Z. Wahrscheinlichkeitstheorie verw. Gebiete
35, 299–314 (1976)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

Moment Inequalities and the Strong Laws of Large Numbers

F. Móricz

Bolyai Institute, University of Szeged, Aradi vértanuk tere 1
H-6720 Szeged, Hungary

0. Introduction

Let $\{\xi_k\}$ be a sequence of random variables. It is not assumed that the ξ_k's are mutually independent or that they are identically distributed. Set

$$S_{b,n}=\sum_{k=b+1}^{b+n}\xi_k \qquad (S_{b,0}=0)$$

and

$$M_{b,n}=\max_{1\leqq k\leqq n}|S_{b,k}| \quad (b\geqq 0,\ n\geqq 1).$$

Thus $M_{b,n}$ is the largest magnitude for the n consecutive partial sums formed from the n consecutive ξ_k's commencing with ξ_{b+1}. Furthermore, for each vector $\xi_{b,n}=(\xi_{b+1},\ldots,\xi_{b+n})$ of n consecutive ξ_k's, let $F_{b,n}$ denote the joint distribution function. In statements about $\xi_{0,n}$ only, the abbreviated notation S_n, M_n, F_n, etc. will be used.

The object of this paper is to provide bounds on $E(M_{b,n}^\gamma)$ in terms of given bounds on $E|S_{b,n}|^\gamma$, where $\gamma>0$. We emphasize that it is not assumed that the ξ_k's are independent. The only restrictions on the dependence will be those imposed on the assumed bounds for $E|S_{b,n}|^\gamma$. These assumed bounds are guaranteed under a suitable dependence restriction, e.g., mutual independence, martingale differences, weak multiplicativity of finite order, or the like.

Bounds on $E(M_{b,n}^\gamma)$ are of use in deriving bounds on the tail distribution of the maximum of certain partial sums in order to study convergence properties of S_n as $n\to\infty$. For development of such results under various dependence restrictions, the theorems of this paper reduce the problem of placing appropriate bounds on $E(M_{b,n}^\gamma)$ to the easier problem of placing appropriate bounds on $E|S_{b,n}|^\gamma$.

The problem posed above is treated essentially in a setting close to that of Serfling [12], whose results are contained as special cases in our Theorems 1 and 4. The applications made by us are also in close relation with those presented by Serfling [13].

1. The Main Result: the Case $\alpha > 1$

In the following, the function $g(F_{b,n})$ denotes a non-negative functional depending on the joint destribution function of $\xi_{b,n}$. Examples are: $g(F_{b,n}) = n^\alpha$ where $\alpha \geqq 1$, or $g(F_{b,n}) = \sum_{k=b+1}^{b+n} a_k^2$, where $\{a_k\}$ is a sequence of numbers. (In most cases, a_k^2 is the finite variance of ξ_k, but this remark plays no role in the theorems stated below.) Throughout the paper we shall assume that the function $g(F_{b,n})$ possesses the following property of rather general nature:

$$g(F_{b,k}) + g(F_{b+k,l}) \leqq g(F_{b,k+l}) \tag{1.1}$$

for all $b \geqq 0$ and $1 \leqq k < k+l$. In the sequel $C, C_1, C_2, \ldots$ will denote positive constants.

Theorem 1. *Suppose that there exists a function $g(F_{b,n})$ satisfying* (1.1) *such that*

$$E\,|S_{b,n}|^\gamma \leqq g^\alpha(F_{b,n}) \qquad (\text{all } b \geqq 0,\ n \geqq 1), \tag{1.2}$$

where $\gamma > 0$ and $\alpha > 1$. Then

$$E(M_{b,n}^\gamma) \leqq C_{\gamma,\alpha}\, g^\alpha(F_{b,n}) \qquad (\text{all } b \geqq 0,\ n \geqq 1). \tag{1.3}$$

Although its specific value will have no importance for us, the constant $C_{\gamma,\alpha}$ may be taken as

$$C_{\gamma,\alpha} = (1 - 2^{(1-\alpha)/\gamma})^{-\gamma} \tag{1.4}$$

if $\gamma > 1$, and $C_{\gamma,\alpha} = 1$ if $0 < \gamma \leqq 1$.

The result (1.3) provides a bound for $E(M_{b,n}^\gamma)$ which is asymptotically optimal as $n \to \infty$, in the sense that it is of the same order of magnitude as the bound assumed for $E\,|S_{b,n}|^\gamma$. In Theorem 1 the bounds may involve parameters of the joint distribution function of $\xi_{b,n}$, a flexibility particularly useful with non-identically distributed rv's.

Before proving Theorem 1, let us collect some results of its prerequisites. Set

$$A_{b,n} = \left(\sum_{k=b+1}^{b+n} a_k^2 \right)^{\frac{1}{2}},$$

where $\{a_k\}$ is a sequence of numbers.

Theorem A (Erdös-Stečkin). *Let $\gamma > 2$. Suppose that there exists a sequence $\{a_k\}$ of numbers such that*

$$E\,|S_{b,n}|^\gamma \leqq C_\gamma A_{b,n}^\gamma \qquad (\text{all } b \geqq 0,\ n \geqq 1).$$

Then

$$E(M_{b,n}^\gamma) \leqq C_\varepsilon\, C_\gamma\, A_{b,n}^\gamma \qquad (\text{all } b \geqq 0,\ n \geqq 1),$$

where C_ε does not depend on γ for $\gamma \geqq 2 + \varepsilon$, $\varepsilon > 0$.

This result was proved by Erdös [3] for lacunary trigonometric series and $\gamma = 4$, while the general form as stated in Theorem A is due to Stečkin. (In fact, it was an oral communication of Stečkin, which was elaborated by Gapoškin [4], pp. 29–31.) A possible generalization of Theorem A, due to Tjurnpü [14], reads as follows.

Theorem B. *Let $\gamma>\delta>1$ and let $\{a_k\}$ be a sequence of numbers such that*

$$E\,|S_{b,n}|^\gamma \leqq C_{\gamma,\delta}\left(\sum_{k=b+1}^{b+n}|a_k|^\delta\right)^{\gamma/\delta} \qquad (\textit{all } b\geqq 0,\ n\geqq 1).$$

Then

$$E(M_{b,n}^\gamma)\leqq C_{\gamma,\delta}^*\left(\sum_{k=b+1}^{b+n}|a_k|^\delta\right)^{\gamma/\delta} \qquad (\textit{all } b\geqq 0,\ n\geqq 1).$$

Another interesting result can be found in Serfling [12].

Theorem C. *Let $\gamma>2$ and suppose that*

$$E\,|S_{b,n}|^\gamma\leqq g^{\frac{1}{2}\gamma}(n) \qquad (\textit{all } b\geqq 0,\ n\geqq 1),$$

where $g(n)$ is non-decreasing, $2g(n)\leqq g(2n)$, and $g(n+1)/g(n)\to 1$ as $n\to\infty$. Then

$$E(M_{b,n}^\gamma)\leqq C\,g^{\frac{1}{2}\gamma}(n) \qquad (\textit{all } b\geqq 0,\ n\geqq 1),$$

where C may depend on γ, g and the joint distributions of the ξ_k's.

A common generalization of Theorems A and C was found by the author [7].

Theorem D. *Let $\gamma>2$ and let $\{a_k\}$ be a sequence of numbers such that*

$$E\,|S_{b,n}|^\gamma\leqq g^{\frac{1}{2}\gamma}(A_{b,n}^2) \qquad (\textit{all } b\geqq 0,\ n\geqq 1),$$

where $g(x)$ is non-decreasing and $2^\beta\,g(x)\leqq g(2x)$ for $x\geqq 0$, where $2/\gamma<\beta\leqq 1$. Then

$$E(M_{b,n}^\gamma)\leqq C_{\gamma,\beta}\,g^{\frac{1}{2}\gamma}(A_{b,n}^2) \qquad (\textit{all } b\geqq 0,\ n\geqq 1).$$

It is not hard to check that both Theorem B and Theorem D are contained in Theorem 1.

The proof of Theorem 1 (and later, the proofs of Theorems 4 and 5) are based on the "bisection" technique applied by Billingsley [1; p. 102].

Proof of Theorem 1. We are to find a constant $C\geqq 1$, depending only on γ and α, for which

$$E(M_{b,n}^\gamma)\leqq C\,g^\alpha(F_{b,n}) \qquad (b\geqq 0,\ n\geqq 1). \tag{1.5}$$

We shall distinguish two cases: (i) $\gamma>1$ and (ii) $0<\gamma\leqq 1$.

First consider the case $\gamma>1$. The proof goes by induction on n. The result is obvious for $n=1$, since $C\geqq 1$. Assume now as induction hypothesis that the result holds for each integer less than n. We shall prove it for n itself. There exists an integer h, $1\leqq h\leqq n$, such that

$$g(F_{b,h-1})\leqq \tfrac{1}{2}\,g(F_{b,n})<g(F_{b,h}), \tag{1.6}$$

where $g(F_{b,h-1})$ on the left is 0 if $h=1$. Then (1.1) and (1.6) implies

$$g(F_{b+h,n-h})\leqq g(F_{b,n})-g(F_{b,h})<\tfrac{1}{2}\,g(F_{b,n}). \tag{1.7}$$

Now, for $h \leqq k \leqq n$, we have

$$|S_{b,k}| \leqq |S_{b,h}| + |S_{b+h,k-h}| \leqq |S_{b,h}| + M_{b+h,n-h}.$$

Also, for $1 \leqq k < h$, we have $|S_{b,k}| \leqq M_{b,h-1}$, and hence

$$|S_{b,k}| \leqq |S_{b,h}| + (M^{\gamma}_{b,h-1} + M^{\gamma}_{b+h,n-h})^{1/\gamma}$$

for $1 \leqq k \leqq n$. Therefore,

$$M_{b,n} \leqq |S_{b,h}| + (M^{\gamma}_{b,h-1} + M^{\gamma}_{b+h,n-h})^{1/\gamma}$$

and, by Minkowski's inequality,

$$E(M^{\gamma}_{b,n})]^{1/\gamma} \leqq [E\,|S_{b,h}|^{\gamma}]^{1/\gamma} + [E(M^{\gamma}_{b,h-1}) + E(M^{\gamma}_{b+h,n-h})]^{1/\gamma}. \tag{1.8}$$

Since (1.2) holds if n is replaced by $h-1$, and since $h-1<n$, we may apply the induction hypothesis to the rv's $\xi_{b+1}, \ldots, \xi_{b+h-1}$ and conclude by (1.5) that

$$E(M^{\gamma}_{b,h-1}) \leqq C\, g^{\alpha}(F_{b,h-1}) \leqq \frac{C}{2^{\alpha}}\, g^{\alpha}(F_{b,n}). \tag{1.9}$$

Here the last inequality follows by (1.6). We note that if $h=1$, then (1.9) is obvious.

If the indices in (1.2) are restricted to $b+h$ and $1 \leqq k \leqq n-h$, then only the rv's $\xi_{b+h+1}, \ldots, \xi_{b+n}$ are involved. Since $n-h<n$, the induction hypothesis applies to $\xi_{b+h,n-h}$; hence (1.5) yields

$$E(M^{\gamma}_{b+h,n-h}) \leqq C\, g^{\alpha}(F_{b+h,n-h}) \leqq \frac{C}{2^{\alpha}}\, g^{\alpha}(F_{b,n}), \tag{1.10}$$

the last inequality following now by (1.7). (If $h=n$, (1.10) is trivial.)

Finally, in view of (1.2),

$$E\,|S_{b,h}|^{\gamma} \leqq g^{\alpha}(F_{b,h}) \leqq g^{\alpha}(F_{b,n}), \tag{1.11}$$

since $g(F_{b,n})$ is non-decreasing in n by (1.1). Combining inequalities (1.8)–(1.11), we find that

$$[E(M^{\gamma}_{b,n})]^{1/\gamma} \leqq \left(1 + \frac{C^{1/\gamma}}{2^{(\alpha-1)/\gamma}}\right) g^{\alpha/\gamma}(F_{b,n}).$$

If C is large enough, then hence it follows that

$$[E(M^{\gamma}_{b,n})]^{1/\gamma} \leqq C^{1/\gamma}\, g^{\alpha/\gamma}(F_{b,n}),$$

which is equivalent to (1.5). The smallest C satisfying

$$1 + \frac{C^{1/\gamma}}{2^{(\alpha-1)/\gamma}} \leqq C^{1/\gamma}$$

is given by (1.4). This completes the induction step and the proof of (1.3) in case $\gamma > 1$.

In the remaining case $0 < \gamma \leqq 1$, instead of Minkowski's inequality we have to apply the following inequality:

$$E\,|\xi + \eta|^{\gamma} \leqq E\,|\xi|^{\gamma} + E\,|\eta|^{\gamma}.$$

Also, for $0<\gamma\leqq 1$ and $\alpha\geqq 1$, we have

$$E(M_{b,n}^{\gamma})\leqq \sum_{k=b+1}^{b+n} E|\xi_k|^{\gamma}\leqq \sum_{k=b+1}^{b+n} g^{\alpha}(F_{k-1,1})\leqq g^{\alpha}(F_{b,n}).$$

Here we use that

$$u^{\alpha}+v^{\alpha}\leqq(u+v)^{\alpha} \quad \text{for } u\geqq 0,\ v\geqq 0, \text{ and } \alpha\geqq 1.$$

Thus the proof of Theorem 1 is complete.

As a by-product, we obtained the following

Theorem 2. *Suppose that there exists a function* $g(F_{b,n})$ *satisfying* (1.1) *such that*

$$E|S_{b,n}|^{\gamma}\leqq g^{\alpha}(F_{b,n}) \quad (\textit{all } b\geqq 0,\ n\geqq 1),$$

where $0<\gamma\leqq 1$ *and* $\alpha\geqq 1$. *Then*

$$E(M_{b,n}^{\gamma})\leqq g^{\alpha}(F_{b,n}) \quad (\textit{all } b\geqq 0,\ n\geqq 1).$$

2. The Case $\alpha=1$

Let us proceed to the study of the case, when $\gamma>1$ and $\alpha=1$. Then, roughly speaking, a factor $(\log 2n)^{\gamma}$ will occur in the bound (1.3) provided by Theorem 1. Here and in the sequel all logarithms are with base 2.

Theorem 3. *Suppose that there exists a function* $g(F_{b,n})$ *satisfying* (1.1) *such that*

$$E|S_{b,n}|^{\gamma}\leqq g(F_{b,n}) \quad (\textit{all } b\geqq 0,\ n\geqq 1), \tag{2.1}$$

where $\gamma>1$. *Then*

$$E(M_{b,n}^{\gamma})\leqq(\log 2n)^{\gamma}\, g(F_{b,n}) \quad (\textit{all } b\geqq 0,\ n\geqq 1). \tag{2.2}$$

This is a special case of the following more general result. Before its formulation, let us give a recurrence definition. Let $\lambda(n)$ be a positive and non-decreasing function of the natural number n. Set $\Lambda(1)=\lambda(1)$ and, for $n\geqq 2$,

$$\Lambda(n)=\lambda(m)+\Lambda(m-1), \tag{2.3}$$

where m denotes the integer part of $\frac{1}{2}(n+2)$. It is clear that $\Lambda(n)$ is also positive and non-decreasing.

Theorem 4. *Suppose that there exist a function* $g(F_{b,n})$ *satisfying* (1.1), *and a positive and non-decreasing function* $\lambda(n)$ *such that*

$$E|S_{b,n}|^{\gamma}\leqq\lambda^{\gamma}(n)\, g(F_{b,n}) \quad (\textit{all } b\geqq 0,\ n\geqq 1), \tag{2.4}$$

where $\gamma>1$. *Let* $\Lambda(n)$ *be defined by* (2.3). *Then*

$$E(M_{b,n}^{\gamma})\leqq\Lambda^{\gamma}(n)\, g(F_{b,n}) \quad (\textit{all } b\geqq 0,\ n\geqq 1). \tag{2.5}$$

We note that if $\lambda(n)$ equals 1 for all n then $\Lambda(n)\leqq\log 2n$, which follows by $1+\log 2(m-1)\leqq\log 2n$. (This is true, since $n\geqq 2m-2$.) Consequently, Theorem 4 contains Theorem 3 as a particular case (i.e., when $\lambda(n)\equiv 1$). Further, we mention that if $\lambda(n)=n^\beta$ with some $\beta>0$ then $\Lambda(n)\leqq(2n)^\beta/(2^\beta-1)^{1/\beta}$, if $\lambda(n)=(\log 2n)^\beta$ then $\Lambda(n)\leqq(\log 2n)^{\beta+1}$, etc.

The history of Theorems 3 and 4 goes back to Rademacher and Mensov. In the theory of sequences of orthogonal rv's (i.e., $E(\xi_i\,\xi_k)=0$ if $i\neq k$), a basic lemma is

Theorem E (Rademacher-Mensov). *If $\xi_1, \ldots, \xi_n$ are mutually orthogonal rv's with finite variances $\sigma_1^2, \ldots, \sigma_n^2$, then*

$$E(M_n^2)\leqq(\log 4n)^2\sum_{k=1}^{n}\sigma_k^2.$$

The result is given and used, e.g., in Doob ([2], p. 156) and, more recently, in Révész ([10], p. 83). Concerning more general result, Billingsley ([1], p. 102) indicates how to prove

Theorem F. *Suppose that there exist non-negative numbers u_k such that*

$$E\,|S_{b,n}|^\gamma\leqq\left(\sum_{k=b+1}^{b+n}u_k\right)^\alpha\qquad(\textit{all }\ b\geqq 0,\ n\leqq 1),\tag{2.6}$$

where $\gamma\geqq 1$ and $\alpha\geqq 1$. Then

$$E(M_{b,n}^\gamma)\leqq(\log 4n)^\gamma\left(\sum_{k=b+1}^{b+n}u_k\right)^\alpha\qquad(\textit{all }\ b\geqq 0,\ n\geqq 1).$$

When we restrict our attention to situations in which (2.6) is assumed to hold for some $\gamma\geqq 2$ and $\alpha\geqq\frac{1}{2}\gamma$, the above theorem is a special case of the following theorem of Serfling [12], which permits the quantity $\sum_{k=b+1}^{b+n}u_k$ to be replaced by quantities of other types.

Theorem G. *Suppose that there exists a function $h(F_{b,n})$ satisfying*

$$h(F_{b,k})+h(F_{b+k,l})\leqq h(F_{b,k+l})\qquad(\textit{all }\ b\geqq 0,\ 1\leqq k<k+l)\tag{2.7}$$

such that

$$E\,|S_{b,n}|^\gamma\leqq h^{\frac{1}{2}\gamma}(F_{b,n})\qquad(\textit{all }\ b\geqq 0,\ n\geqq 1),$$

where $\gamma\geqq 2$. Then

$$E(M_{b,n}^\gamma)\leqq(\log 2n)^\gamma\,h^{\frac{1}{2}\gamma}(F_{b,n})\qquad(\textit{all }\ b\geqq 0,\ n\geqq 1).$$

Consider now Theorem 4 with $\gamma\geqq 2$, $\lambda(n)\equiv 1$, and $g(F_{b,n})=h^{\frac{1}{2}\gamma}(F_{b,n})$. Since $\frac{1}{2}\gamma\geqq 1$, condition (2.7) implies condition (1.1), and hence Theorem 3 contains Theorem G as a special case. On the other hand, condition (1.1) does not imply, in general, condition (2.7) with $h(F_{b,n})=g^{2/\gamma}(F_{b,n})$ if $\gamma>2$. Thus Theorem 4 is more general than Theorem G, even in the particular case $\gamma>2$ and $\lambda(n)\equiv 1$.

Another important result in the theory of sequences of orthogonal rv's is due to Mensov and Paley (see, e.g., [15], p. 189).

Theorem H (Mensov-Paley). *Let $\xi_1, \ldots, \xi_n$ be mutually orthogonal rv's with finite variances $\sigma_1^2, \ldots, \sigma_n^2$ and such that with probability* 1

$$|\xi_k| \leqq K(<\infty) \quad (k=1,2,\ldots,n).$$

If $\gamma>2$ then

$$E(M_n^\gamma) \leqq C_\gamma K^{\gamma-2} n^{\gamma-2} \sum_{k=1}^{n} \sigma_k^\gamma.$$

This result is a simple consequence of Theorem 4 with $\lambda(n)=n^{(\gamma-2)/\gamma}$ (see the note made after Theorem 4) if we take into account that by another theorem of Paley (see [15], p. 121) *under conditions of Theorem H we have*

$$E|S_{b,l}|^\gamma \leqq \bar{C}_\gamma K^{\gamma-2} l^{\gamma-2} \sum_{k=b+1}^{b+l} \sigma_k^\gamma \quad (\textit{all } b \geqq 0,\ 1 \leqq l \leqq n-b).$$

Thus Theorem 4 contains all theorems from E to H. Theorem 4 was proved by the author [7], apart from a slight modification in the definition of $\Lambda(n)$. (Namely, there $\Lambda(n)$ was defined by $\Lambda(n)=\Lambda(\bar{m})+\lambda(\bar{m})$, where $\bar{m}$ denotes the integer part of $\frac{1}{2}(n+1)$.) For the sake of completeness, we shall present its proof here.

Proof of Theorem 4. Let $n>1$ be given and let m be the integer part of $\frac{1}{2}(n+2)$. Then $n=2m-1$ or $2m-2$. Let $b \geqq 0$. Now, for $m \leqq k \leqq n$, we have

$$|S_{b,k}| \leqq |S_{b,m}| + |S_{b+m,k-m}|,$$

whence, for such k's,

$$|S_{b,k}| \leqq |S_{b,m}| + M_{b+m,n-m}.$$

Since, for $1 \leqq k < m$, we have $|S_{b,k}| \leqq M_{b,m-1}$, thus, for any k between 1 and n, we have

$$|S_{b,k}| \leqq |S_{b,m}| + (M_{b,m-1}^\gamma + M_{b+m,n-m}^\gamma)^{1/\gamma}.$$

Therefore,

$$M_{b,n} \leqq |S_{b,m}| + (M_{b,m-1}^\gamma + M_{b+m,n-m}^\gamma)^{1/\gamma}$$

and, by Minkowski's inequality,

$$[E(M_{b,n}^\gamma)]^{1/\gamma} \leqq [E|S_{b,m}|^\gamma]^{1/\gamma} + [E(M_{b,m-1}^\gamma) + E(M_{b+m,n-m}^\gamma)]^{1/\gamma}. \tag{2.8}$$

Suppose now that the conclusion (2.5) of the theorem is true for $k<n$. Then, by the choice of m, we have

$$E(M_{b,m-1}^\gamma) \leqq \Lambda^\gamma(m-1)\, g(F_{b,m-1})$$

and

$$E(M_{b+m,n-m}^\gamma) \leqq \Lambda^\gamma(n-m)\, g(F_{b+m,n-m}) \leqq \Lambda^\gamma(m-1)\, g(F_{b+m,n-m}).$$

Putting these two inequalities together, by (1.1) we find that

$$E(M_{b,m-1}^\gamma) + E(M_{b+m,n-m}^\gamma) \leqq \Lambda^\gamma(m-1)\, g(F_{b,n}). \tag{2.9}$$

Finally, (2.4) implies

$$E\,|S_{b,m}|^\gamma \leqq \lambda^\gamma(m)\, g(F_{b,m}) \leqq \lambda^\gamma(m)\, g(F_{b,n}). \tag{2.10}$$

Collecting inequalities (2.8)–(2.10), we arrive at

$$[E(M_{b,n}^\gamma)]^{1/\gamma} \leqq [\lambda(m) + \Lambda(m-1)]\, g^{1/\gamma}(F_{b,n}).$$

By (2.3) this gives the wanted (2.5). Therefore, since the conclusion of the theorem is true for $n=1$ by the condition (2.4), it follows by induction for all $n=1,2,\ldots$. This completes the proof.

3. The Case $0<\alpha<1$

Now let us deal shortly with the case $0<\alpha<1$ and $\gamma\geqq 1$. Using the same ideas as in the proof of Theorem 4 we can show

Theorem 5. *Suppose that there exist a function $g(F_{b,n})$ satisfying* (1.1), *and a positive and non-decreasing function $\lambda(n)$ such that*

$$E\,|S_{b,n}|^\gamma \leqq \lambda^\gamma(n)\, g^\alpha(F_{b,n}) \quad (\textit{all } b\geqq 0,\ n\geqq 1), \tag{3.1}$$

where $0<\alpha<1$ and $\gamma\geqq 1$. Then

$$E(M_{b,n}^\gamma) \leqq \Lambda^\gamma(n)\, g^\alpha(F_{b,n}) \quad (\textit{all } b\geqq 0,\ n\geqq 1), \tag{3.2}$$

where $\Lambda(n)$ is defined by $\Lambda(1)=\lambda(1)$ and, for $n\geqq 2$,

$$\Lambda^{\gamma/\alpha}(n) = 2^{(\gamma-\alpha)/\alpha}[\lambda^{\gamma/\alpha}(m) + 2^{(1-\alpha)/\alpha}\Lambda^{\gamma/\alpha}(m-1)]; \tag{3.3}$$

here m is the integer part of $\frac{1}{2}(n+2)$.

We note that the case $\lambda(n)\equiv 1$ is of special interest. Then, as it follows by (3.3), $\Lambda(n)=O(n^{(\gamma+1-2\alpha)/\gamma})$. It remains open, whether these estimates are exact or not, as far as the asymptotic order of magnitude as $n\to\infty$ is concerned.

On the other hand, we have to remark that the case $0<\alpha<1$ and $\gamma\geqq 2$ is somewhat restricted in its application since, if condition (3.1) with $\lambda(n)\equiv 1$ and $g(F_{b,n})=A_{b,n}^2$ were met in this case, we would have $E(S_{b,n}^2)\leqq A_{b,n}^{2\delta}$ for a $\delta<1$, an unrealistic condition in many applications.

Proof of Theorem 5. The proof runs along the same lines as that of Theorem 4. Besides, we will apply the following elementary inequality:

$$(u+v)^s \leqq 2^{s-1}(u^s+v^s) \quad \text{if } u\geqq 0,\ v\geqq 0,\ s\geqq 1. \tag{3.4}$$

We start with the inequality (2.8) obtained in the proof of Theorem 4. Applying (3.4) with $s=\gamma/\alpha\geqq 1$, we get that

$$[E(M_{b,n}^\gamma)]^{1/\alpha} \leqq 2^{(\gamma-\alpha)/\alpha}\{[E\,|S_{b,m}|^\gamma]^{1/\alpha} + [E(M_{b,m-1}^\gamma) + E(M_{b+m,n-m}^\gamma)]^{1/\alpha}\},$$

where $n>1$ is a given integer. Applying (3.4) once more but now with $s=1/\alpha$, it follows that

$$[E(M_{b,n}^{\gamma})]^{1/\alpha} \leqq 2^{(\gamma-\alpha)/\alpha} \{[E\,|S_{b,m}|^{\gamma}]^{1/\alpha} + 2^{(1-\alpha)/\alpha}([E(M_{b,m-1}^{\gamma})]^{1/\alpha} + [E(M_{b+m,n-m}^{\gamma})]^{1/\alpha})\}. \tag{3.5}$$

Suppose that the conclusion (3.2) of the theorem holds for $k<n$. Then we have

$$[E(M_{b,m-1}^{\gamma})]^{1/\alpha} \leqq \Lambda^{\gamma/\alpha}(m-1)\, g(F_{b,m-1})$$

and

$$[E(M_{b+m,n-m}^{\gamma})]^{1/\alpha} \leqq \Lambda^{\gamma/\alpha}(n-m)\, g(F_{b+m,n-m}) \leqq \Lambda^{\gamma/\alpha}(m-1)\, g(F_{b+m,n-m}),$$

since $n=2m-2$ or $2m-1$. By (1.1), hence it follows that

$$[E(M_{b,m-1}^{\gamma})]^{1/\alpha} + [E(M_{b+m,n-m}^{\gamma})]^{1/\alpha} \leqq \Lambda^{\gamma/\alpha}(m-1)\, g(F_{b,n}).$$

In view of (3.1) we have

$$[E\,|S_{b,m}|^{\gamma}]^{1/\alpha} \leqq \lambda^{\gamma/\alpha}(m)\, g(F_{b,m}) \leqq \lambda^{\gamma/\alpha}(m)\, g(F_{b,n}).$$

These last two estimates when put into (3.5) show that

$$[E(M_{b,n}^{\gamma})]^{1/\alpha} \leqq 2^{(\gamma-\alpha)/\alpha} \{\lambda^{\gamma/\alpha}(m) + 2^{(1-\alpha)/\alpha} \Lambda^{\gamma/\alpha}(m-1)\}\, g(F_{b,n}),$$

which is the estimate (3.2), as desired. Since the conclusion (3.2) is obviously true for $n=1$, it is true by induction for all $n=1, 2, \ldots$. This proves Theorem 5.

Before turning to the applications, we make a remark on the validity of our results. Viewing the proofs, it is striking that we use no full power of a probability space. In fact, Minkowski's inequality was applied only, which is available in any (not necessarily finite, even not σ-finite) measure space. Thus, e.g., Theorems 1 and 4 can be stated in a more general form as follows.

Theorem 1'. *Let $(X, \mathscr{A}, \mu)$ be a measure space. Suppose that there exists a function $g(F_{b,n})$ satisfying* (1.1) *such that*

$$\int_X |S_{b,n}|^{\gamma}\, d\mu \leqq g^{\alpha}(F_{b,n}) \qquad (\text{all } b \geqq 0,\ n \geqq 1),$$

where $\gamma>0$ and $\alpha>1$. Then

$$\int_X M_{b,n}^{\gamma}\, d\mu \leqq C_{\gamma,\alpha}\, g^{\alpha}(F_{b,n}) \qquad (\text{all } b \geqq 0,\ n \geqq 1).$$

Theorem 4'. *Let $(X, \mathscr{A}, \mu)$ be a measure space. Suppose that there exist a function $g(F_{b,n})$ satisfying* (1.1), *and a positive and non-decreasing function $\lambda(n)$ such that*

$$\int_X |S_{b,n}|^{\gamma}\, d\mu \leqq \lambda^{\gamma}(n)\, g(F_{b,n}) \qquad (\text{all } b \geqq 0,\ n \geqq 1),$$

where $\gamma>1$. Let $\Lambda(n)$ be defined by (2.3). *Then*

$$\int_X M_{b,n}^{\gamma}\, d\mu \leqq \Lambda^{\gamma}(n)\, g(F_{b,n}) \qquad (\text{all } b \geqq 0,\ n \geqq 1).$$

4. Applications: Strong Convergence and Complete Convergence

Now let us examine some of the consequences of Theorem 1. The consequences of Theorem 3 (which coincides with Theorem G if $\gamma=2$ and $\alpha=1$) are discussed by Serfling [13]. We shall concern the following convergence properties of S_n under moment restrictions of type (1.2): $S_n/A_n^2 \to 0$ *(the strong law of large numbers)*, or more generally $S_n/b_n \to 0$ with probability 1, where $A_n = \left(\sum_{k=1}^{n} a_k^2\right)^{\frac{1}{2}} \to \infty$ $(n \to \infty)$, $\{a_n\}$ and $\{b_n\}$ are given sequences of numbers; furthermore,

$$\sum_{n=1}^{\infty} d_n P\left[\sup_{k \geq n} \frac{|S_k|}{A_k^2} \geq \varepsilon\right] \tag{4.1}$$

and

$$\sum_{n=1}^{\infty} P\left[\frac{|S_n|}{c_n} \geq \varepsilon\right] \tag{4.2}$$

converge for every $\varepsilon > 0$, where $\{c_n\}$ and $\{d_n\}$ are sequences of constants.

Condition (4.1) represents information regarding the *rate of the convergence* in the strong law of large numbers. The larger the d_n's may be chosen, the sharper is the result stated by (4.1). Condition (4.2) asserts that the sequence $\{S_n/c_n\}$ *converges completely to zero* in the sense of Hsu and Robbins [5]. The smaller the c_n's may be chosen, the sharper is the statement. By the Borel-Cantelli lemma, complete convergence implies strong convergence.

Properties (4.1), (4.2), or the like will be obtained as consequences of restrictions imposed upon the absolute γ-th moments, for some $\gamma > 2$, of sums $S_{b,n} = \sum_{k=b+1}^{b+n} \xi_k$. More precisely, throughout this Section we shall assume that (1.2) is satisfied with $\gamma > 2$, $\alpha = \frac{1}{2}\gamma$, and $g(F_{b,n}) = A_{b,n}^2 = \sum_{k=b+1}^{b+n} a_k^2$, where $\{a_k\}$ is a given sequence of numbers. That is, we shall assume that $\{\xi_k\}$ satisfies the moment inequality

$$E\,|S_{b,n}|^{\gamma} \leq C_{\gamma} A_{b,n}^{\gamma} \qquad (\text{all } b \geq 0,\ n \geq 1), \tag{4.3}$$

where $\gamma > 2$. By virtue of Theorem 1, then we have

$$E(M_n^{\gamma}) \leq C_{\gamma}^* A_n^{\gamma} \qquad (A_n = A_{0,n}), \tag{4.4}$$

where $C_{\gamma}^* = C_{\gamma} C_{\gamma,\gamma/2}$. This enables us to derive bounds on the tail distribution of M_n, which play a crucial role in the proofs given below. Applying Markov's inequality, (4.4) gives that

$$P[M_n \geq y] \leq C_{\gamma}^* \left(\frac{A_n}{y}\right)^{\gamma} \tag{4.5}$$

for any $y > 0$.

Beside (4.5), in proving the convergence of series (4.1), we make use of the convergence part of the following assertion, applied widely in the theory of numerical series: *Let $d_k \geq 0$ be the terms of a divergent series with partial sums D_n. Then* *the series*

$$\sum_n \frac{d_n}{D_n(\log D_n)^{1+\delta}}$$

converges or diverges according as $\delta>0$ *or* $\delta\leqq 0$.

We note that the results below are proved, on the base of the probability inequality (4.5), by adaptation of more or less standard arguments [2, 6, and 10]. More exactly, Theorems 6–9 generalize Theorems 3.1, 5.1, 5.3 and the relation (6.5) of Serfling [13].

Theorem 6. *Let* $\gamma>2$. *Suppose that* (4.3) *is satisfied and*

$$A_n\to\infty \qquad (n\to\infty). \tag{4.6}$$

Then, for each $\delta>0$, *we have*

$$P[S_n=o\{A_n(\log A_n)^{1/\gamma}(\log\log A_n)^{(1+\delta)/\gamma}\}]=1. \tag{4.7}$$

Proof. Imitating well-known techniques of argument (e.g., Lamperti [6]), put

$$\lambda(n)=A_n(\log A_n^2)^{1/\gamma}(\log\log A_n^2)^{(1+\delta)/\gamma}.$$

Inequality (4.5) gives that

$$P[M_n\geqq\lambda(n)]\leqq\frac{C_\gamma^*}{\log A_n^2(\log\log A_n^2)^{1+\delta}}. \tag{4.8}$$

Now we define a sequence of positive integers $n_1\leqq n_2\leqq\cdots$ in the following way:

$$A_{n_j-1}^2\leqq 2^j<A_{n_j}^2 \qquad (j=1,2,\ldots). \tag{4.9}$$

This is possible in virtue of (4.6), and obviously $n_j\to\infty$ as $j\to\infty$.

On account of (4.8) and (4.9) we find that

$$\sum_j{}' P[M_{n_j}\geqq\lambda(n_j)]\leqq\sum_j{}'\frac{C_\gamma^*}{\log A_{n_j}^2(\log\log A_{n_j}^2)^{1+\delta}}\leqq\sum_{j=2}^{\infty}\frac{C_\gamma^*}{j(\log j)^{1+\delta}}<\infty,$$

where $\sum_j{}'$ means that the summation is taken only once for equal n_j's. Hence, by the Borel-Cantelli lemma, with probability 1 the inequality

$$M_{n_j}<\lambda(n_j)$$

holds for all j large enough. It is evident, by repeating the above argument with (n_j-1)'s instead of n_j's, that with probability 1

$$M_{n_j-1}<\lambda(n_j-1)$$

for all k large enough, too.

Now, for $n_j\leqq n<n_{j+1}$, we have

$$\lambda(n)\geqq\lambda(n_j) \quad\text{and}\quad |S_n|\leqq M_{n_{j+1}-1},$$

and thus, with probability 1

$$\frac{|S_n|}{\lambda(n)} \leqq \frac{M_{n_{j+1}-1}}{\lambda(n_j)} \leqq \frac{\lambda(n_{j+1}-1)}{\lambda(n_j)} \tag{4.10}$$

for all n large enough. Since by (4.9) the right-hand side of (4.10) is bounded as $j \to \infty$, it follows that

$$P[S_n = O\{A_n (\log A_n^2)^{1/\gamma} (\log\log A_n^2)^{(1+\delta)/\gamma}\}] = 1.$$

Taking into account that δ is an arbitrarily small positive number, this immediately yields (4.7), which was to be proved.

We note that the conclusion (4.7) improves as γ increases. By letting $\gamma \to \infty$, we find that

$$P[S_n = o\{A_n (\log A_n)^\delta\}] = 1$$

for each $\delta > 0$.

It is obvious that (4.7) implies the strong law of large numbers, i.e., $S_n/A_n^2 \to 0$ as $n \to \infty$ with probability 1. The following result characterizes the rate of convergence.

Theorem 7. *Let* $\gamma > 2$. *Suppose that* (4.3) *is satisfied and*

$$A_n \to \infty \quad (n \to \infty) \quad \text{and} \quad a_n^2 \leqq q A_n^2 \quad (n \geqq n_0), \tag{4.11}$$

where $0 < q < 1$. *Then, for each* $\delta > 0$ *and* $\varepsilon > 0$, *we have*

$$\sum_n \frac{a_n^2 A_n^{\gamma-2}}{(\log A_n)^{1+\delta}} P\left[\sup_{k \geqq n} \frac{|S_k|}{A_k^2} \geqq \varepsilon\right] < \infty. \tag{4.12}$$

Proof. Let $p = (1-q)^{-1}$. It is clear that $p > 1$. We begin with proving that (4.11) implies the existence of a strictly increasing sequence $\{n_j\}$ of positive integers such that

$$p^j \leqq A_{n_j}^2 < p^{j+1} \tag{4.13}$$

for all j large enough. Otherwise, for infinitely many n's, we have

$$A_n^2 < p^j \quad \text{and} \quad A_{n+1}^2 \geqq p^{j+1}$$

with suitable j's. Hence

$$\frac{a_{n+1}^2}{A_{n+1}^2} = 1 - \frac{A_n^2}{A_{n+1}^2} > 1 - \frac{1}{p} = q$$

for infinitely many n's, which contradicts (4.11).

By a remark made above, to prove (4.12) it is enough to show that

$$P_n = P\left[\sup_{k \geqq n} \frac{|S_k|}{A_k^2} \geqq \varepsilon\right] \leqq \frac{C}{A_n^\gamma} \tag{4.14}$$

for all n large enough. To this effect, let $j_0=j_0(n)$ be defined by $n_{j_0}<n\leqq n_{j_0+1}$. It is obvious that

$$P_n\leqq\sum_{j=j_0}^{\infty}P\left[\max_{n_j<k\leqq n_{j+1}}\frac{|S_k|}{A_k^2}\geqq\varepsilon\right]\leqq\sum_{j=j_0}^{\infty}P[M_{n_{j+1}}\geqq\varepsilon A_{n_j}^2].$$

The use of (4.5) on the right-hand side yields

$$P_n\leqq C_\gamma^*\,\varepsilon^{-\gamma}\sum_{j=j_0}^{\infty}\frac{A_{n_{j+1}}^{\gamma}}{A_{n_j}^{2\gamma}}\leqq C_\gamma^*\,p^\gamma\,\varepsilon^{-\gamma}\sum_{j=j_0}^{\infty}A_{n_j}^{-\gamma}, \tag{4.15}$$

since by (4.13)

$$A_{n_{j+1}}^2\leqq p^2 A_{n_j}^2.$$

The series on the right of (4.15) is convergent due to $p>1$. Putting

$$C_1=C_\gamma^*\,p^\gamma\,\varepsilon^{-\gamma}(1-p^{-\gamma/2})^{-1},$$

we find that

$$P_n\leqq C_1\,p^{-\gamma j_0/2}\leqq C_1\,p^\gamma A_n^{-\gamma},$$

in accordance with (4.14). Thus Theorem 7 is proved.

We note that if (4.3) holds for γ's arbitrarily large, then we have a conclusion substantially better than (4.12). Namely, in this case we have

$$\sum_n a_n^2 A_n^\alpha P\left[\sup_{k\geqq n}\frac{|S_k|}{A_k^2}\geqq\varepsilon\right]<\infty$$

for any choice of α and $\varepsilon>0$.

Turning now to convergence rates corresponding to the law given by Theorem 6, we can assert

Theorem 8. *Let $\gamma>2$. Suppose that* (4.3) *and* (4.11) *are satisfied. Then, for any choice of α and β satisfying*

$$0\leqq\beta<\alpha\gamma-1, \tag{4.16}$$

we have

$$\sum_n\frac{a_n^2}{A_n^2(\log A_n)^{1-\beta}}P\left[\sup_{k\geqq n}\frac{|S_k|}{A_k(\log A_k)^\alpha}\geqq 1\right]<\infty. \tag{4.17}$$

Proof. Consider a strictly increasing sequence $\{n_j\}$ of positive integers defined by (4.13). The existence of such a sequence is ensured by (4.11), a condition appearing among the assumptions of the theorem.

For a given n large enough, define the integer $j_0=j_0(n)$ such that $n_{j_0}<n\leqq n_{j_0+1}$. We obviously have

$$P\left[\sup_{k\geqq n}\frac{|S_k|}{A_k(\log A_k^2)^\alpha}\geqq 1\right]\leqq\sum_{j=j_0}^{\infty}P\left[\max_{n_j<k\leqq n_{j+1}}\frac{|S_k|}{A_k(\log A_k^2)^\alpha}\geqq 1\right]. \tag{4.18}$$

Applying (4.5), the series on the right-hand side of (4.18) is bounded from above by the series

$$\sum_{j=j_0}^{\infty} \frac{C_\gamma^* A_{n_{j+1}}^\gamma}{A_{n_j}^\gamma (\log A_{n_j}^2)^{\alpha\gamma}}.$$

The same use of (4.5) and (4.13) as in the proof of Theorem 7 now yields the convergence of (4.17). The proof of Theorem 8 is ready.

We note that the least restriction on α, namely $\alpha>0$, occurs if γ may be chosen arbitrarily large. In this case, the relation (4.17) holds for any choice of $\alpha>0$ and $0<\beta<1$.

Finally, we consider the question of norming S_n suitably for S_n/c_n to converge completely to zero. The inequality (4.5) immediately provides the following, slightly stronger conclusion.

Theorem 9. *Let* $\gamma>2$. *Under conditions* (4.3) *and* (4.6) *the sequence*

$$\{a_n^{2/\gamma} M_n / A_n^{(\gamma+2)/\gamma} (\log A_n)^{(1+\delta)/\gamma}\}$$

converges completely to zero for each $\delta>0$.

As a particular case, consider a sequence $\{\varphi_k\}$ of *weakly multiplicative rv's of order r*, i.e., we assume that

$$\sum_{1\leqq k_1<k_2<\ldots<k_r} E^2\{\varphi_{k_1}\varphi_{k_2}\cdots\varphi_{k_r}\}<\infty, \tag{4.19}$$

where the summation is extended over all integers satisfying only the condition $1\leqq k_1<k_2<\cdots<k_r$ and $r\geqq 4$ is an even integer. This is a generalization of the concept of *multiplicativity of order r* defined by

$$E\{\varphi_{k_1}\varphi_{k_2}\cdots\varphi_{k_r}\}=0 \qquad (1\leqq k_1<k_2<\cdots<k_r). \tag{4.20}$$

The condition (4.20) is stronger than (4.19). Even the former includes the case of a sequence of martingale differences and the case of mutually independent rv's (when the expectations in (4.20) exist), etc.

In [8] we proved that (4.3) with $\gamma=r$ is valid for any sequence of weakly multiplicative rv's of order r, whose r-th moments are uniformly bounded. More precisely, the following result holds. (See Theorem 1 there.)

Theorem I. *Let r be an even integer,* $r\geqq 4$. *Let* $\{\varphi_k\}$ *be a sequence of rv's such that* (4.19) *and*

$$E(\varphi_k^r)\leqq K(<\infty) \qquad (k=1,2,\ldots) \tag{4.21}$$

are satisfied, Then, for every sequence $\{a_k\}$ *and for every integer n, we have*

$$E\left(\sum_{k=b+1}^{b+n} a_k\varphi_k\right)^r \leqq C_r A_{b,n}^r \qquad (\textit{all } b\geqq 0,\ n\geqq 1).$$

Hence, via Theorems 6–9, we obtain the following corollaries.

Corollary 1. *Let $r \geqq 4$ be an even integer and let $\{\varphi_k\}$ be a sequence of rv's satisfying* (4.19) *and* (4.21). *Let $\{a_k\}$ be a sequence of numbers with* (4.6). *Then, for each $\delta > 0$,*

$$P\left[\sum_{k=1}^{n} a_k \varphi_k = o\{(A_n (\log A_n)^{1/r} (\log\log A_n)^{(1+\delta)/r}\}\right] = 1.$$

Corollary 2. *Let $\{\varphi_k\}$ be a sequence of rv's satisfying* (4.19) *and* (4.21) *for an even integer $r \geqq 4$. Let $\{a_k\}$ be a sequence of numbers satisfying* (4.11). *Then, for each $\delta > 0$ and $\varepsilon > 0$, we have*

$$\sum_n \frac{a_n^2 A_n^{r-2}}{(\log A_n)^{1+\delta}} P\left[\sup_{l \geqq n} \frac{1}{A_l^2} \left|\sum_{k=1}^{l} a_k \varphi_k\right| \geqq \varepsilon\right] < \infty.$$

Corollary 3. *Under the same conditions as in Corollary* 2, *we have*

$$\sum_n \frac{a_n^2}{A_n^2 (\log A_n)^{1-\beta}} P\left[\sup_{l \geqq n} \frac{1}{A_l^2 (\log A_l)^{\alpha}} \left|\sum_{k=1}^{l} a_k \varphi_k\right| \geqq 1\right] < \infty,$$

provided α and β satisfy (4.16).

Corollary 4. *Under the same conditions as in Corollary* 1, *we have*

$$\sum_n P\left[\frac{a_n^{2/r}}{A_n^{(r+2)/r} (\log A_n)^{(1+\delta)/r}} \max_{1 \leqq l \leqq n} \left|\sum_{k=1}^{l} a_k \varphi_k\right| \geqq \varepsilon\right] < \infty$$

for each $\varepsilon > 0$.

We note that Corollaries 1–4 for a sequence of multiplicative rv's of finite order in the special case $a_1 = a_2 = \cdots = 1$ were proved by Serfling [11]. Corollaries 1 and 3, under somewhat more restricted conditions stipulated on $\{a_k\}$, were proved by the author [9].

References

1. Billingsley, P.: Convergence of Probabilty Measures. New York: Wiley 1968
2. Doob, J.L.: Stochastic Processes. New York: Wiley 1953
3. Erdös, P.: On the convergence of trigonometric series. J. Math. and Phys. Massachusetts Institute of Technology **22**, 37–39 (1943)
4. Gapoškin, V.F.: Lacunary series and independent functions. (Russian) Uspehi Mat. Nauk **21**, 3–82 (1966)
5. Hsu, P.L., Robbins, H.: Complete convergence and the law of large numbers. Proc. Nat. Acad. Sci. USA **33**, 25–31 (1947)
6. Lamperti, J.: Probability. New York: Benjamin 1966
7. Móricz, F.: The generalization of some classical inequalities in the theory of orthogonal series. (Russian) Mat. Zametki **17**, 219–230 (1975)
8. Móricz, F.: On the convergence properties of weakly multiplicative systems. Acta Sci. Math. (Szeged) (to appear)
9. Móricz, F.: The law of the iterated logarithm and related results for weakly multiplicative systems. Analysis Math. (to appear)

10. Révész, P.: The Laws of Large Numbers. New York: Academic Press 1968
11. Serfling, R.J.: Probability inequalities and convergence properties for sums of multiplicative random variables. Technical Report, Florida State University (1969)
12. Serfling, R.J.: Moment inequalities for the maximum comulative sum. Ann. Math. Statist. **41**, 1227–1234 (1970)
13. Serfling, R.J.: Convergence properties of S_n under moment restrictions. Ann. Math. Statist. **41**, 1235–1248 (1970)
14. Tjurnpü, H.: On the generalization of a theorem of S.B. Stečkin. (Russian) Acta Sci. Math. (Szeged) **36**, 369–374 (1974)
15. Zygmund, A.: Trigonometric Series, Vol. II. Cambridge: University Press 1959

Received October 27, 1975; in revised form March 3, 1976