Z. Wahrscheinlichkeitstheorie verw. Gebiete
40, 157–158 (1977)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

Lévy's Downcrossing Theorem

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Let $\{W(t): t \geqq 0\}$ be the standard Wiener process (Brownian motion starting at 0) with all paths continuous. Put

$$M^W(t) = \max_{0 \leqq s \leqq t} W(s), \qquad Y(t) = M^W(t) - W(t).$$

For $\varepsilon > 0$ and $t > 0$, let $d_\varepsilon(t)$ be the number of times that Y crosses down from ε to 0 by time t. Thus $d_\varepsilon(t)$ is the maximum integer n for which we can find points $0 < s_1 < t_1 < s_2 < t_2 < \cdots < s_n < t_n \leqq t$ for which $Y(s_k) = \varepsilon (\forall k)$ and $Y(t_k) = 0 (\forall k)$.

Lévy's Downcrossing Theorem

$$P\{\lim_{\varepsilon \downarrow 0} \varepsilon d_\varepsilon(t) = M^W(t), \forall t \geqq 0\} = 1.$$

Chung and Durrett ([1]) recently published a new proof of this result. They based their argument on a rather involved result on Brownian excursions, and they also used one of Jacobi's theta-function identities. Their interesting proof is therefore not particularly simple.

We now give a proof which is totally elementary, which requires no calculation, which does not even require the reader to know that Y is a reflecting Brownian motion (with M^W as its local time), and which gives a more complete explanation. All of the credit goes to Itô who (in [2]) did it all in a much more general context. For further information on M^W and Y, see [3].

For $a \geqq 0$, define

$$T_a^W = \inf\{t: W(t) = a\} = \inf\{t: M^W(t) = a\},$$
$$T_a^Y = \inf\{t: Y(t) = a\} = \inf\{t: M^Y(t) = a\},$$

where $M^Y(t) = \max_{0 \leqq s \leqq t} Y(s)$. Since $Y(T_a^W) = 0$, the following implications are trivially valid for $0 < b < a$:

$$\begin{aligned}[d_\varepsilon(T_a^W) = 0] &\Leftrightarrow [M^Y(T_a^W) < \varepsilon] \Leftrightarrow [T_a^W < T_\varepsilon^Y] \\ &\Rightarrow [M^W(T_\varepsilon^Y) \geqq a] \Rightarrow [T_b^W < T_\varepsilon^Y] \Leftrightarrow [M^Y(T_b^W) < \varepsilon]. \qquad (1)\end{aligned}$$

Lemma. *For every* $\varepsilon>0$, $\{d_\varepsilon(T_a^W):\ a\geqq 0\}$ *is a Poisson process of rate* ε^{-1}.

Proof of Lemma. Fix $\varepsilon>0$ throughout this proof. Pick $x>0$, $y>0$. Since $Y(T_x^W)=0$, the strong Markov property (of W) applied at time T_x^W makes it clear that

$$P[M^Y(T_{x+y}^W)<\varepsilon]=P[M^Y(T_x^W)<\varepsilon]\,P[M^Y(T_y^W)<\varepsilon],$$

so that for some $\beta(\varepsilon)>0$,

$$P[M^Y(T_a^W)<\varepsilon]=e^{-a\beta(\varepsilon)} \qquad (\forall\, a\geqq 0). \tag{2}$$

It now follows from (1) (let $b\uparrow a$) that for $a>0$,

$$P[d_\varepsilon(T_a^W)=0]=P[M^W(T_\varepsilon^Y)\geqq a]=e^{-a\beta(\varepsilon)},$$

so that $M^W(T_\varepsilon^Y)$ is exponentially distributed with mean $\beta(\varepsilon)^{-1}$.

Since $M^W-Y=W$ is a *martingale*, we have

$$EM^W(t\wedge T_\varepsilon^Y)=EY(t\wedge T_\varepsilon^Y) \qquad (\forall\, t\geqq 0).$$

The function $Y(t\wedge T_\varepsilon^Y)$ is bounded by ε and converges to ε as $t\uparrow\infty$. The function $M^W(t\wedge T_\varepsilon^Y)$ is monotonically increasing in t with limit $M^W(T_\varepsilon^Y)$. Hence $EM^W(T_\varepsilon^Y)=\varepsilon$, and $\beta(\varepsilon)$ is identified as ε^{-1}.

An obvious further use of the strong Markov property now completes the proof of the lemma.

Note. Since $\{c^{-1}W(c^2t):\ t\geqq 0\}$ is a Wiener process for every $c>0$, it follows directly from (2) that $\beta(\varepsilon)=\beta(1)\,\varepsilon^{-1}$.

Since $d_\varepsilon(T_x^W)$ is Poisson distributed with mean $x\varepsilon^{-1}$ and is monotonic in ε, it is easy to show (just from Čebyšev's inequality!) that, with probability 1,

$$\lim_{\varepsilon\downarrow 0}\varepsilon d_\varepsilon(T_x^W)=x,$$

first for each fixed x, and then (by monotonicity) simultaneously for all x. Lévy's theorem is proved.

Acknowledgement. The original version of this paper appealed to (the reader's) pictures for "proof". The referee persuaded me to tighten things up. The paper is thereby much improved; and since the reader can still draw pictures, nothing is lost.

References

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Received February 18, 1977; in revised form April 7, 1977