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Path Properties of Processes with Independent and Interchangeable Increments

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In [14] random processes (r.pr.) with interchangeable increments (ich.incr.) were examined with respect to canonical representations and convergence in distribution. The main purpose of the present paper is to investigate the path properties of such pr. Apart from the extensive litterature on independent (ind.) incr.pr. and from the results in [14], quite few such properties seem to be known (and then only in particular cases), including Bühlmann's proof [4] of the pointwise a.s. continuity, Takacs' generalizations [22] of the "ballot" theorem and Hagberg's extensions [13] of Sparre Andersen's combinatorial results.

As shown in [4, 14], ich.incr.pr. on infinite intervals are merely mixtures of pr. with stationary ind.incr. On finite intervals, however, the class of ich.incr.pr. is much more extensive. In fact (Th. 2.1 in [14]), any pr. of this type on [0, 1] which is continuous in probability is equivalent to a pr. in $D[0,1]$ of the form

$$X(t)=X(0)+\alpha t+\sigma B(t)+\sum_j \beta_j[1_+(t-\tau_j)-t], \qquad t\in[0,1], \tag{0.1}$$

in the sense of a.s. uniform convergence, where

(i) $\alpha, \sigma, \beta_1, \beta_2, \ldots$ are r.v. with $\sum \beta_j^2<\infty$ a.s.,

(ii) B is a Brownian bridge on $[0,1]$,

(iii) $\tau_1, \tau_2, \ldots$ are ind. and uniformly distributed on $[0,1]$,

(iv) the three groups (i)–(iii) of random elements are ind.;

and conversely, any pr. of this form has ich.incr. and almost all its paths lie in $D[0,1]$. Despite its greater generality, it will be shown here that many sample path properties known for ind.incr.pr. generalize in a natural way to this larger class of pr. As examples, we shall consider in §§ 2, 3, 5 extensions of results on rates of growth, Hausdorff measure functions and variation due to Ȟinčin [15], Cogburn, Tucker [5], Blumenthal, Getoor [3], Fristedt, Pruitt [8, 10, 11], Millar [17, 18] and others. Our main tools will be the approximation theorems in § 1.

As implicit in [5, 12], variational results are essentially limit theorems for certain random measures, and they are here explicitly stated as such. This approach leads us to improvements in the ind.incr. case, and in particular we are able in §§ 4–5 to relax the symmetry assumptions of Millar ([17], pp. 60, 68, [18], p. 324). Furthermore, it indicates a close relationship between "weak" and "strong" (or a.s.) convergence theory. This similarity is explored further in § 6, the results of which provide a link between ergodicity and variation.

The paper is expected to be read in conjunction with [14], from which we take over notation and terminology. For brevity, let us further introduce some classes of functions:

$$\mathscr{C} = \{f\colon R \to R_+,\, f \text{ continuous},\, f(0)=0,\, f \not\equiv 0\},$$
$$\mathscr{C}_s = \{f \in \mathscr{C}\colon f \text{ even}\},$$
$$\mathscr{C}_1 = \{f \in \mathscr{C}\colon \exists\, \varepsilon > 0;\, f \text{ concave in } (-\varepsilon, 0), (0, \varepsilon)\},$$
$$\mathscr{C}_2 = \{f \in \mathscr{C}_s\colon f'(0)=0,\, \exists\, \varepsilon>0;\, f \text{ convex},\, f(\sqrt{\cdot}) \text{ concave in } (0,\varepsilon)\},$$
$$\mathscr{C}_2' = \{f \in \mathscr{C}_2\colon \exists\, \varepsilon, c > 0;\, f^c \text{ subadditive in } (0,\varepsilon)\},$$
$$\mathscr{C}_3 = \{f \in \mathscr{C}_s\colon f \uparrow \text{on } R_+,\, f(ax) = O(f(x)) \text{ as } x \to 0,\, a > 1\}.$$

Note that for f in $\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_2'$ or $\mathscr{C}_3$, there exists a well-defined continuous inverse f^{-1} on some interval $[0, \varepsilon]$ into R_+. For any finite partition $\Pi = \{0 = t_1 \leqq \cdots \leqq t_k = s\}$ of $[0, s]$, write $|\Pi|_2^2 = \sum_j (t_j - t_{j-1})^2$, $|\Pi|_\infty = \max_j (t_j - t_{j-1})$. Say that $\Pi_1, \Pi_2, \ldots$ are *nested* if Π_{n+1} is a refinement of Π_n, $n \in N$. For any $f\colon [0, s] \to R$, define $\Pi f \in \mathfrak{N}(R)$ by having its unit atoms at $f(t_j) - f(t_{j-1})$, $j = 1, \ldots, k$. Abbreviate $l = 1 \vee |\log|\cdot||$, $l_2 = l \circ l$, $R_+' = R_+ \setminus \{0\}$.

Let us finally point out that most results in this paper (as well as in [14]) carry over with obvious changes to pr. taking values in R^k, $k > 1$.

List of Abbreviations

r.v.	random variable,
r.e.	random element,
r.m.	random measure,
r.pr.	random process,
pr.	process,
ich.	interchangeable,
ind.	independent,
incr.	increment,
can.	canonical,
lcscH	locally compact second countable Hausdorff.

1. Approximation

Throughout §§ 1–3, let X be an ich. incr. pr. in $D_0[0, 1]$ with can. r.e. α, σ, β. Define

$$\rho_X = \inf\{c > 0\colon \beta |h|^c < \infty\}.$$

If X is the restriction of an ich. incr. pr. in $D_0[0, \infty)$ with can. r.e. $\gamma_\varepsilon, \sigma, \lambda$, then clearly $\rho_X = \inf\{c > 0\colon \lambda |h|^c < \infty\}$ (cf. Th. 5.1 in [14]), so ρ_X generalizes the largest index defined by Blumenthal and Getoor [3] for pr. with stationary ind. incr.

To simply formulations, assume throughout that the basic probability space is rich enough to support any randomizations we need. In the following two theorems we state more than is actually needed in the present paper.

Theorem 1.1. *There exist ich. incr. pr. Y in $D_0[0, \infty)$ and Z in $D_0[0, 1]$ with can. r.e. $(\gamma_\infty, \sigma', \lambda) = (\alpha, \sigma, \beta)$ and $(\alpha', 0, \beta')$ respectively such that $X = Y + Z$ on $[0, 1]$ and*

(i) $\beta' |h|\, l^{-c} < \infty$ *a.s.*, $c > \frac{1}{2}$,

(ii) $\beta' |h| < \infty$ *a.s. whenever* $\beta^2 l^c < \infty$ *a.s. for some* $c > 1$,

(iii) $\rho_Z \leqq \rho_X (1 + \frac{1}{2}\rho_X)^{-1}$ *a.s.*

Proof. We may clearly assume that α, σ, β are non-random. Let ϑ be $N(0,1)$ ind. of X, and note that $B + \vartheta h$ is a Brownian motion on $[0,1]$. Taking $\alpha h + \sigma(B + \vartheta h)$ to Y and $-\sigma \vartheta h$ to Z, we may thus assume from now on that (0.1) holds with $\alpha = \sigma = 0$. Separating positive and negative β_j and noting that the assertions are trivially true when $\beta R < \infty$, we may also assume that $e^{-1} \geqq \beta_1 \geqq \beta_2 \geqq \cdots > 0$. Let ξ be a Poisson pr. on N ind. of X and with intensity $\sum_{j \in N} \delta_j$, and let $\xi_1 \leqq \xi_2 \leqq \cdots$ be its unit atom positions. Define

$$\beta'_j = \beta_j - \beta_{\xi_j}, \ j \in N, \qquad \alpha' = \sum_j \beta'_j, \qquad \beta' = \sum_j \delta_{\beta'_j},$$

$$Y(t) = \sum_j [\beta_{\xi_j} 1_+(t - \tau_j) - \beta_j t] = \sum_j \beta_{\xi_j} [1_+(t - \tau_j) - t] - \alpha' t.$$

Since $\sum_j \delta_{\beta_{\xi_j}}$ is a Poisson pr. on R'_+ with intensity β, it follows by [7] and [14] that α' exists a.s. and that Y has stationary ind. incr. with can. r.e. $(0, 0, \beta)$. Furthermore,

$$Z(t) = X(t) - Y(t) = \alpha' t + \sum_j \beta'_j [1_+(t - \tau_j) - t],$$

so Z has ich. incr. and can. r.e. $(\alpha', 0, \beta')$.

For any $c > \frac{1}{2}$ we get by convexity

$$\beta' |h| \, l^{-c} = \sum_j |h| \, l^{-c}(\beta_j - \beta_{\xi_j}) \leqq \sum_j \left| \beta_j |\log \beta_j|^{-c} - \beta_{\xi_j} |\log \beta_{\xi_j}|^{-c} \right|.$$

Writing $\hat{\beta}_j = \beta_j |\log \beta_j|^{-c}$, $\hat{\beta} = \sum_j \delta_{\hat{\beta}_j}$, it is seen that $\beta_j \sim \hat{\beta}_j |\log \hat{\beta}_j|^c$, so $\beta^2 R < \infty$ implies $\hat{\beta}^2 l^{2c} < \infty$, and hence (i) follows from (ii). To prove (ii), assume that $\beta^2 l^c < \infty$ for some $c > 1$. Choose arbitrary $a \in (0, c-1)$, put $m_n = [n^2 (\log n)^{-a}]$, $n \in N$, and define the measures $\nu^+, \nu^- \in \mathfrak{N}(N)$ by

$$\nu^+ = \sum_j \delta_j + \sum_n \delta_{m_n}, \qquad \nu^- = \sum_j \delta_j - \sum_n \delta_{m_n}$$

with unit atom positions $\{\nu_j^+\}$ and $\{\nu_j^-\}$ in non-decreasing order. Since $\delta_{m_n}[1,k] = 1$ iff $n^2 (\log n)^{-a} < k + 1$, and further $n^2 (\log n)^{-a} \sim k$ iff $n \sim 2^{-a/2} k^{\frac{1}{2}} (\log k)^{a/2}$, we obtain for large k

$$\nu^-[1,k] \leqq k - k^{\frac{1}{2}} (\log k)^{a/3}, \qquad k + k^{\frac{1}{2}} (\log k)^{a/3} \leqq \nu^+[1,k].$$

Hence we get for large k, by the Hartman-Wintner law of the iterated logarithm, $\nu^-[1,k] \leqq \xi[1,k] \leqq \nu^+[1,k]$, or equivalently $\nu_j^+ \leqq \xi_j \leqq \nu_j^-$ for large j. Thus

$$\beta_j - \beta_{\nu_j^+} \leqq \beta_j - \beta_{\xi_j} \leqq \beta_j - \beta_{\nu_j^-}, \qquad j \text{ large},$$

so it suffices to prove the inequalities

$$\sum_j (\beta_{\nu_j^+} - \beta_j) < \infty, \qquad \sum_j (\beta_j - \beta_{\nu_j^-}) < \infty.$$

Expressing these sums as double sums in the differences $\beta_j - \beta_{j+1} > 0$ and reversing the order of summation, it is easily seen that both sums equal $\sum_n \beta_{m_n}$.

We shall make use of the inequality

$$x\,y \leqq x^2\,|\log x|^c + y^2\,|\log y|^{-c}, \tag{1.1}$$

which holds for any fixed $c \in R$ provided $x, y > 0$ are sufficiently small. To prove (1.1), note that $t \sim s\,|\log s|^c$ iff $s \sim t\,|\log t|^{-c}$, so for any $r \in (1, 2)$ and for small x and y

$$\begin{aligned} r\,x\,y &\leqq 2\int_0^x s\,|\log s|^c\,ds + 2\int_0^y t\,|\log t|^{-c}\,dt \\ &\sim \int_0^x \frac{d}{ds}\{s^2\,|\log s|^c\}\,ds + \int_0^y \frac{d}{dt}\{t^2\,|\log t|^{-c}\}\,dt \\ &= x^2\,|\log x|^c + y^2\,|\log y|^{-c}. \end{aligned}$$

Now put in (1.1) $x = x_n$, $y = y_n$, where

$$x_n = n^{\frac{1}{2}}(\log n)^{-a/2}\,\beta_{m_n}, \qquad y_n = n^{-\frac{1}{2}}(\log n)^{a/2}, \qquad n \in N,$$

and note that

$$y_n^2\,|\log y_n|^{-c} \sim 2^c\,n^{-1}(\log n)^{-c+a}$$

is summable since $c - a > 1$. To estimate $x_n^2\,|\log x_n|^c$, write $f(s) = s^2\,|\log s|^{-a}$ and note that

$$m_n - m_{n-1} \geqq f(n) - f(n-1) - 1 \sim f'(n) \sim 2n(\log n)^{-a},$$

so by monotonity for large $n_0 \in N$,

$$\begin{aligned} \infty > \sum_j \beta_j^2\,|\log \beta_j|^c &\geqq \sum_n (m_n - m_{n-1})\,\beta_{m_n}^2\,|\log \beta_{m_n}|^c \\ &\geqq \sum_{n > n_0} n(\log n)^{-a}\,\beta_{m_n}^2\,|\log \beta_{m_n}|^c. \end{aligned} \tag{1.2}$$

In particular it follows that $x_n \to 0$, so for large n

$$\begin{aligned} x_n^2\,|\log x_n|^c &= n(\log n)^{-a}\,\beta_{m_n}^2\,|\log\{n^{\frac{1}{2}}(\log n)^{-a/2}\,\beta_{m_n}\}|^c \\ &\leqq n(\log n)^{-a}\,\beta_{m_n}^2\,|\log \beta_{m_n}|^c, \end{aligned}$$

which is summable by (1.2). Therefore by (1.1), $\sum_n \beta_{m_n} = \sum_n x_n\,y_n < \infty$, and the proof of (ii) is complete.

To prove (iii) for $\rho_X < 2$, let $p \in (0, 2)$ be such that

$$\sum_j \beta_j^p < \infty. \tag{1.3}$$

For $a > \frac{1}{2}$, put $k_j^+ = j + [j^a]$, $k_j^- = j - [j^a]$, and let $q \in \left((2^{-1} + p^{-1})^{-1}, 1 \wedge p\right)$ be arbitrary. As in the proof of (ii) it suffices to show that

$$\sum_j (\beta_{k_j^-} - \beta_j)^q < \infty, \qquad \sum_j (\beta_j - \beta_{k_j^+})^q < \infty, \tag{1.4}$$

provided a is small enough. We may restrict our attention to the first sum in (1.4) since the second sum is always smaller. Write

$$m_n = [n^{1/(1-a)}], \qquad m_n' = [(n + 1/2)^{1/(1-a)}],$$

and note that

$$m_n - m_{n-1} \sim (1-a)^{-1} m_n^a \sim (1-a)^{-1}(m_n - k_{m_n}^-),$$

and similarly for $m'_n - m'_{n-1}$. For sufficiently large $j_0 \in N$ we therefore obtain

$$\sum_{j>j_0} (\beta_{k_j^-} - \beta_j)^q \leqq (1-a)^{-1} \{\sum_n m_n^a (\beta_{m_{n-1}} - \beta_{m_n})^q + \sum_n m_n'^a (\beta_{m'_{n-1}} - \beta_{m'_n})^q\}.$$

We shall prove that the first sum on the right is finite. The argument for the second sum is similar. Abbreviate $\beta_{m_n} = x_n$, $a/(1-a) = b$, and note that (1.3) implies

$$\sum_n n^b x_n^p < \infty \tag{1.5}$$

by monotonity. Writing $r = 1 + p(q^{-1} - 1)$, we get by Hölder's inequality

$$\sum_n n^b (x_{n-1} - x_n)^q \leqq \{\sum_n n^b x_n^{r-1} (x_{n-1} - x_n)\}^q \{\sum_n n^b x_n^p\}^{1-q}, \tag{1.6}$$

and here the second factor is finite by (1.5). Approximating sums by integrals and proceeding as in [6], p. 150, it is seen that the first factor in (1.6) is finite iff $\sum_n n^{b-1} x_n^r < \infty$. Applying Hölder's inequality once more, we get

$$\sum_n n^{b-1} x_n^r \leqq \{\sum_n n^d\}^{1-r/p} \{\sum_n n^b x_n^p\}^{r/p},$$

where

$$d = b - (1 - r/p)^{-1} = a/(1-a) - (q^{-1} - p^{-1})^{-1}.$$

Now $q^{-1} - p^{-1} < \frac{1}{2}$ by assumption, so $d < -1$ for a close to $\frac{1}{2}$, making the first factor finite. The second factor is finite by (1.5).

Theorem 1.2. *If $\beta_1 \geqq \beta_2 \geqq \cdots \geqq 0$ and if $\beta^2 l^c < \infty$ a.s. for some $c > 1$, there exist some ich. incr. pr. $Y_\pm$ in $D_0[0, \infty)$ with can. r. e. $(\gamma_\infty^\pm, \sigma, \lambda_\pm)$, where*

$$\lambda_\pm = \sum_{j=1}^{\infty} [1 \pm (j^{-1} l_2 j)^{\frac{1}{2}}] \delta_{\beta_j},$$

such that $Y_+ - X$ and $X - Y_-$ are a.s. non-decreasing on $[0, 1]$ apart from finitely many downward jumps. We may take $\gamma_0^\pm = \alpha - \beta h$ whenever $\sigma = 0$ and $\beta |h| < \infty$.

Proof. As in the last proof we may assume that (0.1) holds with $\alpha = \sigma = 0$ and non-random $\beta_1 \geqq \beta_2 \geqq \cdots \geqq 0$. Note that

$$\sum_{j=1}^{\infty} (j^{-1} l_2 j)^{\frac{1}{2}} \sim \int_e^n (x^{-1} l_2 x)^{\frac{1}{2}} dx \sim \int_e^n [(\log x)^{-1} + l_2 x](x l_2 x)^{-\frac{1}{2}} dx$$
$$= 2(n l_2 n)^{\frac{1}{2}}.$$

If β' is a Poisson pr. on R'_+ ind. of X and with intensity $\lambda_\pm$ and unit atom positions $\beta'_1 \geqq \beta'_2 \geqq \cdots \geqq 0$, it follows by the Hartman-Wintner law that $\beta'_j \geqq \beta_j$ (or $\beta'_j \leqq \beta_j$) for sufficiently large j. On the other hand, the arguments in the last proof show that $\sum_j |\beta'_j - \beta_j| < \infty$, so we may define $Y_\pm$ on $[0, 1]$ by

$$Y_\pm(t) = \sum_j \beta'_j [1_+(t - \tau_j) - t] + t \sum_j (\beta'_j - \beta_j) = \sum_j [\beta'_j 1_+(t - \tau_j) - \beta_j t].$$

To see that $Y_\pm$ have stationary ind. incr., let $\{c_j\}$ be the centering in [7] and note that

$$Y_\pm(t)=\sum_j [\beta'_j 1_+(t-\tau_j)-c_j t]+t\sum_j (c_j-\beta_j).$$

The asserted monotonity of $Y_\pm - X$ is seen from the formula

$$Y_\pm(t)-X(t)=\sum_j (\beta'_j-\beta_j)\,1_+(t-\tau_j), \qquad t\in[0,1],$$

while the last assertion follows from the fact that no centering is needed when $\beta|h|<\infty$.

2. Local and Uniform Rates of Growth

We first extend some results by Hinčin [15], Millar [17] and Fristedt [8].

Theorem 2.1. (i) $\limsup_{t\to 0} X(t)(2t\, l_2\, t)^{-\frac{1}{2}}=\sigma$ *a.s.*

(ii) *If $f\in\mathscr{C}_2$ and $\sigma=0$, $\beta f<\infty$ a.s., then*

$$\lim_{t\to 0} X(t)/f^{-1}(t\,|\log t|^c)=0, \qquad c>1, \ a.s. \tag{2.1}$$

(iii) *If $f\in\mathscr{C}_1$ and $\sigma=\beta R_-=\alpha-\beta h=0$ a.s., $\beta(h+f)<\infty$ a.s., then*

$$\lim_{t\to 0} X(t)/f^{-1}(t)=0 \quad a.s.$$

Proof. (ii) The argument for Th. 4.3 in [17] yields in case of ind. incr.

$$\lim_{t\to 0} t^{-1}|\log t|^{-c} f\circ X(t)=0 \quad \text{a.s.,}$$

and (2.1) follows from the fact that, by the convexity of $(f^{-1})^2$ near 0,

$$\lim_{a\downarrow 0}\limsup_{t\to 0} f^{-1}(a\,t)/f^{-1}(t)=0.$$

Now Th. 1.1 reduces the proof to the case $\rho_X\leqq 1$, since $\beta f<\infty$ clearly remains valid for the pr. Z there. Next the pr. $Y_\pm$ of Th. 1.2 satisfy $|X(t)|\leqq|Y_+(t)|\vee|Y_-(t)|$ for small $t>0$, so applying (2.1) to $Y_\pm$ completes the proof of (ii).

(i) Use the iterated logarithm for Brownian motion and Th. 1 in [14], and apply Th. 1.1 to reduce to the case $\rho_X\leqq 1$. Then apply (ii) with $f(t)\equiv|t|^{\frac{3}{2}}$.

(iii) This follows from Th. 1 in [8] and Th. 1.2 above.

By Th. 3 in [15], the normalization in (i) is essentially the best possible even for $\sigma=0$. As for (ii) and (iii), we have the following result in the converse direction (cf. Fristedt [8, 9] and Millar [17]).

Theorem 2.2. *If $f\in\mathscr{C}_3$ and $\beta f=\infty$ a.s., then*

$$\limsup_{t\to 0} |X(t)|/f^{-1}(t)=\infty \qquad a.s.$$

Proof. Since $f\in\mathscr{C}_3$, there exists for any $a>0$ some $b>0$ with $a f^{-1}(t)<f^{-1}(b\,t)$ for small $t>0$. The argument of Fristedt [9], p. 180, completes the proof.

We next extend some "lower function" results of Fristedt and Pruitt [10, 11]. (The remaining results of § 3 in [10] and §§ 3–4 in [11] may be similarly extended.) Define for $\beta\,|h|<\infty$

$$g(u)=\sum_j (1-e^{-u\beta_j}), \qquad u\geqq 0. \tag{2.2}$$

Theorem 2.3. *Let α, σ, β be non-random with $\sigma=\beta R_-=0, \alpha=\beta h<\infty, \beta R_+=\infty$.*

(i) *Suppose that $\lim_{\varepsilon\downarrow 0}\liminf_{u\to\infty} g(u)\,u^{-\varepsilon}=\infty$, and let $p>1$. Then for some non-random $a\in R'_+$*

$$\liminf_{t\to 0} X(t)\, g^{-1}(p\,t^{-1}\, l_2\, t)/l_2\, t=a \qquad a.s.$$

(ii) *Suppose that $\lim_{c\to\infty}\limsup_{u\to\infty} g(c\,u)/g(u)>d>1$, and put $p_1=p_2^{-1}=d^{\frac{1}{2}}$. Then for some non-random $a_1, a_2\in R'_+$*

$$\liminf_{t\to 0}\inf_{s\in[0,1-t]}\{X(s+t)-X(s)\}\, g^{-1}(p_1\, t^{-1}\,|\log t|)/|\log t|=a_1 \qquad a.s.,$$

$$\limsup_{t\to 0}\inf_{s\in[0,1-t]}\{X(s+t)-X(s)\}\, g^{-1}(p_2\, t^{-1}\,|\log t|)/|\log t|=a_2 \qquad a.s.$$

If $\lim_{c\to\infty}\liminf_{u\to\infty} g(c\,u)/g(u)=\infty$, then p_1 and p_2 may be replaced by any $p>0$.

Proof. To see that a, a_1, a_2 are non-random (possibly 0 or ∞), note that the events in Lemma 2 of [11] are invariant under finite permutations of the τ_j and apply the Hewitt-Savage $0-1$ law. To prove (i), define $\lambda_\pm$ and $Y_\pm$ as in Th. 1.2 and put

$$g_\pm(u)=\int_0^\infty (1-e^{-ur})\,\lambda_\pm(dr), \qquad u\geqq 0.$$

Then $Y_-(t)\leqq X(t)\leqq Y_+(t)$ for small $t>0$, while for large $u>0$, $g_+(u)/2\leqq g(u)\leqq g_-(u)\,p^{\frac{1}{2}}$ or equivalently $g_-^{-1}(p^{-\frac{1}{2}}\,u)\leqq g^{-1}(u)\leqq g_+^{-1}(2u)$. Hence

$$a\leqq\liminf_{t\to 0} Y_+(t)\, g_+^{-1}(2p\,t^{-1}\, l_2\, t)/l_2\, t<\infty,$$

$$a\geqq\liminf_{t\to 0} Y_-(t)\, g_-^{-1}(p^{\frac{1}{2}}\, t^{-1}\, l_2\, t)/l_2\, t>0$$

a.s. by Th. 1 in [10]. As for (ii), the same method works except that we have to change a finite (but random) number of jump sizes to make Y_+-X and $X-Y_-$ everywhere monotone. Clearly, such changes do not effect the events in Lemma 2 of [11].

3. Hausdorff Measure of the Range

We first extend a result by Blumenthal, Getoor [3] and Millar [17]. Write dim for Hausdorff dimension.

Theorem 3.1. *If $\sigma=0$, and if $\alpha=\beta h$ whenever $\beta\,|h|<\infty$, then*

$$\dim X(A)\leqq \rho_X \dim A \qquad a.s. \tag{3.1}$$

for any Borel set $A\subset[0,1]$.

Proof. Let (α, β) be non-random and put $d = \dim A$. Assume first that $d < 1$ and choose arbitrary $a \in (d, 1]$, $c > a\,\rho_X$. For $n \in N$, cover A by intervals I_{nj}, $j = 1, \ldots, r_n$, such that $\sum_j |I_{nj}|^a < n^{-1}$. If we can show that

$$\sum_j |X(I_{nj})|^c \xrightarrow{P} 0, \tag{3.2}$$

then (3.1) will follow by turning to some suitable sub-sequence. Now $X = X_1 + X_2$ implies

$$\begin{aligned}\sum_j |X(I_{nj})|^c &\leqq \sum_j \{|X_1(I_{nj})| + |X_2(I_{nj})|\}^c \\ &\leqq 2^c \{\sum_j |X_1(I_{nj})|^c + \sum_j |X_2(I_{nj})|^c\},\end{aligned} \tag{3.3}$$

so from the proof of Th. 5.1 in [17] and from Th. 1.1 above it is seen that the assertion may reduced successively, first to the case $\rho_X \leqq 1$, then to the case $\beta\,|h| < \infty$ and finally to the case of non-decreasing pr. But for the latter, (3.2) follows from [17] and Th. 1.2 above.

Next suppose that $d = 1$, assume without loss that $A = [0, 1)$, and let $c > \rho_X$. We intend to show that there exists for any $\varepsilon > 0$ some partition of A into (random) intervals I_j such that $\sum_j |X(I'_j)|^c < \varepsilon$ for any refinement $\{I'_j\}$ of $\{I_j\}$. Now this is true for pr. with ind. incr. according to the proof of Th. 5.1 in [17], and the method of successive reductions works as above, provided we replace $\{I_{nj}\}$ in (3.3) by a common refinement of the partitions used for X_1 and X_2.

We next extend Th. 3 of Fristedt and Pruitt [10]. Let g be given by (2.2) and define f_p for $p > 0$ as the inverse (near 0) of the function $t \to l_2\, t / g^{-1}(p\, t^{-1}\, l_2\, t)$. Put $f = f_1$. Write H for Hausdorff measure with respect to f.

Theorem 3.2. *Let α, σ, β be non-random with $\sigma = \beta R_- = 0$, $\alpha = \beta h < \infty$, $\beta R = \infty$. Then $H(X[0, t]) \equiv c\,t$ a.s. for some non-random $c \in R'_+$.*

Proof. By the Hewitt-Savage 0-1 law, $H(X[0, t]) = a(t)$ is a.s. non-random (possibly 0 or ∞) for each $t \in [0, 1]$, and since a is non-decreasing, this holds a.s. for all t simultaneously. To show that $a(t) \in R'_+$, $t \in (0, 1]$, consider the pr. $Y_\pm$ of Th. 1.2 and change the largest jump sizes if necessary to make $Y_+ - X$ and $X - Y_-$ monotone. (This will not effect the Hausdorff measures.) Then check that Th. 3 in [10] remains true with f_p in place of f and carry through the comparison as in the proof of Th. 2.3. To see that a is linear, note that by definition

$$a(s-) + a(t-) \leqq a(s+t) \leqq a(s) + a(t), \qquad s, t > 0,\ s + t \leqq 1, \tag{3.4}$$

so $a(s+t-) \equiv a(s-) + a(t-)$, proving linearity on $[0, 1)$. For $t = 1$ use (3.4).

4. Variation of Independent Increment Processes

Throughout §§ 4–5, let $\Pi, \Pi_1, \Pi_2, \ldots$ be finite partitions of $[0, 1]$. In § 4 only, let X be a pr. in $D_0[0, 1]$ with (not necessarily stationary) ind. incr. and without fixed jumps. For $\varepsilon > 0$, define $\hat{\gamma}_\varepsilon$: $[0, 1] \to R$, $\hat{\sigma}^2 \in \mathfrak{M}([0, 1])$ and $\hat{\lambda} \in \mathfrak{M}([0, 1] \times R')$ by the Lévy formula

$$\log \mathsf{E}\, e^{iuX(t)} \equiv i\,u\,\hat{\gamma}_\varepsilon(t) - \tfrac{1}{2} u^2\, \hat{\sigma}^2[0, t] + \int_{R'} (e^{iux} - 1 - i\,u\,h_\varepsilon(x))\, \hat{\lambda}([0, t] \times dx),$$

and put $\sigma^2 = \hat{\sigma}^2[0, 1]$, $\lambda = \hat{\lambda}([0, 1] \times \cdot)$. Let $\beta_1, \beta_2, \ldots$ be the jump sizes of X and write $\beta = \sum_j \delta_{\beta_j}$.

Theorem 4.1. *Let $\{\Pi_n\}$ be nested with $|\Pi_n|_\infty \to 0$, and suppose that $(\Pi_n \hat{\gamma}_\varepsilon)^2 R \to 0$ for some $\varepsilon > 0$. Then*

$$(\Pi_n X)^2 \xrightarrow{w} \sigma^2 \delta_0 + \beta^2 \qquad a.s. \text{ in } \mathfrak{M}(R). \tag{4.1}$$

This improves slightly a result by Cogburn and Tucker [5]. By taking account of "time", we may easily restate this and the remaining theorems of §§ 4–5 in terms of convergence in $\mathfrak{M}([0, 1] \times \mathrm{R})$.

Proof. By Minkowski's inequality, we may assume that $\hat{\gamma}_\varepsilon = 0$, and then

$$(\Pi_n X)^2 R \to \sigma^2 + \beta^2 R \qquad \text{a.s.} \tag{4.2}$$

by [5]. To complete the proof, note that for any process X in $D[0, 1]$,

$$\Pi_n X \xrightarrow{v} \beta \qquad \text{a.s. in } \mathfrak{N}(R'). \tag{4.3}$$

We next improve and extend results by Fristedt [8] and Millar [17].

Theorem 4.2. *Let $\sigma^2 = 0$ and $|\Pi_n|_\infty \to 0$, and suppose that either*

(i) *$\{\Pi_n\}$ is nested, $f \in \mathscr{C}_2'$, $f\lambda \in \mathfrak{M}(R)$, $(\Pi_n \hat{\gamma}_\varepsilon) f \to 0$ for some $\varepsilon > 0$, or*

(ii) *$\hat{\gamma}_0 = 0$, $f \in \mathscr{C}_1$, $f\lambda \in \mathfrak{M}(R)$.*

Then $\beta f < \infty$ a.s. and

$$f(\Pi_n X) \xrightarrow{w} f\beta \qquad a.s. \text{ in } \mathfrak{M}(R).$$

Proof. To see that $\beta f < \infty$ a.s., note that β is a Poisson pr. on R' with intensity λ, and conclude by monotone convergence from simple functions that $\mathsf{E}\beta f = \lambda f < \infty$.

Assuming (i), it suffices by (4.3) to prove that

$$\liminf_{u \to 0} \limsup_{n \to \infty} (\Pi_n X_u)\, \mathrm{f} = 0 \qquad \text{a.s..} \tag{4.4}$$

where X_u is obtained from X by deleting all jumps of modulus $> u$. We may therefore assume the properties defining $\mathscr{C}_2'$ to hold on all R. In particular, by subadditivity and Minkowski's inequality, for some $c \in (0, 1)$ and any $x_j, y_j, j \in N$,

$$\begin{aligned}\{\sum_j f(x_j + y_j)\}^c &= \{\sum_j [f^c(x_j + y_j)]^{1/c}\}^c \leqq \{\sum_j [f^c(x_j) + f^c(y_j)]^{1/c}\}^c \\ &\leqq \{\sum_j f(x_j)\}^c + \{\sum_j f(y_j)\}^c,\end{aligned}$$

i.e.

$$|\{\sum_j f(x_j)\}^c - \{\sum_j f(y_j)\}^c| \leqq \{\sum_j f(x_j - y_j)\}^c. \tag{4.5}$$

Since $(\Pi_n \hat{\gamma}_\varepsilon) f \to 0$, we may therefore assume that $\hat{\gamma}_\varepsilon = 0$, and since $\hat{\gamma}_\varepsilon - \hat{\gamma}_u$ is continuous and of bounded variation, implying

$$(\Pi_n(\hat{\gamma}_\varepsilon - \hat{\gamma}_u)) f = o[(\Pi_n(\hat{\gamma}_\varepsilon - \hat{\gamma}_u))\, |h|] \to 0 \qquad \text{as } n \to \infty,\ u > 0,$$

it suffices to prove (4.4) with X_u replaced by $Y_u = X_u - \hat{\gamma}_u$. For symmetric Y_u, the arguments leading to Lemma 4.4 in [17] yield $\mathsf{E}(\Pi_n Y_u) f \leqq \lambda_u f$, where $\lambda_u = 1_{I_u} \lambda$.

Since $\{(\Pi_n Y_u) f\}$ is a reversed supermartingale by Lemma 3.1 in [17], we get by Doob's maximal inequality

$$\mathsf{P}\{\sup_n (\Pi_n Y_u) f > a\} \leqq \sup_n \mathsf{E}(\Pi_n Y_u) f/a \leqq \lambda_u f/a, \qquad a>0.$$

Proceeding as in [17], p. 60, and applying (4.5), we obtain for nonsymmetric Y_u

$$\mathsf{P}\{\sup_n |[(\Pi_n Y_u) f]^c - m_{nu}^c| > a^c\} \leqq 4\lambda_u f/a, \qquad a>0, \tag{4.6}$$

where m_{nu} is a median of $(\Pi_n Y_u) f$. Now it follows as in the proof of Th. 4.1 in [17] that $\mathsf{E}(\Pi_n Y_u) f \leqq 2\lambda_u f$, so

$$\tfrac{1}{2} \leqq \mathsf{P}\{(\Pi_n Y_u) f \geqq m_{nu}\} \leqq \mathsf{E}(\Pi_n Y_u) f/m_{nu} \leqq 2\lambda_u f/m_{nu},$$

and we get $m_{nu} \leqq 4\lambda_u f$. Thus by (4.6), $\sup_n (\Pi_n Y_u) f \xrightarrow{P} 0$ as $u \to 0$, and so (4.4) holds for Y_u and place of X_u.

In case of (ii), we may assume by (4.3) that f is concave on R_+ and R_-. But then $(\Pi_n X) f \leqq \beta f$ by monotonity and concavity, while $\liminf_{n\to\infty} (\Pi_n X) f \geqq \beta f$ since $X \in D[0,1]$, so we get $(\Pi_n X) f \to \beta f$, which by (4.3) completes the proof. (Note that the last proof applies to any pure jump process with bounded variation.)

5. Variation of Interchangeable Increment Processes

In this section, let X be an ich. incr. pr. in $D_0[0,1]$ with can. r.e. α, σ, β.

Theorem 5.1. *Relation* (4.1) *holds if either*

(i) $\{\Pi_n\}$ *is nested with* $|\Pi_n|_\infty \to 0$, *or*

(ii) $\sum_n |\Pi_n|_2^2 < \infty$.

Part (ii) extends and improves results by Rubin and Tucker [21] (who assume $\sum_n |\Pi_n|_\infty^{\frac{1}{3}} < \infty$) and by Millar [18], p. 324. Results by Blumenthal and Getoor [2] show that $|\Pi_n|_\infty \to 0$ is not sufficient in general.

Proof. In case of (i) it suffices by (4.3) to show that

$$\limsup_{n\to\infty} |(\Pi_n X_u)^2 R - \sigma^2| \xrightarrow{P} 0, \qquad u \to 0, \tag{5.1}$$

where X_u is defined as in the proof of Th. 4.2. Using Minkowski's inequality and applying Th. 1.1 and 4.1, the proof of (5.1) may easily be reduced to the case $\sigma = 0$ and $\rho_X \leqq 1$. Repeating the same argument twice with Th. 1.2 in place of Th. 1.1 completes the proof. As for (ii), the assertion follows by (4.3), Minkowski's inequality and Fubini's theorem from the following lemma.

Lemma 5.1. *For non-random* $(\sigma, \beta) \neq 0$ *and* $\alpha = 0$,

$$\mathsf{E}\left\{\frac{(\Pi X)^2 R}{\sigma^2 + \beta^2 R} - 1\right\}^2 = O(|\Pi|_2^2). \tag{5.2}$$

Proof. Abbreviate $\beta_0 = \sigma$, $v = \sigma^2 + \beta^2 R$, $p_j = t_j - t_{j-1}$, $\xi_j = X(t_j) - X(t_{j-1})$, $\eta_{j0} = B(t_j) - B(t_{j-1})$, $\eta_{jk} = 1_+(t_j - \tau_k) - 1_+(t_{j-1} - \tau_k) - p_j$, and write $\sum$ and $\sum'$ for summation over Z_+ and N respectively. By the fact that, for jointly Gaussian variables $\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4$ with zero mean,

$$\mathsf{E}\,\vartheta_1\vartheta_2\vartheta_3\vartheta_4 = \mathsf{E}\,\vartheta_1\vartheta_2\,\mathsf{E}\,\vartheta_3\vartheta_4 + \mathsf{E}\,\vartheta_1\vartheta_3\,\mathsf{E}\,\vartheta_2\vartheta_4 + \mathsf{E}\,\vartheta_1\vartheta_4\,\mathsf{E}\,\vartheta_2\vartheta_3,$$

and by the formulae in [1], pp. 65, 107, we get for $i \neq j$, $k \neq l$,

$$\begin{aligned}
&\mathsf{E}\,\eta_{jk}^2 = p_j(1-p_j), \qquad \mathsf{E}\,\eta_{ik}\eta_{jk} = -p_i p_j,\\
&\mathsf{Var}\,\eta_{j0}^2 = 2(\mathsf{E}\,\eta_{j0}^2)^2 = O(p_j^2), \qquad \mathsf{Cov}(\eta_{i0}^2, \eta_{j0}^2) = 2(\mathsf{E}\,\eta_{i0}\eta_{j0})^2 = 2p_i^2 p_j^2,\\
&\mathsf{Var}\,\eta_{jk}^2 = p_j(1-p_j)^4 + (1-p_j)p_j^4 - p_j^2(1-p_j)^2 = p_j + O(p_j^2), \qquad k \neq 0,\\
&\mathsf{Cov}(\eta_{ik}^2, \eta_{jk}^2) = p_i(1-p_i)^2 p_j^2 + p_i^2 p_j(1-p_j)^2 + (1-p_i-p_j)p_i^2 p_j^2\\
&\qquad\qquad -p_i p_j(1-p_i)(1-p_j) = -p_i p_j(1 + O(p_i+p_j)), \qquad k \neq 0,\\
&\mathsf{Var}\,\eta_{jk}\eta_{jl} = p_j^2(1-p_j)^2 = O(p_j^2), \qquad \mathsf{Cov}(\eta_{ik}\eta_{il}, \eta_{jk}\eta_{jl}) = p_i^2 p_j^2.
\end{aligned}$$

Hence by independence, if β has only finitely many atoms,

$$\begin{aligned}
\mathsf{E}\,\xi_j^2 &= \mathsf{Var}\sum_k \beta_k \eta_{jk} = \sum_k \beta_k^2\,\mathsf{Var}\,\eta_{jk} = v\,p_j(1-p_j),\\
\mathsf{Var}\,\xi_j^2 &= \mathsf{Var}\sum_k \beta_k^2 \eta_{jk}^2 + \sum_{k\neq l}\sum \beta_k\beta_l \eta_{jk}\eta_{jl}\\
&= \sum_k \beta_k^4\,\mathsf{Var}\,\eta_{jk}^2 + 2\sum_{k\neq l}\sum \beta_k^2\beta_l^2\,\mathsf{Var}\,\eta_{jk}\eta_{jl} = p_j \sum_k{}' \beta_k^4 + v^2 O(p_j^2),\\
\mathsf{Cov}(\xi_i^2, \xi_j^2) &= \mathsf{Cov}\Big\{\sum_k \beta_k^2\eta_{ik}^2 + \sum_{k\neq l}\sum \beta_k\beta_l\eta_{ik}\eta_{il},\ \sum_k \beta_k^2\eta_{jk}^2 + \sum_{k\neq l}\sum \beta_k\beta_l\eta_{jk}\eta_{jl}\Big\}\\
&= \sum_k \beta_k^4\,\mathsf{Cov}(\eta_{ik}^2, \eta_{jk}^2) + 2\sum_{k\neq l}\sum \beta_k^2\beta_l^2\,\mathsf{Cov}(\eta_{ik}\eta_{il}, \eta_{jk}\eta_{jl})\\
&= -p_i p_j \sum_k{}' \beta_k^4 + v^2 p_i p_j O(p_i + p_j),
\end{aligned}$$

so

$$\begin{aligned}
\mathsf{E}(\Pi X)^2 R &= \sum_j \mathsf{E}\,\xi_j^2 = v\sum_j p_j(1-p_j) = v(1 - |\Pi|_2^2),\\
\mathsf{Var}(\Pi X)^2 R &= \sum_j \mathsf{Var}\,\xi_j^2 + \sum_{i\neq j}\sum \mathsf{Cov}(\xi_i^2, \xi_j^2)\\
&= \sum_k{}' \beta_k^4 \Big\{\sum_j p_j - \sum_{i\neq j}\sum p_i p_j\Big\} + v^2 O\Big\{\sum_j p_j^2 + \sum_{i\neq j}\sum p_i p_j(p_i+p_j)\Big\}\\
&= \sum_k{}' \beta_k^4 |\Pi|_2^2 + v^2 O(|\Pi|_2^2) = v^2 O(|\Pi|_2^2),
\end{aligned}$$

and (5.2) follows. In case of infinitely many β-atoms, apply (5.2) with β replaced by $\beta_\varepsilon = 1_{I_\varepsilon^c}\beta$, $\varepsilon > 0$, and X by the corresponding pr. X_ε. Since $\beta_\varepsilon^2 R \to \beta^2 R$ and $(\Pi X_\varepsilon)^2 R \to (\Pi X)^2 R$ a.s. as $\varepsilon \to 0$, the truth of (5.2) for β and X follows by Fatou's lemma.

Let us finally point out that Th. 4.2 as well as the results by Blumenthal, Getoor [3] and Monroe [19] on "strong" variation generalize in an obvious way to ich. incr. pr. The details are omitted.

6. Ergodicity and Variation

Th. 5.1 may be considered as the "strong" counterpart to the "weak" Th. 2.2 in [14]. In this section we proceed by giving strong analogues to Th. 1.2, 3.2, 4.2 and 4.1 in [14].

Theorem 6.1. *Let $\xi_1, \xi_2, \ldots$ be ich.r.e. with can.r.m. μ in a lcscH space S, and let π_n be the can.p.pr. of $\xi_1, \ldots, \xi_n$, $n \in N$. Then for any measurable $f: S \to R_+$,*

$$f\pi_n/n \xrightarrow{\mathrm{w}} f\mu \quad \text{a.s. within } \{\mu f < \infty\}.$$

This is an obvious extension of a result by Varadarajan (for $f=1$, see Th. 7.1 in [20]; see also [16], p. 400, and [4]).

For the remainder of this section, let X be an ich.incr.pr. in $D_0[0, \infty)$ with can.r.e. Γ, Λ (or $\gamma_\varepsilon, \sigma, \lambda$). The next two theorems improve and extend results by Rubin and Tucker [21].

Theorem 6.2. *For suitable versions of the can.r.m. μ_p of*

$$\{X(jp) - X((j-1)p) : j \in N\}, \quad p > 0,$$

we have as $p \to 0$

$$\mu_p g_1/p \to \Gamma, \quad g_2 \mu_p/p \xrightarrow{\mathrm{w}} \Lambda \quad \text{a.s.} \tag{6.1}$$

Proof. Let Γ, Λ be non-random. Then (6.1) holds as $p \to 0$ through any sequence by [6], p. 564, and the assertion follows.

Theorem 6.3. *For suitable versions of the can.r.e. $(\alpha_s, \sigma_s, \beta_s)$ of the restrictions of X to $[0, s]$, $s > 0$, we have $\sigma_s^2/s \equiv \sigma^2$, and as $s \to \infty$, for any measurable $f: R \to R_+$,*

$$f(\sigma_s^2 \delta_0 + g_2 \beta_s)/s \xrightarrow{\mathrm{w}} f\Lambda \quad \text{a.s. within } \{\Lambda f < \infty\}, \tag{6.2}$$

$$[\alpha_s - \beta_s(h - g_1)]/s \to \Gamma \quad \text{a.s.} \tag{6.3}$$

Note that the last assertions may also be written

$$f\beta_s/s \xrightarrow{\mathrm{w}} f\lambda \quad \text{a.s. within } \{\lambda f < \infty\}, \tag{6.4}$$

$$(\alpha_s - \beta_s^1 I_\varepsilon^c)/s \to \gamma_\varepsilon, \quad \varepsilon > 0, \quad \text{a.s.} \tag{6.5}$$

Proof. Let Γ and Λ be non-random. By Th. 5.1 in [14], it is possible to *define* $\sigma_s^2 \equiv s\sigma^2$, and we may further take $\alpha_s \equiv X(s)$ and let the β_s be given by a Poisson pr. on $R' \times R_+$ with intensity λ times Lebesgue measure. Then $\{\beta_s f\}$ is a pr. in $D_0[0, \infty)$ with stationary ind.incr., and so $\beta_s f/s \to \mathsf{E}\,\beta_1 f = \lambda f$ a.s. by the law of large numbers ([16], p. 558). By [20], Th. 6.6, this implies (6.4). Finally, (6.5) is merely a restatement of the law of large numbers for the pr. X_ε obtained from X by omitting jumps of modulus $> \varepsilon$.

Taking a statistical point of view, suppose we want to estimate Γ, Λ from the incr. of X. As discussed at length by Rubin and Tucker [21], we may then proceed in two steps, using *either* Th. 6.1–6.2 *or* Th. 5.1 and 6.3. An estimate in one step is given by the following ergodic-variational result.

Theorem 6.4. *For $n \in N$, let $c_n = s_n^{-1} > 0$ and let Π_n be a partition of $[0, s_n]$ into k_n intervals of length p_{nj}, $j = 1, \ldots, k_n$. Let $s_n \to \infty$ and suppose that*

(i) $c_n |\Pi_n|_2^2 \to 0$,

(ii) $\sum_n c_n^2 |\Pi_n|_2^2 < \infty$,

(iii) $\sum_n c_n \sum_n p_{n_j} e^{-\varepsilon/p_{nj}} < \infty, \ \varepsilon > 0.$

Then

$$c_n (\Pi_n X) g_1 \longrightarrow \Gamma \quad a.s., \tag{6.6}$$

$$c_n g_2 (\Pi_n X) \xrightarrow{\mathrm{w}} \Lambda \quad a.s. \tag{6.7}$$

Note that, if $p_{nj} = p_n$, $j = 1, \ldots, k_n$, $n \in N$, then (i)–(iii) reduce to

$$\sum_n k_n^{-1} < \infty, \qquad \sum_n e^{-\varepsilon/p_n} < \infty, \quad \varepsilon > 0.$$

As the proof will reveal, (i) and (ii) suffice if $\sigma = 0$, while (ii) and (iii) suffice if $(\gamma_0, \lambda) = 0$. Moreover, (i) and (ii) imply

$$c_n (\Pi_n X)^2 R \to \sigma^2 + \lambda^2 R \quad \text{a.s.}, \tag{6.8}$$

provided λ has bounded support, (or even provided $\lambda^4 R \equiv \lambda h^4 < \infty$).

Proof. We may assume Γ, Λ to be non-random. Suppose that λ has support in $[-M, M]$. If $\alpha_n, \sigma s_n^{\frac{1}{2}}, \beta_n$ are the can. r.e. of the restriction of X to $[0, s_n]$, we get by Lemma 5.1

$$\mathsf{E}\{c_n [\Pi_n (X - c_n \alpha_n h)]^2 R - (\sigma^2 + c_n \beta_n^2 R)\}^2 = \mathsf{E}(\sigma^2 + c_n \beta_n^2 R)^2 \, O(c_n^2 \, |\Pi_n|_2^2).$$

Using an auxiliary result in [17], p. 56, we obtain

$$\mathsf{E}(\sigma^2 + c_n \beta_n^2 R)^2 \leqq 2\sigma^4 + 2c_n^2 \, \mathsf{E}(\beta_n^2 R)^2 \leqq 2\sigma^4 + 8 c_n \lambda^4 R + 4(\lambda^2 R)^2,$$

so by (ii) and Fubini's theorem,

$$c_n [\Pi_n (X - c_n \alpha_n h)]^2 R - (\sigma^2 + c_n \beta_n^2 R) \to 0 \quad \text{a.s.}$$

By (6.4), $c_n \beta_n^2 R$ may be replaced by $\lambda^2 R$, and furthermore,

$$(c_n \alpha_n)^2 c_n (\Pi_n h)^2 R \to \gamma_\infty^2 c_n |\Pi_n|_2^2 \to 0 \quad \text{a.s.}$$

by (6.5) and (i), so (6.8) follows by Minkowski's inequality.

Assuming $\lambda R < \infty$, $\gamma_0 = 0$, write $X = X_1 + X_2$, where X_1 is the purely discrete part of X and let $\{\xi_{nj}\}$ and $\{\eta_{nj}\}$ be the Π_n-incr. of X_1, X_2. Let $Y(s)$ be the number of X_1-jumps in $[0, s]$, $s > 0$, and note that Y is a Poisson pr. with intensity λR. Now

$$c_n (\Pi_n Y)^2 N \geqq c_n Y(s_n) + \tfrac{1}{2} c_n (\Pi_n Y)^2 (N \setminus 1), \qquad n \in N,$$

and letting $n \to \infty$ it follows by (6.8) and the law of large numbers that

$$c_n (\Pi_n Y)^2 (N \setminus 1) \to 0 \quad \text{a.s.}$$

Hence for any interval I

$$c_n |(\Pi_n X_1)^2 I - \beta_n^2 I| \leqq M^2 c_n (\Pi_n Y)^2 (N \setminus 1) \to 0,$$

so by (6.4),

$$c_n (\Pi_n X_1)^2 \xrightarrow{\mathrm{w}} \lambda^2 \quad \text{a.s.} \tag{6.9}$$

As for X_2, note that for any $\varepsilon > 0$

$$\mathsf{E}(\Pi_n X_2)^2 I_\varepsilon^c = \sum_j \mathsf{E}[\eta_{nj}^2; |\eta_{nj}| > \varepsilon] = \sigma^2 \sum_j p_{nj} \, \mathsf{E}[\vartheta^2; |\vartheta| > \varepsilon \sigma^{-1} p_{nj}^{-\frac{1}{2}}], \tag{6.10}$$

where ϑ is $N(0,1)$. But

$$\mathsf{E}[\vartheta^2; |\vartheta|>\varepsilon p^{-\frac{1}{2}}]\to 1, \qquad p\to\infty,$$

$$\mathsf{E}[\vartheta^2; |\vartheta|>\varepsilon p^{-\frac{1}{2}}]\sim\varepsilon(\pi p/2)^{-\frac{1}{2}}\exp(-\tfrac{1}{2}\varepsilon^2/p)=o[\exp(-\tfrac{1}{4}\varepsilon^2/p)], \quad p\to 0,$$

so

$$\sup_{p>0}\mathsf{E}[\vartheta^2; |\vartheta|>\varepsilon p^{-\frac{1}{2}}]\exp(\tfrac{1}{4}\varepsilon^2/p)<\infty,$$

and hence by (6.8), (6.10), (iii) and Fubini's theorem

$$c_n(\Pi_n X_2)^2 \xrightarrow{w} \sigma^2\delta_0 \quad \text{a.s.} \tag{6.11}$$

Returning to X, fix u, $v>0$ and let $\varepsilon\in(0,u\wedge v)$. Abbreviate $A=(-\infty,-u]\cup[v,\infty)$, $A_\varepsilon=(-\infty,-u+\varepsilon]\cup[v-\varepsilon,\infty)$. My Minkowski's inequality,

$$\begin{aligned}
[(\Pi_n X)^2 A]^{\frac{1}{2}} &= [\sum_j\{(\xi_{nj}+\eta_{nj})^2; \xi_{nj}+\eta_{nj}\in A\}]^{\frac{1}{2}}\\
&\leqq [\sum_j\{\xi_{nj}^2; \xi_{nj}\in A_\varepsilon\}]^{\frac{1}{2}}+[\sum_j\{\eta_{nj}^2; \xi_{nj}\in A_\varepsilon, \eta_{nj}\in I_\varepsilon\}]^{\frac{1}{2}}\\
&\quad +[\sum_j\{\xi_{nj}^2; \xi_{nj}\in A_\varepsilon^c, \eta_{nj}\in I_\varepsilon^c\}]^{\frac{1}{2}}+[\sum_j\{\eta_{nj}^2; \eta_{nj}\in I_\varepsilon^c\}]^{\frac{1}{2}}\\
&\leqq [(\Pi_n X_1)^2 A_\varepsilon]^{\frac{1}{2}}+\varepsilon[(\Pi_n X_1) A_\varepsilon]^{\frac{1}{2}}+(u\vee v-\varepsilon)[(\Pi_n X_2) I_\varepsilon^c]^{\frac{1}{2}}\\
&\quad +[(\Pi_n X_2)^2 I_\varepsilon^c]^{\frac{1}{2}}.
\end{aligned}$$

Multiplying by $c_n^{\frac{1}{2}}$, and the letting $n\to\infty$ and $\varepsilon\to 0$, we obtain

$$\limsup_{n\to\infty} c_n(\Pi_n X)^2 A\leqq\lambda^2 A \quad \text{a.s.} \tag{6.12}$$

by (6.9) and (6.11). Similarly,

$$\liminf_{n\to\infty} c_n(\Pi_n X)^2 A^\circ\geqq\lambda^2 A^\circ \quad \text{a.s.,} \tag{6.13}$$

and by comination of (6.8) with (6.12) and (6.13) for arbitrary u and v,

$$c_n(\Pi_n X)^2 \xrightarrow{w} \sigma^2\delta_0+\lambda^2 \quad \text{a.s.} \tag{6.14}$$

The restrictions $\lambda R<\infty$ and $\gamma_0=0$ may be removed by (6.8) and Minkowski's inequality.

Now (6.14) implies (6.7). To prove (6.6), conclude from the law of large numbers that $c_n(\Pi_n X)^1 R\to\gamma_\infty$ and use (6.14). Finally, for λ with arbitrary support, write $X=X'+X''$ where X'' contains the jumps of modulus $>M$. By the law of large numbers

$$c_n|g_k(\Pi_n X) I-g_k(\Pi_n X') I|\leqq 2c_n(\Pi_n X'') R'\leqq 2c_n\beta_n I_M^c\to 2\lambda I_M^c \quad \text{a.s.,} \qquad k=1,2,$$

for any interval I, so it suffices to apply (6.6) and (6.7) to X' and then let $M\to\infty$.

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