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Spectral Representations and Density Operators for Infinite-Dimensional Homogeneous Random Fields

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A theory of spectral representations and spectral density operators of infinite-dimensional homogeneous random fields is established. Some results concerning the form of the spectral representation are given in the general infinite-dimensional case, while the results pertaining to the density operator are confined to Hilbert space valued fields. The concept of a purely non-deterministic (p.n.d.) field is defined, and necessary and sufficient conditions for the property of p.n.d. are obtained in terms of the spectral density operator. The theory is developed using some isomorphisms induced by families of self-adjoint operators in the linear second order space associated with the field. The method seems to lead to more direct results also in the random process case, and it sheds new light on concepts such as multiplicity of the field and rank of the spectral density operator.

1. Introduction

In this paper we try to establish a theory of spectral representations and density operators for infinite-dimensional homogeneous random fields. Finite-dimensional fields of this type have been treated in a classical paper by Yaglom [16]. With the exception of Sections 2 and 4, where we obtain results on the spectral representation in the general case, our concern will be with fields having their values in a separable Hilbert space. From an application oriented point of view random fields (not necessarily homogeneous) with values in such a space are of interest for example in a functional analytic treatment of random integral equations of a slightly more general nature than those considered in Bharucha-Reid [1, Chapter 4].

Virtually all of our results will follow from the existence of two isomorphisms. For a general infinite-dimensional homogeneous field we study an isomorphism induced by a commuting set of self-adjoint operators, the so-called momentum operators of the field. In the special case of a purely non-deterministic field our theory is deduced from an isomorphism, Tjøstheim [14], defined by a set of self-adjoint operators forming a direct sum of Schrödinger n-systems. The approach based on these two isomorphisms seems to lead to more direct results also in the case of a Hilbert space valued random process, which has been studied before (using different methods) among others by Payen [8] and Kallianpur and Mandrekar [4]. In addition our method sheds new light on concepts such as multiplicity of the field and rank of the spectral density operator.

2. The Spectral Representation

In all of the following Lebesgue measure on R^n will be denoted by the letter m, and for a positive measure ν, ν-a.e. will be used as an abbreviation for the statement "ν-measure almost everywhere".

Let H be the Hilbert space of all complex-valued random variables having a finite second moment, and where the inner product is defined by $(F, G)=E\{F\bar{G}\}$ for F and $G\in H$. Let $(\Omega, \mathscr{B}, P)$ be a probability space. We say that $\{F_\lambda(x,\omega);\ \omega\in\Omega,\ \lambda\in L,\ x\in R^n\}$ is a second order random field over R^n if for each λ and x, $F_\lambda(x,\cdot)\in H$. Here, L is a parameter set associated with the "dimension" of the field. If $L=R^m$ and $F_\lambda(\cdot,\cdot)$ is linear in λ, the field is said to be m-dimensional. In this paper we will be concerned with the second order theory of the field, and consequently we will suppress the dependence on ω in our notation and write $\{F_\lambda(x);\ \lambda\in L,\ x\in R^n\}$ or simply $F_\lambda(x)$. In the sequel it will always be assumed that $F_\lambda(x)$ is continuous in quadratic mean (q.m.) in x for each $\lambda\in L$, and that $F_\lambda(x)$ is homogeneous, such that for $\lambda,\mu\in L$ and $x, y, z\in R^n$

$$E\{F_\lambda(x+z)\overline{F_\mu(y+z)}\}=E\{F_\lambda(x)\overline{F_\mu(y)}\}.$$

The second order theory of $F_\lambda(x)$ is developed in the Hilbert space $H(F)\subset H$. This space is the closure in H of the linear hull of the set of elements $\{F_\lambda(x);\ \lambda\in L,\ x\in R^n\}$. It will be assumed that $H(F)$ is separable. The following conditions imply separability: L is a Hausdorff space satisfying the second countability axiom, and $F_\lambda(x)$ is continuous in q.m. relative to the topology of L for each $x\in R^n$. This is easily proved using the technique of the proof of Lemma 2.1 of [3].

Let $\{U(y);\ y\in R^n\}$ be the unitary strongly continuous shift group of $F_\lambda(x)$ defined by $U(y)F_\lambda(x)=F_\lambda(x-y)$ and denote by $(\ ,\ )$ the usual inner product in R^n. It is well known that the spectral representation

$$F_\lambda(x)=\int_{R^n}\exp\{-i(x,u)\}\,d\Phi_\lambda(u) \tag{1}$$

where $\{\Phi_\lambda;\ \lambda\in L\}$ are random measures over the Borel sets $B(R^n)$ of R^n, follows from the spectral representation of $\{U(y);\ y\in R^n\}$ in $H(F)$. As it stands, the decomposition (1) is not very useful for our purposes. In the following theorem more information about the structure of this representation will be obtained.

Theorem 2.1. *Let $\{F_\lambda(x);\ \lambda\in L,\ x\in R^n\}$ be a homogeneous random field. Then there exists a cardinal number M which is uniquely determined by $F_\lambda(x)$ and which may be countably infinite, such that for each λ and x with probability one*

$$F_\lambda(x)=\sum_{j=1}^{M}\int_{R^n}\exp\{-i(x,u)\}\,g_\lambda^j(u)\,d\Phi_j(u) \tag{2}$$

where $\{\Phi_j; j=1, \ldots, M\}$ *are random measures over* $B(R^n)$ *such that for* Δ, Δ_1 *and* $\Delta_2 \in B(R^n)$

i) $E\{\Phi_j(\Delta_1)\overline{\Phi_k(\Delta_2)}\} = \delta_{kj} E|\Phi_j(\Delta_1 \cap \Delta_2)|^2 < \infty$,

ii) *For each* λ, $g_\lambda^j \in L^2_{v_j}(R^n)$ *where* v_j *is the measure defined by* $v_j(\Delta) = E|\Phi_j(\Delta)|^2$.

Proof. Let the k-th shift group $\{U_k(t);\ t \in (-\infty, \infty)\}$ be defined by

$$U_k(t) F_\lambda(x_1, \ldots, x_k, \ldots, x_n) = F_\lambda(x_1, \ldots, x_k - t, \ldots, x_n).$$

Under our assumptions $\{U_k(t);\ t \in (-\infty, \infty)\}$ is unitary strongly continuous and from Stone's theorem there exists a self-adjoint operator P_k in $H(F)$ such that

$$U_k(t) = \exp\{itP_k\}.$$

It is clear that the system $\{P_1, \ldots, P_n\}$ forms a commuting family of operators in $H(F)$, that is, if $\{E_k(s),\ s \in (-\infty, \infty)\}$ is the resolution of identity associated with P_k, then $E_k(s)E_j(t) = E_j(t)E_k(s)$ for $j, k = 1, \ldots, n$ and $s, t \in (-\infty, \infty)$. From a well known result by von Neumann [7], [2, pp. 127–134] (note that the results in [7, 2] are given for a single operator P, but it is not difficult to extend the technique to a commuting family of self-adjoint operators using the definition [10, p. 315) of operator functions of type $v(P_1, \ldots, P_n)$. See also Maurin [5, p. 193].) it follows that the system $\{P_1, \ldots, P_n\}$ induces a realization of $H(F)$ as a direct integral of Hilbert spaces. More precisely,

$$H(F) \overset{V}{\longleftrightarrow} \hat{H} = \int_{R^n} \hat{H}(u)\, dv(u) \tag{3}$$

where v is a positive finite measure over R^n and V is a unitary transformation taking $\hat{H}$ onto H such that if $F \leftrightarrow \{\hat{F}(u)\}$ for $F \in H(F)$, then

$$v(P)F = v(P_1, \ldots, P_n) F \leftrightarrow \{v(u)\hat{F}(u)\}$$

where v is a complex-valued v-measurable function on R^n such that

$$\int_{R^n} |v(u)|^2 \, \|\hat{F}(u)\|_u^2 \, dv(u) < \infty$$

$\|\ \|_u$ being the norm in $\hat{H}(u)$. Moreover, the isomorphism (3) is unique in the following sense: If $H(F) \leftrightarrow \hat{H}' = \int \hat{H}'(u)\, dv'(u)$ is another realization having the same properties, then the measures v and v' are equivalent and $\hat{H}(u)$ and $\hat{H}'(u)$ are isomorphic v-a.e.

Let $\{\hat{\Phi}_j(u);\ j = 1, \ldots, d(u) = \dim \hat{H}(u)\}$ be an orthonormal basis in $\hat{H}(u)$, and consider the realization $F_\lambda(0) \leftrightarrow \hat{F}_\lambda(0) = \{\hat{F}_\lambda(0; u)\}$. Decomposing $\hat{F}_\lambda(0; u) \in \hat{H}(u)$ according to the basis $\hat{\Phi}_j(u)$ in $\hat{H}(u)$, we have

$$\hat{F}_\lambda(0; u) = \sum_{j=1}^{d(u)} (\hat{F}_\lambda(0; u), \hat{\Phi}_j(u))_u \hat{\Phi}_j(u) \tag{4}$$

where $(\ ,\ )_u$ is the inner product in $\hat{H}(u)$. Consider the v-measurable sets $A_i = \{u \in R^n;\ \dim \hat{H}(u) = i\}$; $i = 0, \ldots,$ and denote by A the subset of R^n obtained by deleting from $\bigcup_{i=0}^{\infty} A_i$ those A_i with v-measure zero. From the uniqueness properties of the isomorphism (3) it follows that $M = \sup_{u \in A} \{\dim \hat{H}(u)\}$ is uniquely determined by the field $F_\lambda(x)$. Let $\Delta \in B(R^n)$ and denote by $\{\Phi_j; j=1, \ldots,\}$ the random measures defined by

$$\Phi_j(\Delta) \overset{V}{\longleftrightarrow} \{\chi_\Delta(u)\,\hat{\Phi}_j(u)\} \tag{5}$$

where $\chi_\Delta(u)=1$ for $u \in \Delta$ and zero otherwise, and where we put $\hat{\Phi}_j(u)=0$ for $j>d(u)$. (Note that $\Phi_j(\Delta) \in H(F)$ for all $\Delta \in B(R^n)$ due to the finiteness of v.) Clearly $E|\Phi_j(\Delta)|^2=0$ for $j>M$ and all $\Delta \in B(R^n)$. Using standard arguments, it is not difficult to show that (4) and (5) imply that $F_\lambda(0)$ can be represented in $H(F)$ as

$$F_\lambda(0)=\sum_{j=1}^{M} \int_{R^n} g_\lambda^j(u)\, d\Phi_j(u) \tag{6}$$

where

$$g_\lambda^j(u)=(\hat{F}_\lambda(0;u), \hat{\Phi}_j(u))_u. \tag{7}$$

But $F_\lambda(x)=\exp\left\{-i \sum_{k=1}^{n} x_k P_k\right\} F_\lambda(0)=\exp\{-i(x,P)\}\, F_\lambda(0)$, and it follows that as elements of $H(F)$, that is, with probability one

$$F_\lambda(x)=\sum_{j=1}^{M} \int_{R^n} \exp\{-i(x,u)\}\, g_\lambda^j(u)\, d\Phi_j(u)$$

and the representation (2) is established.

From the defining relation (5) we have

$$\begin{aligned} E\{\Phi_j(\Delta_1)\overline{\Phi_k(\Delta_2)}\} &= \int_{R^n} (\chi_{\Delta_1}(u)\,\hat{\Phi}_j(u), \chi_{\Delta_2}(u)\,\hat{\Phi}_k(u))_u\, dv(u) \\ &= \int_{\Delta_1 \cap \Delta_2} (\hat{\Phi}_j(u), \hat{\Phi}_k(u))_u\, dv(u) = \delta_{kj} v_j(\Delta_1 \cap \Delta_2) \leqq v(\Delta_1 \cap \Delta_2) \end{aligned} \tag{8}$$

from which property i) follows. Property ii) is a direct consequence of the fact that $F_\lambda(0) \in H(F)$, that is, $\{\hat{F}_\lambda(0;u)\} \in \hat{H}$. ||

Theorem 2.1 is the key representation of this paper and most of our subsequent results will be deduced from it. The cardinal number M of Equation (2) will be called the *momentum multiplicity of the field*. As will become clear in the following, it corresponds to the concept of rank of a wide sense stationary process as defined in [12]. If v is equivalent to Lebesgue measure and if $d(u)=M$ v-a.e. on R^n, the field is said to have *uniform momentum multiplicity*.

It should be noted that the functions $\{g_\lambda^j; j=1, \ldots, M\}$ appearing in (2) are highly non-unique. This is because they are strongly dependent on the choice of measure v in the isomorphism (3).

The measure v is unique up to equivalence. The following definition then makes sense: The field $\{F_\lambda(x); \lambda \in L, x \in R^n\}$ will be said to have an *absolutely continuous spectral distribution* if the measure v is absolutely continuous with respect to Lebesgue measure on R^n. In the absolute continuous case it is possible to obtain a representation analogous to (2), but where the non-uniqueness of the functions $\{g_\lambda^j; j=1, \ldots, M\}$ is to a large degree eliminated. To show this, let m be Lebesgue measure on R^n and define k by $k=(dv/dm)^{\frac{1}{2}}$. Furthermore, let $F \overset{V}{\longleftrightarrow} \{\hat{F}(u)\}$ be the representation of $F \in H(F)$ using the isomorphism (3). Then it is easy to construct a realization

$$H(F) \overset{W}{\longleftrightarrow} \int_{R^n} \hat{H}(u)\, dm(u) \tag{9}$$

where $F \overset{W}{\longleftrightarrow} \{k(u)\hat{F}(u)\}$. Using the isomorphism W we obtain a representation

$$F_\lambda(x) = \sum_{j=1}^{M} \int_{R^n} \exp\{-i(x,u)\}\, g_\lambda^{j,0}(u)\, d\Phi_j^0(u) \tag{10}$$

where M is as in (2), $g_\lambda^{j,0} \in L^2_{v_j}(R^n)$ and $v_j(\Delta) = E|\Phi_j^0(\Delta)|^2 \leqq m(\Delta)$ for a Borel set Δ of finite m-measure. Denote by $L^2(R^n)$ the space of complex-valued Borel functions which are square integrable with respect to m. Let h_λ^j be the function defined by $h_\lambda^j = g_\lambda^{j,0}(dv_j/dm)^{\frac{1}{2}}$. Then from (10) we obtain a representation

$$F_\lambda(x) = \sum_{j=1}^{M} \int_{R^n} \exp\{-i(x,u)\}\, h_\lambda^j(u)\, d\Phi_j(u) \tag{11}$$

where $E|\Phi_j(\Delta)|^2 = m(\Delta)$ for $j = 1, \dots, M$ and where the functions $\{h_\lambda^j \in L^2(R^n);\ \lambda \in L,\ j = 1, \dots, M\}$ are uniquely determined up to unitary equivalence.

3. The Spectral Density Operator

In this section we put some additional assumptions on the parameter set L and the mapping $\lambda \to F_\lambda(x)$ from L to $H(F)$. To be exact, we will assume that L is a separable Hilbert space and that the field $\{F_\lambda(x);\ \lambda \in L,\ x \in R^n\}$ can be represented with probability one (as random variables in $H(F)$) as

$$F_\lambda(x) = (\lambda, F(x)) \tag{12}$$

where for each $x \in R^n$, $F(x)$ is an L-valued random variable, and $(\ ,\ )$ is the inner product in L. Moreover, it will be supposed that $E\{\|F(0)\|^2\} < \infty$. We start by proving the following lemma.

Lemma 3.1. *Let $F_\lambda(x) = (\lambda, F(x))$ be as defined above with λ and $F(x) \in L$, a separable Hilbert space. In the representation (2) induced by the isomorphism*

$$H(F) \leftrightarrow \int \hat{H}(u)\, dv(u),$$

the functions $\{g_\lambda^j;\ \lambda \in L, j = 1, \dots, M\}$ can be represented v-a.e. on R^n as $g_\lambda^j(u) = (\lambda, g^{u,j})$ where $g^{u,j} \in L$.

Proof. From (7) and (12) it follows that the mappings $\lambda \to g_\lambda^j(u)$; $j = 1, \dots, M$ define linear functionals v-a.e. on R^n. If it can be proved that these mappings are continuous, the conclusion of the lemma follows from Riesz's representation theorem.

Let $\{\lambda_k;\ k = 1, \dots\}$ be an orthonormal basis of L. Then, as is not difficult to prove

$$\sum_{k=1}^{\infty} E|F_{\lambda_k}(0)|^2 = E\|F(0)\|^2 < \infty.$$

Consider the isomorphism $H(F) \leftrightarrow \hat{H} = \int \hat{H}(u)\,d\nu(u)$ defined in (3). It follows from (4) and (7) and from the definition of the inner product in $\hat{H}$ that

$$\sum_{k=1}^{\infty} E|F_{\lambda_k}(0)|^2 = \sum_{k=1}^{\infty} \sum_{j=1}^{M} \int_{R^n} |g^j_{\lambda_k}(u)|^2\,d\nu(u) < \infty. \tag{13}$$

Thus, $\int |g^j_{\lambda_k}(u)|^2\,d\nu(u)$ is bounded independent of j and k, and hence there exists a positive function C on R^n such that ν-a.e. on R^n, $C(u)$ is finite and $|g^j_{\lambda_k}(u)|^2 \leqq C(u)$. Let L^0 be the linear manifold in L consisting of all finite linear combinations $\sum_{i=1}^{q} a_i \lambda_{k_i}$, where q is an integer and $\{a_i; i=1, \dots, q\}$ are complex numbers. For an arbitrary element $\lambda = \sum_{i=1}^{q} a_i \lambda_{k_i} \in L^0$ we have ν-a.e.

$$|g^j_\lambda(u)|^2 = \left| \sum_{i=1}^{q} a_i\, g^j_{\lambda_{k_i}}(u) \right|^2 \leqq C(u) \sum_{i=1}^{q} |a_i|^2 = C(u)\, \|\lambda\|^2.$$

The ν-a.e. continuity of the mappings $\lambda \to g^j_\lambda(u)$ now follows from the denseness of L^0 in L. ||

The next lemma will be needed in our discussion of the concept of rank.

Lemma 3.2. *The vectors* $\{g^{u,j}; j=1, \dots, M\}$ *defined in Lemma* 3.1 *are linearly independent* ν-a.e. *on* R^n.

Proof. Let $\Delta \in B(R^n)$ be such that $\nu(\Delta) > 0$ and g^{u,j_k}; $k=1, \dots, q$ are well defined for $u \in \Delta$. Assume that $g^{u,j_k} \neq 0$ and $\sum_{k=1}^{q} a_k g^{u,j_k} = 0$ for $u \in \Delta$ and some complex numbers $\{a_k; k=1, \dots, q\}$. Then for all $\lambda \in L$ and $u \in \Delta$

$$\sum_{k=1}^{q} a_k (\lambda, g^{u,j_k}) = \sum_{k=1}^{q} a_k g^{j_k}_\lambda(u) = 0.$$

Hence, using the defining Equation (7) it follows that

$$\left(\hat{F}_\lambda(0; u), \sum_{k=1}^{q} a_k \hat{\Phi}_{j_k}(u) \right)_u = 0$$

Here $\hat{\Phi}_{j_k}(u) \neq 0$ since $g^{u,j_k} \neq 0$. But $\hat{F}_\lambda(0; u)$ spans $\hat{H}(u)$ as λ runs through L. Thus $\sum_{k=1}^{q} a_k \hat{\Phi}_{j_k}(u) = 0$ and we have $a_1 = \dots = a_q = 0$ using the orthonormality of $\{\hat{\Phi}_{j_k}(u); k=1, \dots, q\}$. ||

Consider a homogeneous field having an absolutely continuous spectral distribution as defined in the preceding section. (Note that in the special case of a Hilbert space valued process it is easy to show that our concept of absolute continuity is equivalent to the corresponding concept as defined in [4, p. 6].) Then the field can be represented as in (11). Let the field be Hilbert space valued. Then, using exactly the same proofs as for Lemmas 3.1 and 3.2, it is shown that the functions $\{h^j_\lambda \in L^2(R^n); \lambda \in L, j=1, \dots, M\}$ can be represented for Lebesgue measure m-a.e. by linearly independent elements $h^{u,j} \in L$ such that $h^j_\lambda(u) = (\lambda, h^{u,j})$.

Denote by B_u the Hermitian bilinear functional on L defined by

$$B_u(\lambda_1, \lambda_2) = \sum_{j=1}^{M} (\lambda_1, h^{u,j})\overline{(\lambda_2, h^{u,j})}$$

where λ_1 and $\lambda_2 \in L$, and where the sum is finite m-a.e. since for $\lambda \in L$

$$E|F_\lambda(0)|^2 = \int_{R^n} \sum_{j=1}^{M} |(\lambda, h^{u,j})|^2 \, dm(u) < \infty .$$

The functional B_u is continuous in both arguments λ_1 and λ_2, and thus it defines m-a.e. a linear bounded operator $f(u)$ in L by

$$B_u(\lambda_1, \lambda_2) = (\lambda_1, f(u)\lambda_2). \tag{14}$$

Clearly $f(u)$ can be represented m-a.e. as

$$f(u)\lambda = \sum_{j=1}^{M} (\lambda, h^{u,j}) h^{u,j} \tag{15}$$

where the sum may be interpreted as a limit in the weak topology of L if M is infinite. The operator $f(u)$ (parametrized by $u \in R^n$) will be called the *spectral density operator* of the field. From the uniqueness properties of the representation (11) it follows that $f(u)$ is uniquely determined up to unitary equivalence. (Compare [4, p. 6] for random process case.) Using (15) and an expression analogous to (13) but with $h^j_{\lambda_k}$ instead of $g^j_{\lambda_k}$, it is not difficult to prove that $f(u)$ is a self-adjoint, positive and nuclear operator in L m-a.e. on R^n. Lemmas 3.1 and 3.2 can now be used to give the following relation between the isomorphism $H(F) \leftrightarrow \int \hat{H}(u)\, dm(u)$ given in (9) and the spectral density operator $f(u)$.

Theorem 3.1. *Let $\{F_\lambda(x) = (\lambda, F(x));\ \lambda \in L,\ x \in R^n\}$ be a Hilbert space valued field with an absolutely continuous spectral distribution and let $H(F) \leftrightarrow \int \hat{H}(u)\, dm(u)$ be the direct integral decomposition given in* (9). *If $f(u)$ is the spectral density operator of the field, then for m-measure a.e. on R^n*

$$\operatorname{rank}\{f(u)\} = \dim \hat{H}(u). \tag{16}$$

Proof. From the analogy of Equation (7) valid in the absolutely continuous case it follows that

$$(\lambda, h^{u,j}) = (\hat{F}_\lambda(0; u), \hat{\Phi}_j(u))_u$$

m-a.e. Since $\{\hat{F}_\lambda(0; u);\ \lambda \in L\}$ span $\hat{H}(u)$, it follows that m-a.e., $h^{u,j} = 0$ iff $\hat{\Phi}_j(u) = 0$ for $j = 1, \ldots, M$ and the conclusion follows from Lemma 3.2 and (15). ||

As is well known, the rank of a linear operator is invariant under unitary transformations and rank $\{f(u)\}$ is therefore uniquely determined by the field m-a.e. In the finite-dimensional stochastic process case the rank of the process is sometimes, Rozanov [11, p. 39], defined as the rank of the spectral density matrix (when the latter has constant rank). From (16) it is clear that momentum multiplicity is a natural generalization of rank defined in this way. Note, however, that the concept of momentum multiplicity does not even require absolute continuity for its definition. See also discussion in Section 4 of [12].

4. The Purely Non-Deterministic Case

In this section we will treat general homogeneous fields $\{F_\lambda(x);\ \lambda \in L,\ x \in R^n\}$ as defined in Section 2. Some of the results that will be obtained here were reported in [14]. Furthermore, for the purely non-deterministic random process case we refer to [12] and [13]. In the next section we will apply the results obtained to Hilbert space valued fields.

Let $\{F_\lambda(x);\ \lambda \in L,\ x \in R^n\}$ be a second order homogeneous random field. Let $\{S_k(t);\ k=1, \ldots, n,\ t \in (-\infty, \infty)\}$ be the half-spaces in R^n given by

$$S_k(t) = \{x = (x_1, \ldots, x_n) \in R^n:\ x_k \leqq t\}$$

and let $S(y) = \bigcup_{k=1}^{n} S_k(y_k)$ for $y = (y_1, \ldots, y_n) \in R^n$. Denote by H_t^k and $H(F, y)$ the subspaces of $H(F)$ generated by the set of elements $\{F_\lambda(x);\ \lambda \in L,\ x \in S_k(t)\}$ and $\{F_\lambda(x);\ \lambda \in L,\ x \in S(y)\}$ respectively. Let $s \leqq t$ and let $H_k(s, t] = H_t^k \cap H_s^{k\perp}$ where the symbol $^\perp$ denotes the operation of taking orthogonal complements. Similarly we define $H_k(-\infty, t] = H_t^k \cap H_{-\infty}^{k\perp}$, $H_k(s, \infty) = H_s^{k\perp}$ and $H_k(-\infty, \infty) = H_{-\infty}^{k\perp}$, where $H_{-\infty}^k = \bigcap_{s \in (-\infty, \infty)} H_s^k$. Let $\Delta = (s_1, t_1] \times \cdots \times (s_n, t_n]$ where s_k may be $-\infty$ and $t_k + \infty$. The space $H_{in}(\Delta) = \bigcap_{k=1}^{n} H_k(s_k, t_k]$ may be interpreted as the space spanned by the innovations received by the field in the set $\Delta \subseteq R^n$. The field $F_\lambda(x)$ will be said to be purely non-deterministic (p.n.d.) if $H(F)$ is spanned by the totality of innovations received by the field. More precisely, let I be an index set and

$$\Delta^\alpha = (s_1^\alpha, t_1^\alpha] \times \cdots \times (s_n^\alpha, t_n^\alpha], \qquad \alpha \in I.$$

Then $F_\lambda(x)$ is said to be p.n.d. if for every collection of sets Δ^α, $\alpha \in I$, for which $\bigcup_{\alpha \in I} \Delta^\alpha = R^n$, we have that the Hilbert space generated by $\bigcup_{\alpha \in I} H_{in}(\Delta^\alpha)$ equals $H(F)$.

If $F_\lambda(x)$ is p.n.d., then $H_{in}(R^n) = \bigcap_{k=1}^{n} H_k(-\infty, \infty) = H(F)$. Thus for each k, $H_{-\infty}^{k\perp} = H(F)$, or $H_{-\infty}^k = \bigcap_{s \in (-\infty, \infty)} H_s^k = \{0\}$, and if $E_k(t)$ is the projection operator on H_t^k, then the chain of spaces $\{H_t^k;\ t \in (-\infty, \infty)\}$ defines a self-adjoint operator Q_k having $\{E_k(t);\ t \in (-\infty, \infty)\}$ as its resolution of identity. The operator Q_k will be called the k-th *coordinate operator* of the field.

In order to formulate our next result we have to introduce the concept of Schrödinger n-system (see Putnam [9, p. 81]).

Let $\{p_1, \ldots, p_n;\ q_1, \ldots, q_n\}$ be the system of self-adjoint operators in $L^2(R^n)$ defined by

$$p_k f(x) = -i \frac{\partial f}{\partial x_k}(x) \quad \text{and} \quad q_k f(x) = x_k f(x) \tag{17}$$

where i is the imaginary unit. A system of operators $\{P_1, \ldots, P_n;\ Q_1, \ldots, Q_n\}$ in a separable Hilbert space H will be said to form a direct sum of Schrödinger n-systems if there exists an M, which may be countably infinite, and an orthogonal direct sum decomposition

$$H=\sum_{j=1}^{M}\oplus H_j \quad \text{and} \quad \{P_1,\dots,P_n;\, Q_1,\dots,Q_n\}\doteq\sum_{j=1}^{M}\oplus\{P_1^j,\dots,P_n^j;\, Q_1^j,\dots,Q_n^j\}$$

such that

$$P_k^j=V_j\, p_k\, V_j^* \quad \text{and} \quad Q_k^j=V_j\, q_k\, V_j^* \tag{18}$$

where V_j is a unitary transformation taking $L_j^2(R^n)$ onto H_j. Here $L_j^2(R^n)$ is the j-th "copy" of $L^2(R^n)$.

Theorem 4.1. *Let $\{F_\lambda(x);\, \lambda\in L,\, x\in R^n\}$ be a random field which is homogeneous and p.n.d. Then the system of operators $\{P_1,\dots,P_n;\, Q_1,\dots,Q_n\}$ formed by the momentum-coordinate operators of the field is a direct sum of Schrödinger n-systems in $H(F)$.*

Proof. Let $\{U_k(t)=\exp(itP_k);\, t\in(-\infty,\infty)\}$ be the k-th shift group of the field and let $\{E_k(t);\, t\in(-\infty,\infty)\}$ be the resolution of identity associated with Q_k. Then, as in the random process case [12] the following commutation relation holds for t, $s\in(-\infty,\infty)$

$$E_k(t)\, U_k(s)=U_k(s)\, E_k(t+s).$$

Using this relation and the definition of operators P_k and Q_k; $k=1,\dots,n$ it is deduced that the system $\{P_1,\dots,P_n;\, Q_1,\dots,Q_n\}$ satisfies the so-called Weyl commutation relations [9, p. 81] in $H(F)$ and the theorem follows by a result of von Neumann [6]. ||

It is not difficult to verify that if $\{P_1,\dots,P_n;\, Q_1,\dots,Q_n\}$ forms a direct sum of Schrödinger n-systems, this is true also for the system $\{-Q_1,\dots,-Q_n;\, P_1,\dots,P_n\}$. Thus there exists a unitary transformation $W=\sum_{j=1}^{M}\oplus W_j$ taking $\sum_{j=1}^{M}\oplus L_j^2(R^n)$ onto $H(F)=\sum_{j=1}^{M}\oplus H_j(F)$ and such that

$$P_k^j=W_j\, q_k\, W_j^* \quad \text{and} \quad -Q_k^j=W_j\, p_k\, W_j^*. \tag{19}$$

This isomorphism represents a diagonalization of the momentum operators $\{P_1,\dots,P_n\}$. We recall that the representation (2) was obtained essentially by diagonalizing these operators. Using the isomorphism (19) we have:

Theorem 4.2. *Let $\{F_\lambda(x);\, \lambda\in L,\, x\in R^n\}$ be a homogeneous and p.n.d. field and let Δ_1 and Δ_2 be Borel sets of finite Lebesgue measure m. Then there exists a cardinal number M which is uniquely determined by $F_\lambda(x)$ and which may be countably infinite, such that for each λ and x with probability one*

$$F_\lambda(x)=\sum_{j=1}^{M}\int_{R^n}\exp\{-i(x,u)\}\, h_\lambda^j(u)\, d\Phi_j(u) \tag{20}$$

where $\{\Phi_j;\, j=1,\dots,M\}$ are random measures over Borel sets of finite m-measure such that

i) $E\{\Phi_j(\Delta_1)\,\Phi_k(\Delta_2)\}=\delta_{kj}\, m(\Delta_1\cap\Delta_2)$,
ii) *The functions $\{h_\lambda^j;\, \lambda\in L,\, j=1,\dots,M\}$ belong to $L^2(R^n)$.*

Proof. Let M be the uniquely determined cardinal number defined by the isomorphism $W=\sum_{j=1}^{M}\oplus W_j$ in (19). Let $\Delta\in B(R^n)$ have finite m-measure and define Φ_j; $j=1,\ldots,M$ by

$$\Phi_j(\Delta)=W_j\,\chi_\Delta$$

where $\chi_\Delta\in L^2(R^n)$ is one for $u\in\Delta$, zero otherwise. The decomposition $H(F)=\sum_{j=1}^{M}\oplus H_j(F)$ determined by W induces a decomposition $F_\lambda(0)=\sum_{j=1}^{M}\oplus F_\lambda^j(0)$. Let $h_\lambda^j\in L_j^2(R^n)$ be the element corresponding to $F_\lambda^j(0)$ using the isomorphism W_j. Then, using straightforward arguments

$$F_\lambda^j(0)=\int_{R^n} h_\lambda^j(u)\,d\Phi_j(u)$$

and the rest of the proof follows as in the last part of the proof of Theorem 2.1. ‖

From the isomorphism $H(F)\overset{W}{\longleftrightarrow}\sum_{j=1}^{M}\oplus L_j^2(R^n)$ it is trivial to obtain a realization $H\leftrightarrow\int\hat{H}(u)\,dm(u)$ analogous to (9) where now $\dim\hat{H}(u)=M$ for $u\in R^n$. The following corollary immediately results.

Corollary 4.1. *A homogeneous and p.n.d. field $\{F_\lambda(x);\lambda\in L, x\in R^n\}$ has an absolutely continuous spectral distribution. Furthermore, the momentum multiplicity of the field is uniform and equal to the number M appearing in the Schrödinger sum decomposition of the field.*

The isomorphism $V=\sum_{j=1}^{M}\oplus V_j$ in (18) brings about a diagonalization of the coordinate operators $\{Q_1,\ldots,Q_n\}$, and we obtain a coordinate representation of the field dual to (20). More precisely, let $S_-(y)=\bigcap_{k=1}^{n}S_k(y_k)$. Then $F_\lambda(y)$ can be represented with probability one for each λ and y as (compare Theorem 7 of [15])

$$F_\lambda(y)=\sum_{j=1}^{M}\int_{S_-(y)}\tilde{h}_\lambda^j(y-x)\,dZ_j(x) \tag{21}$$

where $\{Z_j, j=1,\ldots,M\}$ are random measures over Borel sets of finite m-measure such that

i) $E\{Z_j(\Delta_1)\overline{Z_k(\Delta_2)}\}=\delta_{kj}\,m(\Delta_1\cap\Delta_2)$,
ii) $U(y)\,Z_j(\Delta)=Z_j(\Delta-y)$,
iii) $H_-(F,y)=\sum_{j=1}^{M}\oplus H_-(Z_j,y)$ where $H_-(Z_j,y)$ is the Hilbert space generated by the linear hull of the set of random variables $Z_j(\Delta)$, when Δ runs through all Borel sets of finite m-measure such that $\Delta\subset S_-(y)$, and where $H_-(F,y)$ is the space generated by the set of elements $\{F_\lambda(x),\lambda\in L, x\in S_-(y)\}$.

For the random process case a proof of this result is given in Sections 3 and 4 of [12]. The proof in the random field case is similar [14] and is therefore omitted. Going from the isomorphism V to W corresponds to a Fourier transformation of the elements in $\sum_{j=1}^{M} \oplus L_j^2(R^n)$. If we utilize the fact that $\tilde{h}_\lambda^j(y-x)=0$ for $x \notin S_-(y)$, it follows that the functions h_λ^j and $\tilde{h}_\lambda^j$ of (20) and (21) respectively are related as follows

$$\tilde{h}_\lambda^j(x)=\left(\frac{1}{2\pi}\right)^{n/2} \int_{R^n} \exp\{-i(x,u)\}\, h_\lambda^j(u)\, dm(u) \tag{22}$$

and

$$h_\lambda^j(u)=\left(\frac{1}{2\pi}\right)^{n/2} \int_{R_+^n(0)} \exp\{i(u,x)\}\, \tilde{h}_\lambda^j(x)\, dm(x) \tag{23}$$

where (22) and (23) are interpreted as identities in L^2-sense, and where $R_+^n(0)$ is the subset of R^n defined by

$$R_+^n(0)=\bigcap_{k=1}^{n} R_+^{n,k}(0) \quad \text{with} \quad R_+^{n,k}(0)=\{x=(x_1,\ldots,x_n)\in R^n\colon x_k \geqq 0\}.$$

5. Some Necessary and Sufficient Conditions for p.n.d.

It will be our purpose here to obtain necessary and sufficient conditions for a homogeneous Hilbert space valued field to be p.n.d. It will be demonstrated that such conditions can in fact be derived virtually directly from the theory established in the preceding sections. Thus, our first theorem essentially consists of collecting and restating the results of Section 4.

Theorem 5.1. *Let* $\{F_\lambda(x)=(\lambda, F(x));\, \lambda\in L,\, x\in R^n\}$ *be a homogeneous Hilbert space valued random field. Then* $F_\lambda(x)$ *is p.n.d. of uniform momentum multiplicity M iff*

i) $F_\lambda(x)$ *has an absolutely continuous spectral distribution*

ii) *The spectral density operator* $f(u)$ *is such that* $\operatorname{rank}\{f(u)\}=M$ *m-a.e. on* R^n.

iii) *The functions* h_λ^j *in* (11) *can be represented in* $L^2(R^n)$ *as in* (23).

Proof. Assume that conditions i), ii) and iii) are fulfilled. From ii) and Theorem 3.1 it follows that $F_\lambda(x)$ has uniform momentum multiplicity M. Using i) and iii) it is deduced that $F_\lambda(x)$ can be represented as in (21), and the sufficiency part of the theorem follows using standard arguments.

The necessity of condition i) was demonstrated in Corollary 4.1. Condition ii) follows from the same corollary and Theorem 3.1, while condition iii) results from the fact that (21) and (23) are valid for a p.n.d. field. ||

Condition iii) may be reformulated into a statement concerning the factorization of the spectral density operator of the field. Let $F_\lambda(x)=(\lambda, F(x))$ be a field of momentum multiplicity M and having an absolutely continuous spectral distribution. Denote by $e_1, \ldots, e_M$ an orthonormal system of elements in L and let $h^{u,j}$; $j=1,\ldots,M$ be as in (15). Then m-a.e. in R^n we can define an operator $p(u)$ in L by

$$p(u)\,\lambda=\sum_{j=1}^{M} (\lambda, h^{u,j})\, e_j. \tag{24}$$

$\left(\text{Here } \sum_{j=1}^{M} |(\lambda, h^{u,j})|^2 < \infty \ m\text{-a.e. and } p(u) \text{ is well-defined.}\right)$ Let $f(u)$ be the spectral density operator of the field. It follows from (15) and (24) that for arbitrary elements λ_1 and λ_2 in L

$$(\lambda_1, f(u)\,\lambda_2) = (p(u)\,\lambda_1, p(u)\,\lambda_2). \tag{25}$$

Hence $f(u) = p^*(u)\,p(u)$ m-a.e. Since $f(u)$ is nuclear, it follows at once that $p(u)$ is a Hilbert Schmidt operator m-a.e.

Assume that the field is p.n.d. Using the same technique as in the proof of Lemma 3.1, we have that $\tilde{h}_\lambda^j$ of (21) can be represented as $\tilde{h}_\lambda^j(x) = (\lambda, \tilde{h}^{x,j})$ for some elements $\tilde{h}^{x,j} \in L$, this being valid m-a.e. If we define m-a.e. the operator $\tilde{p}(x)$ by

$$\tilde{p}(x)\,\lambda = \sum_{j=1}^{M} (\lambda, \tilde{h}^{x,j})\, e_j \tag{26}$$

it follows from (23) and (24) that $p(u)$ can be represented as

$$p(u) = \left(\frac{1}{2\pi}\right)^{n/2} \int_{R_+^n(0)} \exp\{i(u, x)\}\, \tilde{p}(x)\, dx \tag{27}$$

where the interpretation of operator integrals of this type is discussed for example in [4, p. 8].

Conversely assume that for a field having an absolutely continuous spectral distribution, the operator $p(u)$ of (24) and (25) can be represented as in (27). Then iii) of Theorem 5.1 holds, and we have proved the following theorem (which extends the main continuous-time result (Theorem 6.7) of [4] from random processes to random fields).

Theorem 5.2. *Let* $\{F_\lambda(x) = (\lambda, F(x));\ \lambda \in L, x \in R^n\}$ *be a Hilbert space valued homogeneous random field. Then* $F_\lambda(x)$ *is p.n.d. of uniform momentum multiplicity* M *iff*

i) $F_\lambda(x)$ *has an absolutely continuous spectral distribution*

ii) *The spectral density operator* $f(u)$ *has* $\operatorname{rank}\{f(u)\} = M$ *m-a.e.*

iii) $f(u) = p^*(u)\,p(u)$ *where* $p(u)$ *can be represented as in* (27).

In the discrete-time random process case an alternative characterization of the p.n.d. property is given by Payen [8, Proposition 9]. In our last theorem we present a similar result valid for random fields.

Theorem 5.3. *Let* $\{F_\lambda(x);\ \lambda \in L, x \in R^n\}$ *be as in Theorems* 5.1 *and* 5.2. *Then* $F_\lambda(x)$ *is p.n.d. of uniform momentum multiplicity* M *iff*

i) $F_\lambda(x)$ *has an absolutely continuous spectral distribution*

ii) *The spectral density operator can be represented m-a.e. as* $f(u) = \sum_{j=1}^{M} \rho_j(u)\, Q_j(u)$

where $Q_j(u)$ *is a projector on a subspace of dimension one spanned by an element* $h_j(u) \in L$ *such that* $h_j(u)$ *can be represented as*

$$h_j(u) = \left(\frac{1}{2\pi}\right)^{n/2} \int_{R_+^n(0)} \exp\{i(u, x)\}\, \tilde{h}_j(x)\, dx$$

and $\rho_j(u) = \|h_j(u)\|^2$.

Proof. The proof of the theorem follows directly from Lemma 3.1 and Theorem 5.1 if $h_j(u)$ is identified with $h^{u,j}$ of (15). ‖

6. An Extension

We have considered the possibility of extending our theory to fields taking values in the dual space of a countably Hilbert space as defined in Gel'fand and Vilenkin [2, p. 57]. Such a field is given with probability one as $F_\lambda(x)=\langle F(x)|\lambda\rangle$ where $\langle F(x)|\lambda\rangle$ is the value of the functional $F(x)\in L'$ at the point $\lambda\in L$. The topology of the countably Hilbert space L is given by a countable compatible collection of inner products $(\ ,\ )_j$, where it can always be arranged so that $(\lambda,\lambda)_k \leqq (\lambda,\lambda)_j$ for $k\leqq j$ and $\lambda\in L$. If we denote by L_j the completion of L in the norm $\|\ \|_j$ and by L'_j the corresponding dual space, then $L'=\bigcup_{j=1}^{\infty} L'_j$ and $L=\bigcap_{j=1}^{\infty} L_j$ when considered as abstract sets. Assume that there exists a k such that with probability one $F(x)$ can be extended to a continuous linear functional $F^{(k)}(x)\in L'_k$ and such that $E\,\|F(x)\|^2_{-k}<\infty$, where for $F\in L'_k$,

$$\|F\|^2_{-k}=\sup_{\lambda\in L_k,\ \|\lambda\|_k=1} |\langle F|\lambda\rangle|^2.$$

Under these somewhat restrictive assumptions we have been able to carry over (the changes being mostly of notational character) the theory of Hilbert space valued fields as outlined in Sections 3 and 5 to fields of type $F_\lambda(x)=\langle F(x)|\lambda\rangle$.

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