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# On Brownian Slow Points

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**Summary.** It is shown that, for a Wiener process $X_t$, both the quantities $\inf_t \overline{\lim}_{h\to 0+} |X_{t+h}-X_t|/\sqrt{h}$ and $\sup_t \underline{\lim}_{h\to 0+} (X_{t+h}-X_t)/\sqrt{h}$ are almost surely equal to 1.

## 1. Introduction

Let $W_t$, $t \geqq 0$, be standard Brownian motion. In the 1932 paper where they showed that almost every Brownian path is nowhere differentiable ([7]), Paley, Wiener, and Zygmund proved the stronger result that, for each $\varepsilon > 0$,

$$P\left(\overline{\lim_{h\to 0+}} \frac{|W_{t+h}-W_t|}{h^{\frac{1}{2}+\varepsilon}} = \infty \ \forall t\right) = 1,$$

and in 1963 A. Dvoretzky [1] improved this by establishing

$$P\left(\overline{\lim_{h\to 0+}} \frac{|W_{t+h}-W_t|}{\sqrt{h}} > c_0 \ \forall t\right) = 1, \tag{1.1}$$

for a positive constant $c_0$. The natural question, whether (1.1) holds for all constants, was settled by J.P. Kahane in 1974 [2]. The answer is no. Kahane showed

$$P\left(\exists t\colon \overline{\lim_{h\to 0}} \frac{|W_{t+h}-W_t|}{\sqrt{|h|}} < c_1\right) = 1, \tag{1.2}$$

for a constant $c_1 < \infty$.

Here, note that $h$ may be allowed to approach 0 from either the left or the right, giving a better result. (The two sided verison of (1.1) is of course weaker than (1.1).) Kahane calls those $t$ which satisfy

$$\overline{\lim_{h\to 0}} \frac{|W_{t+h}-W_t|}{\sqrt{|h|}} < \infty$$

slow points.

The law of the iterated logarithm implies that the slow points almost surely have Lebesgue measure 0, but Kahane has proved ([2, 3]) that their Hausdorff dimension a.s. equals 1 and that the Hausdorff dimension of those slow points which are also zeros of $W_t$ is a.s. $\frac{1}{2}$, so that the slow points are a fairly thick set. Kahane has very recently given another, simpler, proof of (1.2) and the two sided version of (1.1), together with related results for other Gaussian processes, in [4].

Following Kahane, we will call a point $t$ *slow from the right* if

$$\overline{\lim_{h\to 0+}} \frac{|W_{t+h}-W_t|}{\sqrt{h}} < \infty.$$

In Sect. 2 we investigate the question: How slow from the right can a point be? It is shown that

$$\inf_t \overline{\lim_{h\to 0+}} \frac{|W_{t+h}-W_t|}{\sqrt{h}} = 1 \quad \text{a.s.} \tag{1.3}$$

The proof that the expression to the left of the equality in (1.3) is no smaller than 1 is a refinement of Dvoretzky's proof in [1], while the proof that it is no larger than 1 is not related to Kahane's arguments.

Let $z$ be the smallest positive 0 of $M(-\frac{1}{2},\frac{1}{2},x^2/2)$, where $M$ is the confluent hypergeometric function ($z \approx 1.3069$). In Sect. 4 it is shown that

$$P\left(\inf_t \overline{\lim_{h\to 0}} \frac{|W_{t+h}-W_t|}{\sqrt{|h|}} < z\right) = 0, \tag{1.4}$$

but we cannot prove

$$P\left(\inf_t \overline{\lim_{h\to 0}} \frac{|W_{t+h}-W_t|}{\sqrt{|h|}} = z\right) = 1.$$

Nonetheless, this is probably true. Not only does (1.4) hold, but also it is shown in Sect. 4 that, if $X_t$ and $Y_t$ are independent Brownian motions, and if $D_r^X = \left\{t\colon \overline{\lim_{h\to 0+}} \frac{|X_{t+h}-X_t|}{\sqrt{h}} < r\right\}$ and $D_r^Y$ is defined similarly, then $D_r^X \cap D_r^Y$ is almost surely empty if $r<z$ and not empty if $r>z$.

S. Orey and J. Taylor have shown how rapid a point can be by proving (see [5])

$$\sup_t \overline{\lim_{h\to 0+}} \frac{|W_{t+h}-W_t|}{\sqrt{2h\log\frac{1}{h}}} = 1 \quad \text{a.s.}$$

This is equivalent to

$$\sup_t \overline{\lim_{h\to 0+}} \frac{W_{t+h}-W_t}{\sqrt{2h\log\frac{1}{h}}} = 1 \quad \text{a.s.,} \tag{1.5}$$

giving a global upper bound for the lim sup, as $h\to 0+$, of $W_{t+h}-W_t$. In his book ([5], p. 148) F. Knight asks for a global lower bound for this lim sup. Precisely, Knight asks for a function $\phi(h)>0$ such that, almost surely, for all $t$

$$\overline{\lim_{h\to 0+}}\,(W_{t+h}-W_t)/\phi(h)\geqq -1 \quad\text{and}\quad \overline{\lim_{h\to 0+}}\,(W_{t+h}-W_t)/\phi(h)=-1$$

for some $t$. In Sect. 3 we come pretty close to solving this problem, and do give a lower bound in the sense that (1.5) gives an upper bound. We prove

$$\inf_t \overline{\lim_{h\to 0+}} \frac{W_{t+h}-W_t}{\sqrt{h}}=-1 \qquad \text{a.s.} \tag{1.6}$$

We do not know if the $\overline{\lim}$ equals $-1$ for some $t$.

The two sided version of (1.6) is essentially known and not hard to prove. It is

$$\inf_t \overline{\lim_{h\to 0}} \frac{W_{t+h}-W_t}{\sqrt{|h|}}=0 \qquad \text{a.s.} \tag{1.7}$$

Note that the existence of times for which $W_t$ has a local maximum shows that 0 can not be replaced by a larger number in (1.7). Only strict maxima need be considered, and, these being countable, the proof of (1.7) can be completed by examining the behavior of $W_t$ around an absolute maximum. See [5] for a treatment of such ideas, which yields sharper results than (1.7).

Define the sets

$$A_c=\{\exists t\in[0,1]\colon |W_{t+h}-W_t|<c\sqrt{h}\;\forall h\in(0,1]\}$$

and

$$B_c=\{\exists t\in[0,1]\colon (W_{t+h}-W_t)>c\sqrt{h}\;\forall h\in(0,1]\}.$$

We will prove that $P(A_c)=0$ if $c<1$ and $P(A_c)>0$ if $c>1$, implying (1.3), and that $P(B_c)=0$ if $c>1$ and $P(B_c)>0$ if $c<1$, implying $\sup_t \underline{\lim}_{h\to 0+} (W_{t+h}-W_t)/\sqrt{h} =1$ a.s., which is equivalent to (1.6).

For the remainder of the paper the qualifier a.s. will usually be omitted.

## 2. Proof of (1.3)

First the following lemma is established. Let $\wedge$ denote minimum.

**Lemma 2.1.** *If the nonnegative random variable $X$ satisfies*

$$\lim_{n\to\infty} nP(X\geqq n)/EX\wedge n=0, \tag{2.1}$$

*then $EX^p<\infty$, $0<p<1$.*

*Proof.* Note that the example $P(X\geqq t)=t^{-1}$, $t\geqq 1$, shows that (2.1) can hold and $EX=\infty$.

Now for any nonnegative random variable $Z$ and any $p>0$, $EZ^p$ is finite or infinite depending on whether $\sum_{n=1}^{\infty} 2^{np} P(Z \geqq 2^n)$ is finite or infinite. Let $\gamma_n = 2^n P(X \geqq 2^n)$. Then (2.1) implies

$$\lim_{n\to\infty} \gamma_n \Big/ \sum_{i=1}^{n-1} \gamma_i = 0.$$

For $\varepsilon>0$ let $N(\varepsilon)=N$ satisfy $\gamma_n \Big/ \sum_{i=1}^{n-1} \gamma_i < \varepsilon$, $n>N$. Put $\sum_{i=1}^{N} \gamma_i = y$. Then $\gamma_{N+1} < \varepsilon y$ and $\gamma_{N+k} < \varepsilon \left(y + \sum_{j=1}^{k-1} \gamma_{N+j}\right)$. If we put $\alpha_{N+k} = y(1+\varepsilon)^k$, then $\alpha_{N+1} \geqq \varepsilon y$ and $\alpha_{N+k} \geqq \varepsilon \left(y + \sum_{j=1}^{k-1} \alpha_{N+j}\right)$, and so, by induction, $\gamma_{N+k} \leqq \alpha_{N+k} = y(1+\varepsilon)^k$, $k \geqq 1$. Since $\varepsilon$ is arbitrary, this implies $EX^p < \infty$, $p<1$, as claimed.

Let $\tau_r(W) = \tau_r = \inf\{t \geqq 1 \colon |X_t| \geqq r\sqrt{t}\}$. Precise information concerning the moments of $\tau_r$ may be found in Shepp, [8]. For our purposes the following lemma suffices. The notation $P_{a,b}$ and $E_{a,b}$ will signify probability and expectation associated with $W_t$ given $W_a = b$.

**Lemma 2.2.** *If $r>1$ there is a $p=p(r)<1$ such that $E\tau_r^p = \infty$. Furthermore $E_{1,0}\tau_1 = \infty$.*

*Proof.* First the well known proof of the second statement will be supplied. For a stopping time $T$ we have

$$E W_T^2 = ET \quad \text{if } ET < \infty, \tag{2.2}$$

and applying this to the Wiener process $W_{t+1}$ under $P_{1,0}$ yields

$$E_{1,0} W_{\tau_1}^2 = E_{1,0}(\tau_1 - 1) \quad \text{if } E_{1,0}(\tau_1 - 1) < \infty.$$

Since $P_{1,0}(W_{\tau_1}^2 = \tau_1) = 1$, this implies $E_{1,0}(\tau_1 - 1) = \infty$, so $E_{1,0}\tau_1 = \infty$.

To prove the rest of the lemma, fix $r>1$ and define

$$\gamma_1 = \inf\{t \geqq 1 \colon W_t = 0 \text{ or } |W_t| \geqq r\sqrt{t}\},$$

and in general

$$\gamma_{2k} = \inf\{t \geqq \gamma_{2k-1} \colon |W_t| \geqq \sqrt{t}\},$$

and

$$\gamma_{2k+1} = \inf\{t \geqq \gamma_{2k} \colon W_t = 0 \text{ or } |W_t| \geqq r\sqrt{t}\}.$$

Note that on $\{W_{\gamma_{2k-1}} = 0\}$, if $\lambda > 0$,

$$P(\gamma_{2k} - \gamma_{2k-1} > \lambda \gamma_{2k-1} \mid W_{\gamma_{2k-1}}) = P_{1,0}(\tau_1 > 1 + \lambda)$$

using the strong Markov property and Brownian scaling. Furthermore, if $\varepsilon = P_{1,1}(W_t = 0 \text{ before } |W_t| = r\sqrt{t})$, then, on $\{\gamma_{2k-2} < \tau_r\}$, we have, for $k \geqq 2$ and $\lambda > 0$,

$$\begin{aligned}P(\gamma_{2k}-\gamma_{2k-1}>\lambda\gamma_{2k-2}\,|\,W_{\gamma_{2k-2}})\\ &\geqq P(\gamma_{2k}-\gamma_{2k-1}>\lambda\gamma_{2k-1}\,|\,W_{\gamma_{2k-2}}, W_{\gamma_{2k-1}}=0)\,P(W_{\gamma_{2k-1}}=0\,|\,W_{\gamma_{2k-2}})\\ &=P_{1,0}(\tau_1>1+\lambda)\,\varepsilon.\end{aligned}$$

Thus, since $\{\gamma_{2k-2}<\tau_r\}=\{\gamma_{2k-3}<\gamma_{2k-2}\}$, this gives

$$\begin{aligned}E(\gamma_{2k}-\gamma_{2k-1})^p &\geqq \varepsilon E_{1,0}(\tau_1-1)^p\cdot E\gamma_{2k-2}^p\, I(\gamma_{2k-2}<\tau_r)\\ &\geqq \varepsilon E_{1,0}(\tau_1-1)^p\, E(\gamma_{2k-2}-\gamma_{2k-3})^p\end{aligned}$$

and iteration gives

$$E(\gamma_{2k}-\gamma_{2k-1})^p\geqq(\varepsilon E_{1,0}(\tau_1-1)^p)^{k-1}\,E(\gamma_2-\gamma_1)^p.$$

Pick $p<1$ such that $\varepsilon E_{1,0}(\tau_1-1)^p>1$. This is possible since $E_{1,0}\tau_1=\infty$. Then $E\tau_r^p\geqq E\gamma_{2k}^p\geqq E(\gamma_{2k}-\gamma_{2k-1})^p\to\infty$ as $k\to\infty$.

Next put $M_w=M=\max\limits_{0\leqq t\leqq 1}|W_t|$, and $T_r^w=T_r=\inf\{t\geqq 1\colon |W_t|=M+r\sqrt{t}\}$.

**Lemma 2.3.** *If* $c<1$, $ET_c<\infty$.

*Proof.* For $t\geqq 1$ the equality (2.2) gives

$$\begin{aligned}ET_c\wedge t&=EW^2_{T_c\wedge t}\\ &\leqq E(M+c\sqrt{T_c\wedge t})^2\\ &=EM^2+2c\,EM\sqrt{T_c\wedge t}+c^2\,ET_c\wedge t.\end{aligned}$$

Thus

$$\begin{aligned}(1-c^2)\,ET_c\wedge t&\leqq EM^2+2c\,EM\sqrt{T_c\wedge t}\\ &\leqq EM^2+2c(EM^2)^{\frac12}(ET_c\wedge t)^{\frac12}.\end{aligned}$$

Since $EM^2$ is finite, $ET_c\wedge t$ must stay bounded as $t\to\infty$, so $ET_c<\infty$.

As has been mentioned, the following theorem implies (1.3).

**Theorem 2.1.** *If* $c<1$, $P(A_c)=0$, *and if* $c>1$, $P(A_c)>0$.

*Proof.* Fix $c<1$, and for a subinterval $[a,b]=I$ of $[0,1]$ let $\Delta_I=\{\exists t\in I\colon |X_{t+h}-X_t|<c\sqrt{h}, 0<h\leqq 1\}$. Note that, if $M_I=\max\limits_{a\leqq t\leqq b}|W_t-W_a|$, then $\Delta_I\subset\{|W_{a+h}-W_a|<M_I+c\sqrt{h}, b-a\leqq h\leqq 1\}$, by a geometrical argument. Thus, conditioning on $W_a$ and changing scale, we have

$$P(\Delta_I)\leqq P(T_c\geqq(b-a)^{-1}),$$

and especially, if $I$ has length $n^{-1}$, $P(\Delta_I)\leqq P(T_c\geqq n)$. Divide $[0,1]$ into intervals $I_k$ of length $n^{-1}$. Then $P(A_c)\leqq\sum P(\Delta_{I_k})\leqq nP(T_c\geqq n)$. Lemma 2.3 gives $ET_c<\infty$, so $nP(T_c\geqq n)\to 0$, proving $P(A_c)=0$.

Now fix $c>1$ and put $\Gamma_n=\{\exists t\in[0,1]\colon |W_{t+h}-W_t|<c\sqrt{h}, n^{-1}\leqq h\leqq 1\}$. Note that $\Gamma_n\subseteq\Gamma_m$ if $n\geqq m$. We will show that $\varliminf\limits_{n\to\infty}P(\Gamma_n)>0$, implying $P\left(\bigcap\limits_{n=1}^{\infty}\Gamma_n\right)=P(A_c)>0$. Put $v_{0,n}=v_0=0$, and, if $i\geqq 1$,

$$v_{i,n}=v_i=(v_{i-1}+1)\wedge\inf\{t\geqq v_{i-1}+n^{-1}\colon |W_t-W_{v_{i-1}}|\geqq c\sqrt{t-v_{i-1}}\}.$$

Then

$$P(v_{i+1}-v_i=1\,|\,W_{v_i})=P(\tau_c\geqq n), \tag{2.3}$$

and

$$E(v_{i+1}-v_i\,|\,W_{v_i})=n^{-1}\,E\tau_c\wedge n. \tag{2.4}$$

Of course $v_k\geqq 1$ if $v_i-v_{i-1}=1$ for some $i\leqq k$. Thus

$$\begin{aligned}\varphi_{n,m}&=P(v_{i+1}-v_i=1 \text{ for some } i\leqq m \text{ such that } v_i\leqq 1\}\\&=\sum_{i=1}^{m} P(\tau_c\geqq n)\,P(v_i\leqq 1)\\&\geqq m P(\tau_c\geqq n)\,P(v_m\leqq 1).\end{aligned}$$

Let $\{n_k\}_{k=1}^{\infty}$ be a sequence of integers approaching infinity such that $n_k P(\tau_c\geqq n_k)/E\tau_c\wedge n_k\geqq\alpha>0$ for all $k$, such a choice being possible by Lemmas 2.1 and 2.2. We also assume

$$E\tau_c\wedge n_k/n_k\leqq 1/6.$$

Let the integer $m_k$ satisfy

$$1/3\leqq(m_k/n_k)\,E\tau_c\wedge n_k\leqq 1/2. \tag{2.5}$$

By (2.4),

$$E v_{m_k,n_k}=(m_k/n_k)\,E\tau_c\wedge n_k,$$

so

$$P(v_{m_k,n_k}\geqq 1)\leqq 1/2,$$

and, using the left inequality in (2.5), we have

$$\begin{aligned}P(\Gamma_{n_k})&\geqq\varphi_{n_k,m_k}\geqq m_k P(\tau_c\geqq n_k)\,P(v_{m_k,n_k}\leqq 1)\\&\geqq m_k P(\tau_c\geqq n_k)/2\geqq m_k\alpha E\tau_c\wedge n_k/2n_k\geqq\alpha/6.\end{aligned}$$

## 3. Proof of (1.6)

The arguments involving $B_c$ are very similar to those of the last section and proofs will just be sketched. Let $\eta_a=\inf\{t\geqq 0\colon W_t\leqq a\sqrt{t}\}$. Precise information on the moments of $\eta_a$ has been supplied by Novikov in [6]. Here we need only the following analog of Lemma 2.2.

**Lemma 3.1.** *If* $r<1$ *there is a* $q=q(r)<1$ *such that* $E\eta_r^q=\infty$. *Furthermore* $E_{1,2}\eta_1=\infty$.

*Proof.* That $E_{0,1}\eta_1=\infty$ follows from $E_{0,1}\eta_1=E_{0,1}(W_{\eta_1}-1)^2$ if $E_{0,1}\eta_1<\infty$, because $P_{0,1}(W_{\eta_1}^2=\eta_1)=1$. Since $P_{1,2}(\eta_1-1>\lambda)>P_{0,1}(\eta_1>\lambda)$, $\lambda>0$, we get $E_{1,2}\eta_1\geqq E_{0,1}\eta_1=\infty$.

The proof of the first assertion of Lemma 3.1 can be patterned on the proof of the first assertion of Lemma 2.2. The analogs of the times $\gamma_i$ here are

$$\tilde{\gamma}_1 = \inf\{t \geqq 1: W_t = 2\sqrt{t} \text{ or } W_t \leqq r\sqrt{t}\},$$

and in general

$$\tilde{\gamma}_{2k} = \inf\{t \geqq \tilde{\gamma}_{2k-1}: W_t \leqq \sqrt{t}\},$$

and

$$\tilde{\gamma}_{2k+1} = \inf\{t \geqq \tilde{\gamma}_{2k}: W_t = 2\sqrt{t} \text{ or } W_t \leqq r\sqrt{t}\}.$$

Now let $M^- = \min_{0 \leqq t \leqq 1} W_t$, and let $U_r = \inf\{t \geqq 1: W_t = r\sqrt{t-1} + M^-\}$.

**Lemma 3.2.** *If* $c > 1$, $EU_c < \infty$.

*Proof.* For each $t > 1$, (2.2) gives

$$\begin{aligned} EU_c \wedge t &= E(W_{U_c \wedge t})^2 \\ &\geqq E(c\sqrt{U_c \wedge t - 1} + M^-)^2, \end{aligned}$$

and the rest of the proof resembles the proof of Lemma 2.3.

**Theorem 3.1.** *If* $c > 1$, $P(B_c) = 0$ *and, if* $c < 1$, $P(B_c) \neq 0$.

*Proof.* Note that if $[a, b] = I$ is a subinterval of $[0, 1]$, and if $M_I^- = \min_{a \leqq t \leqq b} W_t - W_a$, then

$$\begin{aligned} &\{\exists t \in [a, b]: W_{t+h} - W_t > c\sqrt{h}\, \forall h \in (0, 1]\} \\ &\subseteq \{W_{t+h} - W_t > c\sqrt{h - (b-a)} + M_I^-, b - a \leqq h \leqq 1\}, \end{aligned}$$

and the rest of the proof that $P(B_c) = 0$, $c > 1$, follows from Lemma 3.2 just like the proof that $P(A_c) = 0$, $c < 1$, followed from Lemma 3.3. Furthermore, the proof that $P(B_c) > 0$, $c < 1$, is almost the same as the proof that $P(A_c) > 0$, $c > 1$.

## 4. Independent Wiener Processes

The arguments in this section are similar to those of Sect. 2, but we use more of Shepp's results in [8]. Fix $r > 0$, and for $0 < t < 2$ and $|s| < r\sqrt{t}$ let $f_{t,s}$ be the continuous version of the density of $W_2 I(\tau_r > 2)$ under $P_{t,s}$. Of course $f$ vanishes off $(-\sqrt{2}r, \sqrt{2}r)$. Then if $\alpha(r) = \alpha = 2/P_{1,0}(\tau_r < 2)$, we have

$$f_{1,y}(s)/f_{1,0}(s) \leqq \alpha, \quad -\sqrt{2}r < s < \sqrt{2}r. \tag{4.1}$$

To see this let $I$ be a closed subinterval of $(-r\sqrt{2}, r\sqrt{2})$ and define the set $F \subset \{(t, s): 1 \leqq t \leqq 2, |s| < r\sqrt{t}\}$ by $(t, s) \in F$ if $g(t, s) \geqq g(1, y)$, where

$$g(a, b) = P_{a,b}(W_2 \in I \text{ and } \tau_r > 2).$$

Then $F$ is a closed set containing a curve joining $(1, y)$ and the midpoint of $I$. Let $v$ be the first time $(t, W_t) \in F$. Using the Strong Markov Property, we get $g(1, 0) \geqq g(1, y)\, P_{1,0}(v < \tau_r \wedge 2)$. For $y > 0$ we have $P_{1,0}(v < \tau_r \wedge 2) \geqq P_{1,0}(\tau_r < 2, W_{\tau_r} > 0)$ with a similar formula for $y < 0$, so $g(1, 0) \geqq \alpha g(1, y)$, implying (4.1).

Similarly, we can prove that for each $y\in(-r,r)$ there is a $K(r,y)=K>0$ such that

$$f_{1,y}(s)/f_{1,0}(s)\geqq K,\ -\sqrt{2}r<s<\sqrt{2}r. \tag{4.2}$$

Shepp shows in [8] that if $z$ is as in Sect. 1 and $r>z$, there exists $\gamma(r)=\gamma\in(0,\frac{1}{2})$ such that $E_{1,0}\tau_r^\gamma=\infty$, and so, using (4.2) and conditioning on $W_{\tau_r\wedge 2}$, we have $E_{1,y}\tau_r^\gamma=\infty$ for each $y\in(-r,r)$, implying

$$\overline{\lim_{\lambda\to\infty}}\ P(\tau_r>\lambda)\,\lambda^p=\infty \qquad \text{for each } p>\gamma. \tag{4.3}$$

Now let $X_t$ and $Y_t$ be independent Wiener processes. Put $\theta_r=\tau_r(X)\wedge\tau_r(Y)$. Then $P(\theta_r>\lambda)=P(\tau_r>\lambda)^2$, so, by (4.3), $\overline{\lim}_{\lambda\to\infty} P(\theta_r>\lambda)\,\lambda^{2p}=\infty$ if $p>\gamma$. In particular, there is an $\alpha=\alpha(r)<1$ such that $E\theta_r^\alpha=\infty$. Now, methods similar to those employed in Sect. 2 show that, for $r>z$, $P(D_r^X\cap D_r^Y\neq\emptyset)=1$. Note the set corresponding to $A_c$ is

$$\{\exists t\in[0,1]\colon |X_{t+h}-X_t|\vee|Y_{t+h}-Y_t|<r\sqrt{h}\ \forall h\in(0,1])\},$$

and that

$$\theta_r=\inf\{t\geqq 1\colon |X_{t+h}-X_t|\vee|Y_{t+h}-Y_t|\geqq r\sqrt{h}\}.$$

The sets $A_r$ and $D_r$ are defined in Sect. 1.

Shepp also proves that, if $s<z$, there exists a $\delta=\delta(s)>\frac{1}{2}$ such that $E_{1,0}\tau_s^\delta<\infty$. Conditioning on $X_{\tau_s\wedge 2}$ and using (4.1), this gives

$$\lim_{\lambda\to\infty}\ \sup_{y\in(-s,s)} P_{1,y}(\tau_s>\lambda)\,\lambda^\delta<\infty. \tag{4.4}$$

Now fix $r\in(0,z)$ and let $s=(r+z)/2$. Put

$$\Gamma=\Gamma_{k,n}=\{\exists t\in[(k/n,(k+1)/n]\colon |X_{t+h}-X_t|\vee|Y_{t+h}-Y_t|<r\sqrt{h}\ \forall h\in(0,1]\}.$$

Let $M$ be the smallest integer such $(s-r)M\geqq r$. Define the events $C_{j,k,n}=C_j$, $-M\leqq j\leqq M$, and $G_{i,k,n}=G_i$, $-M\leqq i\leqq M$, by

$$C_j=\{(t,X_t)\in\{(t,x)\colon |x-\alpha_j|\leqq s\sqrt{t-(k/n)}\},(k+1)/n\leqq t\leqq(k/n)+1\},$$

where $\alpha_j=X_{(k+1)/n}+(s-r)j/\sqrt{n}$, and

$$G_i=\{(t,Y_t)\in\{(t,x)\colon |x-\beta_i|\leqq s\sqrt{t-(k/n)}\},(k+1)/n\leqq t\leqq(k/n)+1\},$$

where $\beta_i=Y_{(k+1)/n}+(s-r)j/\sqrt{n}$.

Conditioning on $X_{(k+1)/n}$, and using Brownian scaling, we see both $P(C_j)$ and $P(G_i)$ are maximized by $\sup_{y\in(-r,r)} P_{1,y}(\tau_s>n)$ so that $P(C_j\cap G_i)=O(n^{-2\delta})=o(n^{-1})$ by (4.4). A geometrical argument gives $\Gamma\subset\bigcup_{i,j} C_j\cap G_i$, so that $P(\Gamma)=o(n^{-1})$, yielding

$$P\left(\bigcup_{k=0}^{n-1} \Gamma_{k,n}\right) = o(1),$$

which implies

$$P(D_r^X \cap D_r^Y) = 0.$$

These arguments easily generalize to $n$ independent Wiener processes, with the aid of the results in [8]. Let $z_n$ be the smallest positive zero of $M(-1/n, 1/2, x^2/2)$, where $M$ is the confluent hypergeometric function. Then we have

**Theorem 4.1.** $\bigcap_{i=1}^{n} D_r^{X_i}$ *is a.s. empty if* $r < z_n$ *and not empty if* $r > z_n$.

A proof of (1.4) can be made which is very similar to that of Theorem 4.1. Here it is convenient to work with a Brownian motion $Z_t$, $t \in (-\infty, \infty)$. We note that $S_t = Z_{(k+1)/n+t} - Z_{(k+1)/n}$, $t \geqq 0$, and $R_t = Z_{k/n-t} - Z_{k/n}$, $t \geqq 0$, are independent Wiener processes. Furthermore

$$\left\{\exists t \in \left[\frac{k}{n}, \frac{k+1}{n}\right] : |Z_{t+h} - Z_t| < r\sqrt{|h|} \ \forall h \in (0,1] \cup [-1,0)\right\}$$

can be shown to be contained in a set defined in terms of $S_t$ and $R_t$ in a manner similar to the way $\bigcup C_i \cap G_j$ was defined in terms of $X_t$ and $Y_t$ earlier, and thereby shown to have probability equal to $o(1/n)$ if $r < z$, from which we get

$$P(\exists t \in [0,1] : |Z_{t+h} - Z_t| < r\sqrt{|h|} \ \forall h \in (0,1] \cup [-1,0)) = 0,$$

which is equivalent to (1.4).

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**Added in Proof.** Priscilla Greenwood and Edwin Perkins have independently and differently proved (1.3) in a paper in the May 1983 Ann. Prob. See Perkin's paper in this issue in regard to the question after (1.4). Perkins and the author have settled (yes) the question after (1.6) recently.