $\mathscr{H}\times\mathscr{F}^*$-measurable,

$$E[\phi(Z_t,\theta_t)|\mathscr{M}_t](\omega)=E^{t,X_t(\omega)}[\phi(Z_t(\omega),\theta_0)] \quad \text{for } P\text{-a.a. } \omega.$$

It is enough to show this for $\phi=1_{A\times B}$, $A\in\mathscr{H}$, $B\in\mathscr{F}^*$. In this case, for all W $\mathscr{M}_t$-measurable,

$$E[W\phi(Z_t,\theta_t)]=E[W1_A(Z_t)P^{t,X_t}(B)]$$

and

$$E^{t,X_t(\omega)}[\phi(Z_t(\omega),\theta_0)]=E^{t,X_t(\omega)}\{1_A[Z_t(\omega)]\times 1_B\}$$
$$=1_A[Z_t(\omega)]\times P^{t,X_t(\omega)}(B) \quad \text{for } P\text{-a.a. } \omega.$$

Lemma 6.2. *For all* $\mathscr{M}_t$-measurable random variable W_t,

$$E_n[W_t\times(Y1_{\Omega_0})\circ\theta_t]=E_n[[X_t\in E];\ W_t\times K^{t,X_t,Z_t}(Y)]=EV_t$$

where V_t is defined in Ω_0 by

$$V_t(\omega)=\Big[\int_{[0,t]} W_t\circ b_r\,d\eta(r)\Big](\omega)\times E^{t,X_t(\omega)}[Z_t(\omega)\times Y],$$

η denoting the distribution on $[0,t]$ determined by $\eta([0,r])=n(r,t)$.

Proof. We want to show that

$$E\int_{[0,t]}[W_t\times(Y1_{\Omega_0})\circ\theta_t]\circ b_r\,dn_r=E\int_{[0,t]}\{W_t\times 1_{[X_t\in E]}\times K^{t,X_t,Z_t}(Y)\}\circ b_r\,dn_r$$
$$=EV_t$$

$$E\int_{[0,t]}[W_t\times(Y1_{\Omega_0})\circ\theta_t]\circ b_r\,dn_r=E\Big\{Y\circ\theta_t\int_{[0,t]}W_t\circ b_r\,dn_r\Big\}$$
$$=E\Big\{Y\circ\theta_t\times n_t\times\int_{[0,t]}W_t\circ b_r\,d\eta(r)\Big\}$$

by Assumption 5.1

$$=EV_t \text{ by Lemma 6.1.}$$

On the other side,

$$E\int_{[0,t]}\{W_t\times 1_{[X_t\in E]}\times K^{t,X_t,Z_t}(Y)\}\circ b_r\,dn_r$$
$$=E\int_{[0,t]}W_t\circ b_r\times K^{t,X_t,Z_t}(Y)\,dn_r \quad \text{by Assumption 5.2}$$
$$=E\frac{V_t}{E^{t,X_t}(Z_t)}\times n_t \quad \text{by Assumption 5.1}$$
$$=EV_t \quad \text{by Lemma 6.1.}$$

Proposition 6.1. $K_{s,t-s}(\delta,f)=\dfrac{E[f(X_t)(n_t-n_s)]+(1-En_t)f(\delta)}{1-En_s}$ *for all* $s\leqq t\in[0,\infty)$.

Proof. We must show that

$$E_n\{[X_s=\delta]; f(X_t)\} = E[f(X_t)(n_t-n_s)] + (1-En_t)f(\delta)$$

$$\begin{aligned} E_n\{[X_s=\delta]; f(X_t)\} &= E_n[f(X_t)] - E_n\{[X_s\in E]; f(X_t)\} \\ &= E_n\{[X_t\in E]; f(X_t)\} + P_n[X_t=\delta]\times f(\delta) \\ &\quad - E_n\{[X_s\in E]; f(X_t)\} \end{aligned}$$

$$E_n\{[X_t\in E]; f(X_t)\} = E[f(X_t)\times n_t] \qquad \text{by Lemma 6.2.}$$

Also $E_n\{[X_s\in E_s]; f(X_t)\} = E[f(X_t)\times n_s]$ by Lemmas 6.1 and 6.2. Finally

$$\begin{aligned} P_n[X_t=\delta] &= 1 - P_n[X_t\in E] \\ &= 1 - En_t \qquad \text{by Lemma 6.2.} \end{aligned}$$

Theorem 6.1. *Under P_n^μ, $(X_t)_{t\geqq 0}$ is conditionally Markov with respect to $\{\mathcal{M}_t\}_{t\geqq 0}$ given $(Z_t)_{t\geqq 0}$ with transition function K and entrance law given by $P_n[X_t\in A] = E([X_t\in A];\, n_t)$ for all $A\in\mathscr{E}$.*

Proof. The P_n-distribution of X_t is given in Lemma 6.2 considering $W_t=1_A(X_t)$, $A\in\mathscr{E}$ and $Y=1_{\Omega_0}$. Notice that then

$$V_t(\omega) = 1_A(X_t)\times E^{t,X_t(\omega)}[Z_t(\omega)]$$

so that $P_n[X_t\in A] = E([X_t\in A]; n_t)$ by Lemma 6.1.

The conditional Markov property is

$$E_n[W_t\times(Y\circ\theta_t)] = E_n[W_t\times K^{t,X_t,Z_t}(Y)]$$

for all bounded random variables W_t and Y, $\mathcal{M}_t$- and $\mathscr{F}^*$-measurable respectively.

It was proved before that

$$E\int_{[0,t]} [W_t\times(Y\circ\theta_t)]\circ b_r\, dn_r = E\int_{[0,t]} [W_t\times K^{t,X_t,Z_t}(\omega)]\, b_r\, dn_r$$

$W_t\circ b_r(\omega) = W_t([\delta])$ for all ω if $r>t$, because $[X_t=\delta]$ is an atom of $\mathscr{F}_{t^+}$ by right continuity of the elements of Ω.

Then

$$E\int_{(t,\infty]} [W_t\times(Y\circ\theta_t)]\circ b_r\, dn_r = W_t([\delta])\, E^\mu \int_{(t,\infty]} Y\circ\theta_t\circ b_r\, dn_r.$$

Also

$$\begin{aligned} E\int_{(t,\infty]} [W_t\times K^{t,X_t,Z_t}(Y)]\circ b_r\, dn_r &= W_t([\delta])\times K^{t,\delta,Z_t}(Y)\times E(1-n_t) \\ &= W_t([\delta])\times K^{t,\delta,Z_t}(Y)\times P_n[X_t=\delta] \\ &= W_t([\delta])\times E_n\{[X_t=\delta];\; Y\circ\theta_t\}. \end{aligned}$$

Thus the proof is concluded if

$$E_n\{[X_t=\delta];\ Y\circ\theta_t\}=E\int_{(t,\infty]} Y\circ\theta_t\circ b_r\,dn_r$$

which follows from $X_t(b_r(\omega))\in E$ for all $r\in[0,t]$ and all $\omega\in\Omega_0$.

7. The Post-Birth Process

The next results will refer to the process $(X_{\alpha+t})_{t>0}$ where $\alpha=\inf\{t: X_t\in E\}$. In the case of a comultiplicative functional n constructed from a coterminal time L, $X_{\alpha+t}\circ b_L=X_{L+t}$ and the P_n^μ-mass on $b_L(\omega)$ is the P^μ-mass on ω so that $(X_{\alpha+t})_{t>0}$ under P_n^μ has the same distribution of $(X_{L+t})_{t>0}$ under P^μ.

Let $\alpha_i=\inf\left\{\frac{k}{2^i}: \alpha\leqq\frac{k}{2^i},\ k\in\mathbb{N}\right\}$ for all $i\in\mathbb{N}$.

Proposition 7.1. *α and α_1 for all i are stopping times with respect to $\{\mathscr{F}_t^0\}_{t\in[0,\infty)}$.*

Proof. $[\alpha\leqq t]=[X_t\in E]\in\mathscr{F}_t^0$.

$$[\alpha_i\leqq t]=\bigcup_{\frac{k}{2^i}<t}\left[\alpha_i=\frac{k}{2^i}\right]\quad\text{and}\quad\left[\alpha_i=\frac{k}{2^i}\right]=\left[\alpha\leqq\frac{k}{2^i}\right]\cap\left[\alpha>\frac{k-1}{2^i}\right]\in\mathscr{F}^0_{\frac{k}{2^i}}.$$

Proposition 7.2. *Under P_n^μ, $(X_{\alpha_i+t})_{t\geqq 0}$ is, given $(Z_t)_{t\geqq 0}$, Markov with respect to $\{\mathscr{M}_{\alpha_i+t}\}_{t\geqq 0}$ with K as transition function.*

Proof. It is enough to show that for all s and all bounded random variables W $\mathscr{M}_{\alpha_i+s}$-measurable,

$$E_n(W\times Y\circ\theta_{\alpha_i+s})=E_n(W\times K^{\alpha_i+s,X_{\alpha_i+s},Z_{\alpha_i+s}}(Y)).$$

Let $J_k=1_{\left[\alpha_i=\frac{k}{2^i}\right]}$ for each $k\in\mathbb{N}$.

$$\begin{aligned}E_n[W\times Y\circ\theta_{\alpha_i+s}]&=\sum_{k=0}^{\infty}E_n[WJ_kY\circ\theta_{\alpha_i+s}]\\&=\sum_{k=0}^{\infty}E_n[WJ_kY\circ\theta_{\frac{k}{2^i}+s}]\qquad\text{by Theorem 6.1}\\&=\sum_{k=0}^{\infty}E_n[WJ_kK^{\frac{k}{2^i}+s,X_{\frac{k}{2^i}+s},Z_{\frac{k}{2^i}+s}}(Y)]\\&=\sum_{k=0}^{\infty}E_n[WJ_kK^{\alpha_i+s,X_{\alpha_i+s},Z_{\alpha_i+s}}(Y)]\\&=E_n[W\times K^{\alpha_i+s,X_{\alpha_i+s},Z_{\alpha_i+s}}(Y)].\end{aligned}$$

Let us now suppose P a homogeneous transition function and (X_t) a right process under P^μ (as defined in [8] or [2]).

Theorem 7.1. *If $(n_t)_{t \geqq 0}$ is right continuous, then $(X_{\alpha+t})_{t>0}$ is, given $(Z_t)_{t \geqq 0}$, Markov with respect to $\{\mathscr{F}_{\alpha+t^+}\}_{t>0}$ with transition function K.*

Proof.

$$E_n[f(X_{\alpha+t}) | \mathscr{F}_{\alpha+s^+}] = \lim_{i \to \infty} E_n[f(X_{\alpha_i+t}) | \mathscr{F}_{\alpha_i+s^+}] = \lim_{i \to \infty} K^{X_{\alpha_i+s}, Z_{\alpha_i+s}}[f(X_{t-s})]$$

by Proposition 7.2 for all continuous f.

Then the theorem is proved if we show that

a) for all sequences of $\{\mathscr{F}_{t^+}\}$-stopping times $\{T_i\}_{i \in \mathbb{N}}$, if $T_i \downarrow T \geqq \alpha$, then, for all r, all continuous f and P_n-a.a. ω,

$$E^{X_{T_i}(\omega)}[f(X_r)[Z_{T_i}(\omega)]_0] \to E^{X_T(\omega)}[f(X_r)[Z_T(\omega)]_0]$$

and

b) for all $s \in (0, \infty)$, $P_n\{\omega: E^{X_{\alpha+s}(\omega)}[Z_{\alpha+s}(\omega)]_0 = 0\} = 0$.

a) The P^μ-a.s. right continuity of (n_t) gives

$$f(X_{r+T_i}) n_{T_i} \to f(X_{r+T}) n_T \; P^\mu\text{-a.s.} \quad \text{as } T_i \downarrow T.$$

Then

$$E[f(X_{r+T_i}) n_{T_i} | \mathscr{F}_{T_i}] \to E[f(X_{r+T}) n_T | \mathscr{F}_T]$$

i.e.,

$$E^{X_{T_i}(\omega)}[f(X_r)[Z_{T_i}(\omega)]_0] \to E^{X_t(\omega)}[f(X_r)[Z_T(\omega)]_0]$$

for P^μ-a.a. ω.

Finally, since

$$1_{\{\omega: E^{X_{T_i}(\omega)}[f(X_r)[Z_{T_i}(\omega)]_0] \to E^{X_T(\omega)}[f(X_r)[Z_T(\omega)]_0],\ T(\omega) \geqq \alpha\}} \circ b_s$$
$$= 1_{\{\omega: E^{X_{T_i}(\omega)}[f(X_r)[Z_{T_i}(\omega)]_0] \to E^{X_T(\omega)}[f(X_r)[Z_T(\omega)]_0],\ T(\omega) \geqq \alpha\}}$$

for all $s \in \mathbb{R}_+$,

$$P_n\{\omega: E^{X_{T_i}(\omega)}[f(X_r)[Z_{T_i}(\omega)]_0] \to E^{X_T(\omega)}[f(X_r)[Z_T(\omega)]_0]\}$$
$$= P\{\omega: E^{X_{T_i}(\omega)}[f(X_r)[Z_{T_i}(\omega)]_0] \to E^{X_T(\omega)}[f(X_r)[Z_T(\omega)]_0]\}$$

if $T \geqq \alpha$.

b) Let $\mathscr{F}_T^\mu$ denote the completion of $\mathscr{F}_T$ with respect to P^μ for all time T. Let $(Y_t)_{t \geqq 0}$ be a $\{\mathscr{F}_t^\mu\}$-optional process such that for all finite $\{\mathscr{F}_t^\mu\}$-optional time T, $Y_T = E(n_T | \mathscr{F}_T^\mu)$. The existence of such a process is proved in [4], for instance.

By the same argument used in a) for all sequences $\{T_i\}_{i \in \mathbb{N}}$ of $\{\mathscr{F}_t^\mu\}$-stopping times, if $T_i \downarrow T$, then $Y_{T_i} \downarrow Y_T$.

Then, by a theorem of Mertens [3], $(Y_t)_{t>0}$ is a right continuous process.

Thus $S = \inf\{t > \alpha, Y_t = 0\}$ is a $\{\mathscr{F}_t^\mu\}$-stopping time and $Y_S = 0$.

$$\begin{aligned} P_n[S<\infty] &= E \int_{[0,\infty]} 1_{[S<\infty]} \circ b_s \, dn_s \\ &= E([S<\infty], n_\alpha) \\ &\leqq E([S<\infty], n_S) \\ &= E([S<\infty, Y_S=0]; n_S) \\ &= E([S<\infty, Y_S=0]; Y_S)=0 \end{aligned}$$

Then $P_n[Y_{\alpha+s}=0$ for some $s>0]=0$.
This implies $P_n(\{\omega: E^{X_{\alpha+s}(\omega)}[Z_{\alpha+s}(\omega)]_0=0\})=0$ for all $s\in(0,\infty)$ because

$$\begin{aligned} &P_n(\{\omega: Y_s(\omega) \neq E^{X_{\alpha+s}(\omega)}[Z_{\alpha+s}(\omega)]_0\}) \\ &\quad = E[\{\omega: Y_s(\omega) \neq E^{X_{\alpha+s}(\omega)}[Z_{\alpha+s}(\omega)]_0\}; n_\alpha] \\ &\quad \leqq P(\{\omega: Y_s(\omega) \neq E^{X_{\alpha+s}(\omega)}[Z_{\alpha+s}(\omega)]_0\})=0. \end{aligned}$$

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Received October 7, 1977