Z. Wahrscheinlichkeitstheorie verw. Gebiete
64, 341–357 (1983)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

Functions of Discrete Probability Measures: Rates of Convergence in the Renewal Theorem

Rudolf Grübel

Universität Essen-GHS, Fachbereich 6 (Mathematik), Universitätsstr. 3, D-4300 Essen, Federal Republic of Germany

Summary. Let ν and τ be finite measures on the set of integers such that the Fourier transform of ν is an analytic function of the τ transform. The central result shows how ν may be approximated by linear combinations of convolution powers of τ. Applications are given to renewal theory, infinitely divisible measures and age-dependent branching processes.

1. Introduction

Let ν and τ be finite complex-valued measures on $\mathbb{Z}$, the set of integers. We regard τ as known and we assume that the Fourier transform of ν is a known function Ψ of the transform of τ.

This situation arises in several different parts of probability theory, for example in renewal theory, in connection with infinite divisibility and random sums, and in many applied models.

We briefly sketch the situation in the first example, the details are given in Sect. 4 below: Let $(X_i)_{i \in \mathbb{N}}$ be an i.i.d. sequence of random variables on some probability space $(\Omega, \mathfrak{A}, P)$ with finite first moment μ_1 and $P(X_1 \in \mathbb{N})=1$. Regard these variables as successive life-times in some self-renewing aggregate, and let u_n be the probability of a renewal at time n. If we take τ as the stationary distribution of the renewal process and define ν as the (signed) measure with mass $u_n - u_{n-1}$ at n then under an additional aperiodicity assumption the above relation holds with $\Psi(z)=(\mu_1 z)^{-1}$.

Even if Ψ is as simple as in this example it is normally not possible to obtain any (useful) explicit expression of the "output" measure ν except for a few special "input" measures τ.

Confining oneself to asymptotic results certain analytic methods apply very well, the use of Abelian and Tauberian theorems for example suggests itself. Another method which has proved to be fruitful in this context makes use of the theory of commutative Banach algebras: τ may be thought of as an element of some convolution algebra $\mathfrak{H}_1$ of summable sequences characterized by a certain asymptotic behaviour. Then ν has the same property if $\nu \in \mathfrak{H}_1$, that is if Ψ "operates in $\mathfrak{H}_1$" – this leads to the Gelfand theory of commutative Banach algebras.

The literature on this subject is extensive, important articles are, among others, those of Borovkov [2], Chover, Ney and Wainger [4], Essén [11] and Rogozin [17], [18]. The results obtained may roughly be distinguished by the type of $\mathfrak{H}_1$ involved. The first type imposes one-sided conditions on the asymptotic behaviour of the elements of $\mathfrak{H}_1$ such as O- and o-conditions or summability conditions. The second deals with spaces of sequences which are asymptotically equal to a given sequence, normally the sequence of atoms or tails of the input measure. Here the results only apply to measures which behave smoothly in a certain sense, which leads for example to the notion of subexponential probability distributions.

We present a new variant of this Banach algebra method. Its starting point is the simple idea that a zero sequence may be called smooth if its differences decrease faster than the sequence itself, a concept which may be iterated in a natural way. Our main result shows that such smoothness assumptions lead to expansions of the output in terms of input convolution powers.

This result is given in Sect. 2 below, Sect. 3 contains the proof. It is applied to renewal theory in Sect. 4 where we show that it yields a simpler and unified derivation of old results on the rate of convergence in the discrete renewal theorem, but also new ones as for example: If $P(X_1=n+1)-P(X_1=n)=o(n^{-\gamma})$ for some $\gamma>3$, then

$$u_n-\frac{1}{\mu_1}=\frac{1}{\mu_1}(6P(Y_1>n)-4P(Y_1+Y_2>n)+P(Y_1+Y_2+Y_3>n))+o(n^{-\gamma}),$$

where Y_1, Y_2, Y_3 are independent and distributed according to the stationary distribution of the renewal process. An estimate of u_n-1/μ_1 using the tail of a random sum of such Y_i's has recently been given by Ney [16].

Some other applications and complementary remarks constitute the last section.

All measures in this article are assumed to be concentrated on $\mathbb{Z}$, the generalization to other lattices $h\mathbb{Z}$, $h>0$, is obvious. A corresponding treatment of absolutely continous measures is in preparation.

2. The Main Result

Let $\mathfrak{H}$ denote the set of all absolutely summable sequences of complex numbers,

$$\mathfrak{H}=\{a\in\mathbb{C}^{\mathbb{Z}}: \sum_{n\in\mathbb{Z}}|a_n|<\infty\},$$

which is identified in the obvious manner with the set of all $\mathbb{C}$-valued finite measures on $\mathbb{Z}$, for example we write ν_n for $\nu(\{n\})$ and $\nu([n,\infty))$ for $\sum_{m=n}^{\infty}\nu_m$.

For any $a \in \mathfrak{H}$ denote the corresponding Fourier transform by $\hat{a}$,

$$\hat{a}(\theta) = \sum_{n \in \mathbb{Z}} a_n e^{in\theta} \quad \text{for all } \theta \in \mathbb{R}.$$

For any two elements a and b of $\mathfrak{H}$ we define their convolution $a * b = ((a * b)_n)_{n \in \mathbb{Z}}$ by

$$(a * b)_n = \sum_{m \in \mathbb{Z}} a_m b_{n-m} \quad \text{for all } n \in \mathbb{Z}.$$

Then n-fold convolution of $a \in \mathfrak{H}$ with itself will be denoted by a^{*n}, $n \in \mathbb{N}$.

The operator Δ: $\mathbb{C}^{\mathbb{Z}} \to \mathbb{C}^{\mathbb{Z}}$ is defined by

$$(\Delta a)_n = a_n - a_{n-1} \quad \text{for all } n \in \mathbb{Z},$$

as usual the corresponding powers Δ^k for $k \in \mathbb{N}$ are defined by induction, Δ^0 denotes the identity. On $\mathfrak{H}$ define Σ by

$$(\Sigma a)_n = \begin{cases} \displaystyle\sum_{m=n+1}^{\infty} a_m, & \text{if } n \geqq 0, \\ \displaystyle -\sum_{m=-\infty}^{n} a_m, & \text{if } n < 0. \end{cases}$$

So $(\Sigma \nu)_n$ equals the right tail $\nu([n+1, \infty))$ of the measure ν if n is nonnegative.

In the terminology of [4] our result has a local and a global part ((i) and (ii) respectively).

Theorem. *Let ν and τ be finite $\mathbb{C}$-valued measures on $\mathbb{Z}$ with $\Sigma\tau \in \mathfrak{H}$; $d = \tau(\mathbb{Z})$. Let $G \subset \mathbb{C}$ be open with $\hat{\tau}(\mathbb{R}) \subset G$, let Ψ: $G \to \mathbb{C}$ be analytic and assume*

$$\hat{\nu}(\theta) = \Psi(\hat{\tau}(\theta)) \quad \textit{for all } \theta \in \mathbb{R}.$$

(i) *Suppose that for some $k \in \mathbb{N}$, $\gamma > 1$,*

$$(\Delta^k \tau)_n = O(n^{-\gamma-k}) \quad \textit{as } n \to \infty, \qquad \sum_{n=1}^{\infty} n^k |(\Delta^{k-1} \tau)_n| < \infty.$$

Then

$$\nu(\{n\}) = \sum_{j=1}^{k} c(k, j)\, \tau^{*j}(\{n\}) + O(n^{-\gamma-k}) \quad \textit{as } n \to \infty,$$

where

$$c(k, j) = \frac{1}{j!} \sum_{l=0}^{k-j} \frac{\Psi^{(j+l)}(d)}{l!} (-d)^l, \quad 1 \leqq j \leqq k.$$

(ii) *Suppose that for some $k \in \mathbb{N}_0$, $\gamma > 1$,*

$$(\Delta^k \tau)_n = O(n^{-\gamma-k}) \quad \textit{as } n \to \infty,$$

and

$$\sum_{n=1}^{\infty} n |\tau(\{n\})| < \infty, \quad \textit{if } k = 0, \qquad \sum_{n=1}^{\infty} n^k |(\Delta^{k-1} \tau)_n| < \infty, \quad \textit{if } k > 0.$$

Then

$$\nu([n,\infty)) = \sum_{j=1}^{k+1} d(k,j)\,\tau^{*j}([n,\infty)) + O(n^{-\gamma-k}) \quad \text{as } n\to\infty,$$

where

$$d(k,j) = \frac{1}{j!}\sum_{l=0}^{k+1-j} \frac{\Psi^{(j+l)}(d)}{l!}(-d)^l, \quad 1\leqq j\leqq k+1.$$

The same statements hold if O is replaced by o throughout.

Since $\hat{\tau}(\mathbb{R})$ is a connected subset of $\mathbb{C}$ we may assume without loss of generality that G is connected, but note that G may not be simply connected.

The proof of the theorem is given in the next section.

Obviously the transition $\tau_n \to \tau_{-n}$ yields a corresponding result on the asymptotic behaviour of $\nu(\{n\})$ and the left tail $\nu((-\infty, n])$ as $n\to -\infty$.

3. Proof of the Main Result

For any $k\in\mathbb{N}_0$, $\gamma>1$, we define the spaces $\mathfrak{D}(k,\gamma)$ and $\mathfrak{D}_0(k,\gamma)$ by

$$\mathfrak{D}(k,\gamma) = \{a\in\mathfrak{H}\colon (\Delta^k a)_n = O(n^{-\gamma-k})\}$$

and

$$\mathfrak{D}_0(k,\gamma) = \{a\in\mathfrak{H}\colon (\Delta^k a)_n = o(n^{-\gamma-k})\},$$

for all $a\in\mathfrak{D}(k,\gamma)$ we set

$$\|a\|_{k,\gamma} = \sum_{n\in\mathbb{Z}} |a_n| + \sup_{n\in\mathbb{N}} n^{\gamma+k}|(\Delta^k a)_n|.$$

Our plan is to prove that analytic functions operate in these spaces. Once this is done the proof of the theorem will be easy.

Note that $\mathfrak{H}$, endowed with the norm $\|.\|$,

$$\|a\| = \sum_{n\in\mathbb{Z}} |a_n| \quad \text{for all } a\in\mathfrak{H},$$

is a Banach algebra with respect to convolution.

Proposition 1. (i) *$\mathfrak{D}(k,\gamma)$ and $\mathfrak{D}_0(k,\gamma)$ are linear spaces and closed with respect to convolution.*

(ii) *$\|.\|_{k,\gamma}$ is a norm on $\mathfrak{D}(k,\gamma)$ (and therefore also on $\mathfrak{D}_0(k,\gamma)$). Endowed with this norm $\mathfrak{D}(k,\gamma)$ and $\mathfrak{D}_0(k,\gamma)$ are complete.*

(iii) *There exists a constant $C(k,\gamma)$ such that*

$$\|a*b\|_{k,\gamma} \leqq C(k,\gamma)\,\|a\|_{k,\gamma}\,\|b\|_{k,\gamma} \quad \textit{for all } a,b\in\mathfrak{D}(k,\gamma).$$

Proof. For all $a\in\mathfrak{D}(k,\gamma)$, $n\in\mathbb{N}$, we define

$$C_0(a,n) = \sup_{m\geqq n} m^{\gamma+k}|(\Delta^k a)_m|,$$

$$m_n = \begin{cases} n/2, & \text{if } n \text{ is even,} \\ (n-1)/2, & \text{otherwise.}\end{cases}$$

Now let $a, b \in \mathfrak{D}(k, \gamma)$. Summing partially k times we obtain

$$(\Delta^k(a * b))_n = A(n) + B(n) + \sum_{j=1}^{k} D(n,j)\, E(n,j)$$

with

$$A(n) = \sum_{m=-\infty}^{m_n} a_m (\Delta^k b)_{n-m}, \qquad B(n) := \sum_{m=m_n+1}^{\infty} (\Delta^k a)_m b_{n-m},$$

$$D(n,j) = (\Delta^{k-j} a)_{m_n}, \qquad E(n,j) := (\Delta^{j-1} b)_{n-m_n-1}.$$

It is easy to see that for any $m \geqq n$ with suitable constants $C_i(k, \gamma)$, $1 \leqq i \leqq 4$,

$$m^{\gamma+k} |A(m)| \leqq C_1(k,\gamma)\, C_0(b, n-m_n) \sum_{l \in \mathbb{Z}} |a_l|,$$

$$m^{\gamma+k} |B(m)| \leqq C_2(k,\gamma)\, C_0(a, m_n+1) \sum_{l \in \mathbb{Z}} |b_l|$$

and

$$m^{\gamma+k-j} |D(m,j)| \leqq C_3(k,\gamma)\, C_0(a, m_n+1), \qquad 1 \leqq j \leqq k,$$
$$m^{\gamma+j-1} |E(m,j)| \leqq C_4(k,\gamma)\, C_0(b, n-m_n), \qquad 1 \leqq j \leqq k.$$

Using these estimates and the assumption $\gamma > 1$ it is easy to prove that $\mathfrak{D}(k, \gamma)$ and $\mathfrak{D}_0(k, \gamma)$ are closed with respect to convolution and that a constant exists which satisfies (iii).

In order to prove completeness of $(\mathfrak{D}(k,\gamma), \|\cdot\|_{k,\gamma})$ let $(a^{(j)})_{j \in \mathbb{N}}$ be a Cauchy sequence in this space. It is also a Cauchy sequence in $(\mathfrak{H}, \|\cdot\|)$ then, so there exists an $a \in \mathfrak{H}$ with $\lim_{j \to \infty} \|a - a^{(j)}\| = 0$. Now we choose a subsequence $(a^{(j_i)})_{i \in \mathbb{N}}$ such that

$$\|a^{(j_i)} - a^{(j_{i+1})}\|_{k,\gamma} \leqq 2^{-i} \qquad \text{for all } i \in \mathbb{N}.$$

Because of $\|\Delta a\| \leqq 2\|a\|$ Δ and with it Δ^k are continous on $(\mathfrak{H}, \|\cdot\|)$ which implies

$$\lim_{i \to \infty} \|\Delta^k(a^{(j_i)} - a)\| = 0,$$

especially

$$|(\Delta^k(a^{(j_i)} - a))_n| \leqq \sum_{l=i}^{\infty} |(\Delta^k(a^{(j_l)} - a^{(j_{l+1})})_n|.$$

This gives

$$\sup_{n \in \mathbb{N}} n^{\gamma+k} |(\Delta^k(a^{(j_i)} - a))_n| \leqq \sum_{l=i}^{\infty} \|a^{(j_l)} - a^{(j_{l+1})}\|_{k,\gamma} \leqq 2^{-i+1},$$

which yields $a \in \mathfrak{D}(k, \gamma)$ and $\lim_{i \to \infty} \|a^{(j_i)} - a\|_{k,\gamma} = 0$. Since $(a^{(j_i)})_{j \in \mathbb{N}}$ is a Cauchy sequence this implies

$$\lim_{j \to \infty} \|a^{(j)} - a\|_{k,\gamma} = 0.$$

Finally the mapping $\Phi\colon \mathfrak{D}(k,\gamma) \to \mathbb{R}$ defined by

$$\Phi(a) = \limsup_{n \to \infty} n^{\gamma+k} |(\Delta^k a)_n| \qquad \text{for all } a \in \mathfrak{D}(k, \gamma)$$

is continous with respect to $\|\cdot\|_{k,\gamma}$ and we have $\mathfrak{D}_0(k,\gamma)=\Phi^{-1}(\{0\})$, so $(\mathfrak{D}_0(k,\gamma), \|\cdot\|_{k,\gamma})$ is also complete. All other assertions of the proposition are immediate. □

Let $\mathfrak{H}_0$ denote the space of sequences with only a finite number of non-vanishing terms,

$$\mathfrak{H}_0=\{a\in\mathbb{C}^{\mathbb{Z}}:\ \#\{n\in\mathbb{Z}: a_n\neq 0\}<\infty\}.$$

Evidently we have $\mathfrak{H}_0\subset\mathfrak{D}(k,\gamma)$ for all $k\in\mathbb{N}_0$, $\gamma>1$. The following lemma shows that in a certain sense $\mathfrak{H}_0$ is weakly dense in $(\mathfrak{D}(k,\gamma), \|\cdot\|_{k,\gamma})$.

Lemma 1. *If* $k\in\mathbb{N}_0$, $\gamma>1$ *and* $a\in\mathfrak{D}(k,\gamma)$, *then there exists a sequence* $(b_l)_{l\in\mathbb{N}}\subset\mathfrak{H}_0$ *such that*

$$\lim_{l\to\infty}\|(a-b_l)^{*2}\|_{k,\gamma}=0,\qquad \lim_{l\to\infty}\|a-b_l\|=0. \tag{1}$$

Proof. For all $l>k$ we define $b_l=(b_{ln})_{n\in\mathbb{Z}}$ by

$$b_{ln}=\begin{cases} a_n-\sum_{j=0}^{k-1}(-1)^j\binom{l-1-n}{j}(\Delta^j a)_{l-1}, & \text{if } 0\leqq n\leqq l-1,\\ a_n, & \text{if } -(l-1)\leqq n<0,\\ 0, & \text{if } |n|\geqq l\end{cases}\qquad \Big(\sum_{\emptyset}=0\Big).$$

Then we have for every $i\in\{0,\dots,k-1\}$

$$(\Delta^i(a-b_l))_n=\begin{cases}\sum_{j=0}^{k-1-i}(-1)^j\binom{l-1-n}{j}(\Delta^{j+i}a)_{l-1}, & \text{if } i\leqq n\leqq l-1,\\ (\Delta^i a)_n, & \text{if } n\geqq l-1,\\ 0, & \text{if } n=-1,\end{cases} \tag{2}$$

and

$$(\Delta^k(a-b_l))_n=\begin{cases}0, & \text{if } k\leqq n\leqq l-1,\\ (\Delta^k a)_n, & \text{if } n\geqq l.\end{cases} \tag{3}$$

Firstly we show

$$\lim_{l\to\infty}\|a-b_l\|=0. \tag{4}$$

We have

$$\sum_{n\in\mathbb{Z}}|(a-b_l)_n|\leqq\sum_{|n|\geqq l}|a_n|+\sum_{j=0}^{k-1}\sum_{n=0}^{l-1}\binom{l-1-n}{j}|(\Delta^j a)_{l-1}|.$$

Since $\mathfrak{D}(k,\gamma)\subset\mathfrak{H}$ it follows that $\lim_{l\to\infty}\sum_{|n|\geqq l}|a_n|=0$. Because of

$$\sum_{n=0}^{l-1}\binom{l-1-n}{j}|(\Delta^j a)_{l-1}|\leqq l^{j+1}|(\Delta^j a)_{l-1}|$$

(4) will follow if we can show

$$(\Delta^j a)_n=o(n^{-j-1})\qquad\text{for all } j\in\{0,\dots,k-1\}, \tag{5}$$

but this is an immediate consequence of $(\Delta^k a)_n = O(n^{-\gamma-k})$ $(\gamma > 1)$ and

$$|(\Delta^j a)_n| \leqq \sum_{m=n}^{\infty} |(\Delta^{j+1} a)_m| \quad \text{for all } n \in \mathbb{N}.$$

Since Δ^k is continuous on $(\mathfrak{H}, \|.\|)$ (4) implies

$$\lim_{l\to\infty} \sum_{n\in\mathbb{Z}} |(\Delta^k((a-b_l)^{*2}))_n| = 0,$$

hence

$$\lim_{l\to\infty} |(\Delta^k((a-b_l)^{*2}))_n| = 0 \quad \text{for all } n \in \mathbb{Z},$$

so it remains to prove

$$\lim_{l\to\infty} \sup_{n\geqq n_0} n^{\gamma+k} |(\Delta^k((a-b_l)^{*2}))_n| = 0$$

for some $n_0 \in \mathbb{N}$ independent of l.

We have

$$(\Delta^k((a-b_l)^{*2}))_n = \left(\sum_{m=-\infty}^{-1} + \sum_{m=0}^{n} + \sum_{m=n+1}^{\infty}\right) (\Delta^k(a-b_l))_m (a-b_l)_{n-m}. \tag{6}$$

Consider the last term. Using (3) it follows that

$$\sup_{n\geqq k} n^{\gamma+k} \left| \sum_{m=n+1}^{\infty} (\Delta^k(a-b_l))_m (a-b_l)_{n-m} \right|$$
$$\leqq (\sup_{n\in\mathbb{N}} n^{\gamma+k} |(\Delta^k a)_n|)(\sum_{n\in\mathbb{Z}} |(a-b_l)_n|),$$

so $a \in \mathfrak{D}(k, \gamma)$ and (4) yield

$$\lim_{l\to\infty} \sup_{n\geqq k} n^{\gamma+k} \left| \sum_{m=n+1}^{\infty} (\Delta^k(a-b_l))_m (a-b_l)_{n-m} \right| = 0.$$

The corresponding statement holds for the first term too since

$$\sum_{m=-\infty}^{-1} (\Delta^k(a-b_l))_m (a-b_l)_{n-m} = \sum_{m=n+1}^{\infty} (\Delta^k(a-b_l))_m (a-b_l)_{n-m}$$

by (2) and partial summation.

The corresponding asymptotic behaviour of the middle term in (6) will follow from

$$\lim_{l\to\infty} \sup_{2k+1\leqq n<2l} (n+1)^{\gamma+k} \left| \sum_{m=0}^{n} (\Delta^k(a-b_l))_m (a-b_l)_{n-m} \right| = 0 \tag{7}$$

and

$$\lim_{l\to\infty} \sup_{n\geqq 2l} (n+1)^{\gamma+k} \left| \sum_{m=0}^{n} (\Delta^k(a-b_l))_m (a-b_l)_{n-m} \right| = 0, \tag{8}$$

which we prove now.

Using (3) we obtain

$$\sum_{m=0}^{n} (\Delta^k(a-b_l))_m (a-b_l)_{n-m} = \left(\sum_{m=0}^{k-1} + \sum_{m=l}^{n}\right) (\Delta^k(a-b_l))_m (a-b_l)_{n-m}.$$

The first sum may be transformed to

$$\sum_{m=0}^{k-1} (a-b_l)_m (\Delta^k(a-b_l))_{n-m} + \sum_{j=1}^{k} (\Delta^{j-1}(a-b_l))_{k-1} (\Delta^{k-j}(a-b_l))_{n-k}.$$

Considering separately the cases $n \geqq l$ and $n<l$ the first term is easy to handle. Using (2) and the argument which led to (5) we get

$$(\Delta^{j-1}(a-b_l))_{k-1} = O(l^{-j+1-\gamma})$$

and similarly

$$\sup_{2k+1 \leqq n < l+k} |(\Delta^{k-j}(a-b_l))_{n-k}| = O(l^{j-k-\gamma}).$$

If $n \geqq l+k$ we have $(\Delta^{k-j}(a-b_l))_{n-k} = (\Delta^{k-j}a)_{n-k}$ and it follows that

$$\sup_{l+k \leqq n < 2l} |(\Delta^{k-j}(a-b_l))_{n-k}| = O(l^{j-k-\gamma}).$$

Inserting these estimates we get

$$\lim_{l\to\infty} \sup_{2k+1 \leqq n < 2l} (n+1)^{\gamma+k} \left| \sum_{m=0}^{k-1} (\Delta^k(a-b_l))_m (a-b_l)_{n-m} \right| = 0.$$

The second sum is easily disposed of since we may replace $\Delta^k(a-b_l)$ by $\Delta^k a$ and then use (4).

Thus (7) is proved.

Turning to Equation (8) we define $m_n = n/2$ if n is even and $m_n = (n-1)/2$ if n is odd. We then have

$$\begin{aligned}
&\sum_{m=0}^{n} (\Delta^k(a-b_l))_m (a-b_l)_{n-m} \\
&= \sum_{m=0}^{m_n} (a-b_l)_m (\Delta^k(a-b_l))_{n-m} + \sum_{m=m_n+1}^{n} (\Delta^k(a-b_l))_m (a-b_l)_{n-m} \\
&\quad + \sum_{j=1}^{k} (\Delta^{j-1}(a-b_l))_{n-m_n-1} (\Delta^{k-j}(a-b_l))_{m_n} \\
&= \sum_{m=0}^{m_n} (a-b_l)_m (\Delta^k a)_{n-m} + \sum_{m=m_n+1}^{n} (\Delta^k a)_m (a-b_l)_{n-m} \\
&\quad + \sum_{j=1}^{k} (\Delta^{j-1} a)_{n-m_n-1} (\Delta^{k-j} a)_{m_n}.
\end{aligned}$$

On using $a \in \mathfrak{D}(k, \gamma)$ and (4) for the first two terms and the argument which led to (5) for the last it is easy to see that (8) holds. □

The next lemma is a simple application of the dominated convergence theorem, its proof is therefore omitted.

Lemma 2. *Suppose* $\{a_\eta: 0 \leqq \eta \leqq 2\pi\}$ *is a subset of* $\mathfrak{D}(k,\gamma)(\mathfrak{D}_0(k,\gamma))$ *with*

$$\sup_{0 \leqq \eta \leqq 2\pi} \|a_\eta\|_{k,\gamma} < \infty$$

and such that $\eta \to \hat{a}_\eta(\theta)$ *is absolutely integrable over* $[0, 2\pi]$ *for every* $\theta \in \mathbb{R}$; *define* $f: \mathbb{R} \to \mathbb{C}$ *by*

$$f(\theta) = \int_0^{2\pi} \hat{a}_\eta(\theta)\, d\eta \qquad \text{for all } \theta \in \mathbb{R}.$$

Then $f = \hat{a}$ *for some* $a \in \mathfrak{D}(k,\gamma)(\mathfrak{D}_0(k,\gamma))$.

Proposition 2. *Let* $a \in \mathfrak{D}(k,\gamma)$ *and* $G \subset \mathbb{C}$ *be open with* $\hat{a}(\mathbb{R}) \subset G$. *If* $\Psi: G \to \mathbb{C}$ *is analytic, then there exists a* $b \in \mathfrak{D}(k,\gamma)$ *such that*

$$\hat{b}(\theta) = \Psi(\hat{a}(\theta)) \qquad \text{for all } \theta \in \mathbb{R}.$$

This b *is unique in* $\mathfrak{H}$. *The same holds for* $\mathfrak{D}_0(k,\gamma)$ *in place of* $\mathfrak{D}(k,\gamma)$.

One of the possible proofs of this proposition follows a standard procedure from Banach algebra theory: It firstly characterizes the maximal ideals in $\mathfrak{D}(k,\gamma)$ as being of the form

$$I(\theta_0) = \{a \in \mathfrak{D}(k,\gamma): \hat{a}(\theta_0) = 0\}$$

for some $\theta_0 \in \mathbb{R}$. This implies the existence of a convolution inverse in $\mathfrak{D}(k,\gamma)$ to $a - z d_0$ for every $z \notin \hat{a}(\mathbb{R})$. Integrating the $\mathfrak{D}(k,\gamma)$-valued function $z \to \frac{1}{2\pi i} \Psi(z)(a - z d_0)^{*(-1)}$ over some suitable contour Γ one then obtains the required b.

The choice of Γ seems to be a crucial point in this proof and it has not always been handled correctly in the literature. It is simple if we assume additionally that G is simply connected (see e.g. [3], p. 254), but in many probabilistic applications this is not satisfied. Nevertheless the proposition may be proved in full generality with the method outlined above, but one has to spend some care with the construction of Γ (see e.g. [13], p. 166 and [19], p. 241).

No such difficulties arise in the proof we will give now. It completely dispenses with results from the general theory of Banach algebras, only elementary calculations are used in it.

Proof. Since $\hat{a}(\mathbb{R})$ is compact and contained in G there exists a $\rho > 0$ such that $U_{4\rho}(\hat{a}(\theta)) \subset G$ for all $\theta \in \mathbb{R}$ [1]. Using Lemma 1 we get some $s \in \mathfrak{H}_0$ such that we have with $r := a - s$ and a suitable constant C

---

[1] $U_\rho(z) = \{z' \in \mathbb{C}: |z - z'| < \rho\}$, $\overline{U_\rho(z)} = \{z' \in \mathbb{C}: |z - z'| \leqq \rho\}$ $(z \in \mathbb{C}, \rho > 0)$.

$$\|r\| < \rho, \tag{9}$$

$$\|r^{*j}\|_{k,\gamma} \leqq C \rho^j \qquad \text{for all } j \in \mathbb{N}_0. \tag{10}$$

Because of (9) we can use the Cauchy formula to obtain

$$\Psi(\hat{a}(\theta)) = \frac{1}{2\pi} \int_0^{2\pi} \hat{h}_\eta(\theta)\, \hat{c}_\eta(\theta)\, d\eta \qquad \text{for all } \theta \in \mathbb{R}$$

with

$$\hat{h}_\eta(\theta) = \Psi(\hat{s}(\theta) + 2\rho e^{i\eta}), \qquad \hat{c}_\eta(\theta) = \frac{2\rho e^{i\eta}}{2\rho e^{i\eta} - \hat{r}(\theta)}, \ 0 \leqq \eta \leqq 2\pi, \ \theta \in \mathbb{R}.$$

From this we get the assertion with the help of Lemma 2 if we can show

$$\sup_{0 \leqq \eta \leqq 2\pi} \|c_\eta\|_{k,\gamma} < \infty, \tag{11}$$

$$\sup_{0 \leqq \eta \leqq 2\pi} \|h_\eta\|_{k,\gamma} < \infty, \tag{12}$$

where c_η and h_η denote the sequences of coefficients corresponding to $\hat{c}_\eta$ and $\hat{h}_\eta$ respectively.

The first inequality (11) is immediate from

$$c_\eta = \sum_{j=0}^{\infty} (2\rho)^{-j} e^{-ij\eta} r^{*j}$$

and (10).

Because of $s \in \mathfrak{H}_0$ there exists an $\varepsilon > 0$ and for every $\eta \in [0, 2\pi]$ a function Φ_η analytic on $S_\varepsilon := U_{1+\varepsilon}(0) - \overline{U_{1-\varepsilon}(0)}$ such that

$$\Phi_\eta(e^{i\theta}) = \hat{h}_\eta(\theta) \qquad \text{for all } \eta \in [0, 2\pi], \ \theta \in \mathbb{R},$$

and

$$\sup_{0 \leqq \eta \leqq 2\pi,\, z \in S_\varepsilon} |\Phi_\eta(z)| < \infty.$$

From this we easily obtain (12) with the help of Cauchy's inequality for Laurent coefficients.

The same arguments yield the corresponding result for $\mathfrak{D}_0(k, \gamma)$. □

The proof of the following lemma does not require any new arguments and is therefore omitted.

Lemma 3. *For all $i, j, k \in \mathbb{N}_0$, $\gamma \geqq 0$, with $j \leqq k$, $j \leqq i \leqq 2j$, the sets*

$$\mathfrak{F}(j, k;\, i, \gamma) = \left\{ a \in \mathfrak{H} \colon \sum_{n=1}^{\infty} n^k |(\Delta^j a)_n| < \infty,\ (\Delta^i a)_n = O(n^{-\gamma}) \right\}$$

and

$$\mathfrak{F}_0(j, k;\, i, \gamma) = \{ a \in \mathfrak{F}(j, k;\, i, \gamma) \colon (\Delta^i a)_n = o(n^{-\gamma}) \}$$

are closed with respect to convolution.

Proof of the Theorem. (i) There exists a function Φ analytic on G such that

$$\Psi(z)=\sum_{j=0}^{2k}\frac{\Psi^{(j)}(d)}{j!}(z-d)^j+\Phi(z)(z-d)^{2k+1}$$

for all $z\in G$. This implies

$$\hat{v}(\theta)=\sum_{j=0}^{2k}\frac{\Psi^{(j)}(d)}{j!}(\hat{\tau}(\theta)-d)^j+\Phi(\hat{\tau}(\theta))(\hat{\tau}(\theta)-d)^{2k+1}$$

for all $\theta\in\mathbb{R}$. Since

$$\Phi(\hat{\tau}(\theta))(\hat{\tau}(\theta)-d)^{2k+1}=(\Phi(\hat{\tau}(\theta))(e^{i\theta}-1)^k)\left(\left(\frac{\hat{\tau}(\theta)-d}{e^{i\theta}-1}\right)^{2k+1}(e^{i\theta}-1)^{k+1}\right)$$

we can apply Proposition 2 and Lemma 3 to obtain $O(n^{-\gamma-k})$ - behaviour for the coefficients of the last term. Similarly Lemma 3 yields this behaviour for the coefficients of $(\hat{\tau}(\theta)-d)^j$ if $j\in\{k+1,\ldots,2k\}$. The remaining terms may be rearranged,

$$\begin{aligned}\sum_{j=0}^{k}\frac{\Psi^{(j)}(d)}{j!}(\hat{\tau}(\theta)-d)^j&=\sum_{j=0}^{k}\frac{\Psi^{(j)}(d)}{j!}\sum_{l=0}^{j}\binom{j}{l}\hat{\tau}(\theta)^l(-d)^{j-l}\\&=\sum_{l=0}^{k}\hat{\tau}(\theta)^l\frac{1}{l!}\sum_{j=1}^{k}\frac{1}{(j-l)!}\Psi^{(j)}(d)(-d)^{j-l}\\&=\sum_{l=1}^{k}c(k,l)\,\hat{\tau}(\theta)^l+\text{constant},\end{aligned}$$

so the assertion follows by comparison of coefficients.

(ii) We have

$$\Psi(z)=\sum_{j=0}^{2k+1}\frac{\Psi^{(j)}(d)}{j!}(z-d)^j+\Phi(z)(z-d)^{2k+2}\qquad\text{for all } z\in G$$

with some function Φ analytic on G, so

$$\frac{\hat{v}(\theta)-\hat{v}(0)}{e^{i\theta}-1}=\sum_{j=1}^{2k+1}\frac{\Psi^{(j)}(d)(\hat{\tau}(\theta)-d)^j}{j!(e^{i\theta}-1)}+\Phi(\hat{\tau}(\theta))\frac{(\hat{\tau}(\theta)-d)^{2k+2}}{e^{i\theta}-1}.$$

As in the local case proved above we may use Lemma 3 and Proposition 2 to obtain $O(n^{-\gamma-k})$-behaviour for the coefficients of

$$\sum_{j=k+2}^{2k+1}\frac{\Psi^{(j)}(d)(\hat{\tau}(\theta)-d)^j}{j!(e^{i\theta}-1)}+\Phi(\hat{\tau}(\theta))\frac{(\hat{\tau}(\theta)-d)^{2k+2}}{e^{i\theta}-1},$$

rearranging the remaining terms gives

$$\begin{aligned}\sum_{j=1}^{k+1}\frac{\Psi^{(j)}(d)(\hat{\tau}(\theta)-d)^j}{j!(e^{i\theta}-1)}&=\sum_{j=1}^{k+1}\frac{\Psi^{(j)}(d)}{j!}\sum_{l=1}^{j}\binom{j}{l}\left(\frac{\hat{\tau}(\theta)^l-d^l}{e^{i\theta}-1}\right)(-d)^{j-l}\\&=\sum_{l=1}^{k+1}d(k,l)\frac{\hat{\tau}(\theta)^l-d^l}{e^{i\theta}-1},\end{aligned}$$

and the second assertion follows.

Using $\mathfrak{D}_0(k,\gamma)$ and $\mathfrak{F}_0(k,k;k+1,k+\gamma)$ ($\mathfrak{F}_0(1,1;1,\gamma)$ if $k=0$) instead of $\mathfrak{D}(k,\gamma)$ and $\mathfrak{F}(k,k;k+1,k+\gamma)$ ($\mathfrak{F}(1,1;1,\gamma)$ if $k=0$) we obtain the corresponding o-results. □

4. An Application to Renewal Theory

In this section we apply our theorem to obtain results on the rate of convergence in the discrete renewal theorem.

Let $(X_j)_{j\in\mathbb{N}}$ be an i.i.d. sequence of random variables on some probability space $(\Omega, \mathfrak{A}, P)$ with $P(X_1\in\mathbb{Z})=1$; $p_n:=P(X_1=n)$ for all $n\in\mathbb{Z}$.

Assume that the distribution p is aperiodic, i.e.

$$\text{g.c.d. } \{n\in\mathbb{Z}\colon p_n>0\}=1,$$

and that X_1 has a finite first moment $\mu_1>0$.

The corresponding renewal sequence $(u_n)_{n\in\mathbb{Z}}$ is defined by

$$u_n:=E\left(\#\left\{l\in\mathbb{N}_0\colon \sum_{j=1}^{l} X_j=n\right\}\right) \qquad \text{for all } n\in\mathbb{Z},$$

the discrete renewal theorem states

$$\lim_{n\to\infty} u_n=\frac{1}{\mu_1}, \qquad \lim_{n\to-\infty} u_n=0.$$

The corresponding rate of convergence has been investigated by many authors, among those who used the Banach algebra method we mention Borovkov [2], Essén [11], Rogozin [17] and Davies and Grübel [6]. Other approaches, ranging from Fourier techniques to coupling methods, have been applied by Stone and Wainger [20], Kalashnikov [14], Lindvall [15] and Ney [16]; see also the references given in these articles.

If we define the measures ν and τ by

$$\nu:=\Delta u, \qquad \tau:=\frac{1}{\mu_1}\Sigma p,$$

then the aperiodicity assumption yields

$$\hat{\tau}(\theta)\neq 0 \qquad \text{for all } \theta\in\mathbb{R},$$

and the renewal equation gives

$$\mu_1\,\hat{\tau}(\theta)\,\hat{\nu}(\theta)=1 \qquad \text{for all } \theta\in\mathbb{R}$$

(see [10] for the details in the one-sided case $P(X_1\in\mathbb{N})=1$). Thus our theorem is applicable with $G:=\mathbb{C}-\{0\}$, $\Psi(z):=(\mu_1 z)^{-1}$ for all $z\in G$, and we have $d=1$ and $\nu([n+1,\infty))=-u_n+1/\mu_1$.

If $k=0$ the global part of the theorem yields that $EX_1^2<\infty$, $P(X_1>n) = o(n^{-\gamma})(O(n^{-\gamma}))$ for some $\gamma \geqq 2$, implies

$$u_n = \frac{1}{\mu_1} + \frac{1}{\mu_1^2} (\Sigma p)([n+1, \infty)) + o(n^{-\gamma})(O(n^{-\gamma})). \tag{13}$$

This result has also been obtained by Rogozin [17].
If we take $k=1$ we obtain that $EX_1^2<\infty$, $p_n = o(n^{-\gamma})(O(n^{-\gamma}))$ for some $\gamma \geqq 2$, implies

$$u_n = \frac{1}{\mu_1} + \frac{3}{\mu_1^2} (\Sigma p)([n+1, \infty)) - \frac{1}{\mu_1^3} (\Sigma p)^{*2}([n+1, \infty)) + o(n^{-\gamma})(O(n^{-\gamma})). \tag{14}$$

This has also been obtained by Essén ([11], Theorem 3.1). Both authors admit a bigger class of reference sequences than $(n^{-\gamma})_{n \in \mathbb{N}}$, $\gamma \geqq 2$, but see Sect. 5.3 below.

We get new results if we take $k>1$. For example the case $k=2$ yields that $EX_1^2<\infty$, $p_n - p_{n-1} = o(n^{-\gamma})(O(n^{-\gamma}))$ for some $\gamma>3$ implies

$$\begin{aligned} u_n = {} & \frac{1}{\mu_1} + \frac{6}{\mu_1^2} (\Sigma p)([n+1, \infty)) - \frac{4}{\mu_1^3} (\Sigma p)^{*2}([n+1, \infty)) \\ & + \frac{1}{\mu_1^4} (\Sigma p)^{*3}([n+1, \infty)) + o(n^{-\gamma})(O(n^{-\gamma})). \end{aligned} \tag{15}$$

In the one-sided case $P(X_1 \in \mathbb{N}_0)=1$ τ is a probability measure, namely the stationary distribution of the renewal process. If Y_1, Y_2, Y_3 are independent random variables with this distribution we can rewrite (15) as given in the introduction.

Under the assumption $EX_1^3<\infty$ Stone and Wainger [20] obtained that $p_n = o(n^{-\gamma})(O(n^{-\gamma}))$ for some $\gamma>3$ implies

$$u_n = \frac{1}{\mu_1} + \frac{1}{\mu_1^2} (\Sigma p)([n+1, \infty)) - \frac{\mu_2 - \mu_1}{\mu_1^3} (\Sigma p)_n + o(n^{-\gamma})(O(n^{-\gamma})) \tag{16}$$

where $\mu_2 = EX_1^2$.
This may easily be deduced from (14) as we show now (we only consider the o-case).

We have

$$\frac{\hat{\tau}(\theta)^2 - \hat{\tau}(0)^2}{e^{i\theta}-1} = \hat{\tau}_2(\theta)^2 (e^{i\theta}-1)^3 + 2\hat{\tau}(0)\hat{\tau}_1(\theta) + 2\hat{\tau}_1(0)\hat{\tau}(\theta) + p(e^{i\theta}) \tag{17}$$

with some polynomial p and

$$\hat{\tau}_1(\theta) = \frac{\hat{\tau}(\theta) - \hat{\tau}(0)}{e^{i\theta}-1}, \qquad \hat{\tau}_2(\theta) = \frac{\hat{\tau}_1(\theta) - \hat{\tau}_1(0)}{e^{i\theta}-1}.$$

The n-th coefficient on the left side of (17) is $\left(\frac{1}{\mu_1} \Sigma p\right)^{*2}([n+1, \infty))$ if $n \geqq 0$.

Since $\tau_2 \in \mathfrak{F}_0(2,2;3,\gamma)$ we obtain $o(n^{-\gamma})$-behaviour for the coefficients of the first term on the right side of (17) with Lemma 3. Inserting $\hat{\tau}(0)=1$, $\hat{\tau}_1(0) = \frac{1}{2\mu_1}(\mu_2-\mu_1)$ we get (16) from (14).

If $EX_1^4 < \infty$ we may proceed in much the same way to obtain from (15) that $p_n - p_{n-1} = o(n^{-\gamma})(O(n^{-\gamma}))$ for some $\gamma > 4$ implies

$$u_n = \frac{1}{\mu_1} + \frac{1}{\mu_1^2}(\Sigma p)([n+1,\infty)) - \frac{\mu_2-\mu_1}{\mu_1^3}(\Sigma p)_n + \frac{1}{12\mu_1^4}(\mu_1^2 + 9\mu_2^2 - 6\mu_1\mu_2 - 4\mu_1\mu_3)p_n + o(n^{-\gamma})(O(n^{-\gamma})), \tag{18}$$

where $\mu_3 = EX_1^3$.

These results may also be employed to obtain results on asymptotic equality. In a recent paper Embrechts and Omey [9] proved that in the one-sided case $((\Sigma p)^{*2})_n \sim 2\mu_1(\Sigma p)_n$ implies

$$u_n - \frac{1}{\mu_1} \sim \frac{1}{\mu_1^2}(\Sigma p)([n+1,\infty)). \tag{19}$$

We obtain from (13) that the condition

$$(\Sigma p)_n = O(n^{-\gamma}), \quad (\Sigma p([n+1,\infty)))^{-1} = o(n^{\gamma}) \qquad \text{for some } \gamma > 2 \tag{20}$$

implies (19). Consider for example the distribution p with

$$p_n = \begin{cases} \frac{1}{2}(\alpha_{k-1}^{-\gamma} - \alpha_k^{-\gamma}), & \text{if } n = \alpha_k \text{ for some } k \in \mathbb{N}, \\ 1 - \frac{1}{2}\alpha_0^{-\gamma}, & \text{if } n = 1, \\ 0, & \text{otherwise,} \end{cases}$$

where $\alpha_{k+1} = [2^{1/\gamma}\alpha_k]$ for all $k \in \mathbb{N}_0$ and α_0 is chosen large enough to ensure $\alpha_k \uparrow \infty$. We then have

$$\tfrac{1}{2}n^{-\gamma} \leqq (\Sigma p)_{n-1} \leqq n^{-\gamma} \qquad \text{for all } n \geqq \alpha_0,$$

which implies (20), and

$$\lim_{k\to\infty} \frac{(\Sigma p)_{\alpha_k - 1}}{(\Sigma p)_{\alpha_k}} = 2.$$

So this is not contained in the result of Embrechts et al. since it follows from their assumptions that $(\Sigma p)_n \sim (\Sigma p)_{n+1}$. Interestingly our condition (20) implies that $\tau = \frac{1}{\mu_1}\Sigma p$ is a subexponential distribution, i.e.

$$\lim_{n\to\infty} \frac{\tau^{*2}([n,\infty))}{\tau([n,\infty))} = 2, \tag{21}$$

which may be seen as follows: We have

$$\tau^{*2}([n,\infty)) = \sum_{i=1}^{n-1} \tau_i \tau([n-i, n-1]) + \left(\sum_{i=0}^{n-1} \tau_i\right) \tau([n,\infty)) + \tau([n,\infty)),$$

hence (21) will follow from

$$\sum_{i=1}^{[n/2]} \tau_i \tau([n-i, n-1]) = O(n^{-\gamma}),$$
$$\sum_{i=[n/2]+1}^{n-1} \tau_i \tau([n-i, n-1]) = O(n^{-\gamma}). \tag{22}$$

Because of $\gamma > 2$ we have $\sum_{i=1}^{\infty} i\tau_i < \infty$ and $\sum_{i=1}^{\infty} \tau([i, \infty)) < \infty$, using this and

$$\sum_{i=1}^{[n/2]} \tau_i \tau([n-i, n-1]) \leqq \sum_{i=1}^{[n/2]} \tau_i i \sup_{j \geqq n/2} \tau_j$$

for the first,

$$\sum_{i=[n/2]+1}^{n-1} \tau_i \tau([n-i, n-1]) \leqq \sup_{j \geqq n/2} \tau_j \sum_{i=1}^{[n/2]} \tau([i, \infty))$$

for the second part we get (22).

Similarly (16) and (18) yield conditions which imply

$$u_n - \frac{1}{\mu_1} - \frac{1}{\mu_1^2} (\Sigma p)([n+1, \infty)) \sim -\frac{\mu_2 - \mu_1}{\mu_1^3} (\Sigma p)_n$$

or

$$u_n - \frac{1}{\mu_1} - \frac{1}{\mu_1^2} (\Sigma p)([n+1, \infty)) + \frac{\mu_2 - \mu_1}{\mu_1^3} (\Sigma p)_n$$
$$\sim \frac{1}{12\mu_1^4} (\mu_1^2 + 9\mu_2^2 - 6\mu_1\mu_2 - 4\mu_1\mu_3) p_n.$$

5. Further Applications, Concluding Remarks

5.1 Suppose that ν is an infinitely divisible distribution on $\mathbb{Z}$ with Lévy measure τ. Then τ has support $\mathbb{Z} - \{0\}$ and we have

$$\hat{\nu}(\theta) = \exp(\hat{\tau}(\theta) - \hat{\tau}(0)) \quad \text{for all } \theta \in \mathbb{R}.$$

Results concerning the asymptotic behaviour of ν and τ have been obtained by several authors; for example Embrechts, Goldie and Veraverbeke [7] give a necessary and sufficient condition for asymptotic equality of the tails of ν and τ, Embrechts and Hawkes [8] deal with asymptotic equality of ν_n and τ_n. In both articles ν is assumed to be concentrated on the non-negative numbers, in [7] non-discrete measures are considered too.

Our theorem gives in the case $k=1$ that $\sum_{n \in \mathbb{Z}} |n| \tau_n < \infty$, $\tau_n - \tau_{n-1} = o(n^{-\gamma})$ for some $\gamma > 2$ implies

$$\nu_n = \tau_n + o(n^{-\gamma})$$

and

$$\nu([n, \infty)) = (1 - \tau(\mathbb{Z}))\, \tau([n, \infty)) + \tfrac{1}{2} \tau^{*2}([n, \infty)) + o(n^{-\gamma}),$$

if we additionally assume $\sum_{n\in\mathbb{Z}} n^2\tau_n<\infty$, then

$$\nu([n,\infty))=\tau([n,\infty))+E(\tau)\,\tau_n+o(n^{-\gamma})$$

with $E(\tau)=\sum_{n\in\mathbb{Z}} n\tau_n$. The same holds with o replaced by O throughout.

Note that we do not need Proposition 2 here in its full generality since $\hat{\nu}$ depends on $\hat{\tau}$ via an entire function in which case Proposition 1 suffices.

5.2 Let $(Z_n)_{n\in\mathbb{N}_0}$ be an age-dependent branching process (see [1], Chap. IV) with lifetime distribution τ, $\tau(\mathbb{N})=1$, and such that the mean m of the offspring distribution satisfies $m<1$. Then we have ([1], IV 5.(4))

$$EZ_n=\sum_{j=0}^{\infty} m^j(\tau^{*j}([0,n])-\tau^{*(j+1)}([0,n])).$$

If we define ν, G and Ψ by

$$\nu_0=EZ_0, \qquad \nu_n:=E(Z_n-Z_{n-1}) \quad \text{for all } n\in\mathbb{N},$$

$$G=U_{1/m}(0), \qquad \Psi(z):=\frac{1-z}{1-mz} \quad \text{for all } z\in G,$$

then our result applies. If we take $k=0$ for example the global part yields that if τ has a finite mean and $\tau_n=o(n^{-\gamma})(O(n^{-\gamma}))$ for some $\gamma>1$ then

$$EZ_n=\frac{1}{1-m}\,\tau([n,\infty))+o(n^{-\gamma})(O(n^{-\gamma})).$$

A corresponding result on asymptotic equality is [1], IV Theorem 3B(ii).

5.3 In the proof of the theorem we essentially made use of the following properties of the sequence $\alpha=(\alpha_n)_{n\in\mathbb{N}}$, $\alpha_n=n^{-\gamma}$ for all $n\in\mathbb{N}$ $(\gamma>1)$,

$$\sup_{n\in\mathbb{N}}\frac{\alpha_n}{\alpha_{2n}}<\infty, \qquad \sum_{n=1}^{\infty}\alpha_n<\infty, \quad \alpha_n=o(n^{-1}), \qquad \lim_{n\to\infty}\frac{1}{n}\log\alpha_n=0.$$

These conditions are of course satisfied by many other sequences, and it is easy to obtain a corresponding generalization of our main result.

5.4 The application of our result to infinitely divisible sequences occupies a special position even in another respect than that already mentioned in 5.1: If we try to approximate the input measure by convolutions of the output the difficulty appears that the values of the Fourier transform are perhaps not contained in one sheet of the Riemann surface of the logarithm. In this case other results from the theory of commutative Banach algebras still permit approximations as given in our theorem, we refer the reader to [12], § 13 and [5], VIII § 1.3.12.

5.5 At the end let us indicate a limitation of the Banach algebra method: It starts with a subalgebra of the space of summable sequences, which, in renewal theory, translates to the assumption of finiteness of the first moment. To quote from a referee's report, "it would be highly interesting to find similar methods for the infinite mean case".

References

1. Athreya, K.B., Ney, P.E.: Branching Processes. Berlin-Heidelberg-New York: Springer 1972
2. Borovkov, A.A.: Remarks on Wiener's and Blackwell's theorems. Theory Probability Appl. **9**, 303–312 (1964)
3. Borovkov, A.A.: Stochastic Processes in Queuing Theory. New York-Heidelberg-Berlin: Springer 1976
4. Chover, J., Ney, P., Wainger, S.: Functions of probability measures. J. Analyse Math. **26**, 255–302 (1973)
5. Cotlar, M., Cignoli, R.: An Introduction to Functional Analysis. Amsterdam-London: North Holland 1974
6. Davies, L., Grübel, R.: Spaces of summable sequences in renewal theory and the theory of Markov chains. Math. Nachr. **104**, 119–128 (1981)
7. Embrechts, P., Goldie, C.M., Veraverbeke, N.: Subexponentiality and infinite divisibility. Z. Wahrscheinlichkeitstheorie verw. Gebiete **49**, 335–347 (1979)
8. Embrechts, P., Hawkes, J.: A limit theorem for the tails of discrete infinitely divisible laws with applications to fluctuation theory. J. Austral. Math. Soc. Ser. A **32**, 412–422 (1982)
9. Embrechts, P., Omey, E.: Functions of power series. Preprint, 1982
10. Erdös, P., Feller, W., Pollard, H.: A property of power series with positive coefficients. Bull. Amer. Math. Soc. **55**, 201–204 (1949)
11. Essén, M.: Banach algebra methods in renewal theory. J. Analyse Math. **26**, 303–336 (1973)
12. Gelfand, I.M., Raikow, D.A., Schilow, G.E.: Kommutative normierte Algebren. Berlin: Deutscher Verlag der Wissenschaften 1964
13. Hille, E., Phillips, R.S.: Functional Analysis and Semi-groups. Providence: Amer. Math. Soc. Coll. Publ. XXXI 1957
14. Kalashnikov, V.V.: Uniform estimation of the convergence rate in a renewal theorem for the case of discrete time. Theory Probability Appl. **22**, 390–394 (1977)
15. Lindvall, T.: On coupling of discrete renewal processes. Z. Wahrscheinlichkeitstheorie verw. Gebiete **28**, 57–70 (1979)
16. Ney, P.: A refinement of the coupling method in renewal theory. Stochastic Processes Appl. **11**, 11–26 (1981)
17. Rogozin, B.A.: An estimate of the remainder term in limit theorems in renewal theory. Theory Probability Appl. **18**, 662–677 (1973)
18. Rogozin, B.A.: Asymptotic behaviour of the coefficients of functions of power series and Fourier series. Sb. Math. J. **17**, 640–647 (1976)
19. Rudin, W.: Functional Analysis. New Delhi: Tata McGraw-Hill 1974
20. Stone, C., Wainger, S.: One-sided error estimates in renewal theory. J. Analyse Math. **20**, 325–352 (1967)

Received September 10, 1982; in revised form May 13, 1983